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CONTROLLABILITY OF MULTIDIMENSIONAL RIGID BODY

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Abstract
We study global controllability of 'rotating' multidimensional rigid body (MRB) controlled by application of few torques. Study by methods of geometric control leads to analysis of algebraic structure introduced by the quadratic term of Euler-Frahm equation. We discuss problems, which arise in the course of this analysis, suggest solutions for some of them and establish several controllability criteria for damped and non damped cases.

Key words
controllability, multidimensional rigid body, geometric nonlinear control, Lie rank criteria, Navier-Stokes equation

1 Introduction

In recent work [Agrachev and Sarychev, 2005; Agrachev and Sarychev, 2008; Rodrigues, 2006] one proceeded with study of controllability of Navier-Stokes (NS) equation on a 2D domain controlled by means of low-dimensional forcing. Approximate controllability criteria have been obtained for torus, sphere, hemisphere, rectangle and generic Riemannian surface with boundary. To achieve this goal geometric control approach has been employed. According to it one starts with original system controlled by low-dimensional control and constructs a sequence of *Lie extensions* which add to the original system new controlled vector fields. These latter are calculated via iterated Lie-Poisson brackets of the drift vector field (corresponding to zero control), and the controlled vector fields.

In above cited publications one started with small number of original constant controlled vector fields, whose values belong to the set of 'equilibrium points' of Euler equation, representing so-called 'steady flows' ([Arnold and Khesin, 1998]). Computation of double Lie bracket of drift vector field with two such constant controlled vector fields amounts to a new constant field (direction). This defines bilinear operator on the algebra of flows. Extending controlled directions are obtained by iterated application of this operator to small set of original controlled directions. At each step we must guarantee expansion of set of the extending controlled directions.

Tracing these iterations is by no means easy. All cases analyzed in [Agrachev and Sarychev, 2005; Agrachev and Sarychev, 2008] are related to an explicit description of the set of steady flows - a basis in the space of flows - and to specific representation of the above mentioned bilinear operator with respect to this basis. The results so obtained are heavily dependent on choice of initial controlled directions and on geometry of the domain of NS system. In particular the method does not allow to conclude structural stability of controllability property with respect to perturbations of controlled directions and/or domain.

In the present contribution we address controllability issues for a finite-dimensional "relative" of NS system - Euler-Frahm equation for rotation of multidimensional rigid body (MRB) subject to few controlling torques and to possible damping. In particular we investigate how the inertia operator of MRB (which stays for the domain in NS model) and the choice of controlled directions influence the controllability property. We establish several criteria of controllability for damped and non damped multidimensional body controlled by one, two or three torques. We do not only concentrate on a choice of minimal set of (two) controlled torques, which are able to guarantee global controllability of MRB, but also search for criteria which are structurally stable with respect to perturbation of inertia operator of MRB and of (some of the) controlled directions.

2 Euler Equation for Generalized Rigid Body

We follow [Arnold, 1997] for definition of 'generalized rigid body'. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and let left-invariant Riemannian metric on G be defined by scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$.

Introduce $\mathcal{I} : \mathfrak{g} \mapsto \mathfrak{g}^*$ - a symmetric operator, which corresponds to the Riemannian metrics by formula: $\langle \xi, \eta \rangle = \mathcal{I}\xi | \eta$, where $\cdot|\cdot$ is the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. The operator $\mathcal{I}$ is called *inertia operator of generalized rigid body*.

The trajectory of the motion of generalized rigid body is a curve $g(t) \in G$. One can introduce *angular velocity in moving frame*, corresponding to this motion: $\mathfrak{g} \ni \Omega = L_{g^{-1}*}\dot{g}$, where L_g is left translation by g.

The image of angular velocity Ω under $\mathcal{I}$ is *angular momentum in moving frame* $M \in \mathfrak{g}^*$. Energy of the body equals $\langle \Omega, \Omega \rangle = M | \Omega$.

Finally Euler equation for the motion of generalized rigid body is

$$\dot{\Omega} = \mathcal{B}(\Omega, \Omega), \tag{1}$$

where bilinear operator $\mathcal{B} : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$ is defined by formula:

$$\langle [a, b], c \rangle = \langle \mathcal{B}(c, a), b \rangle, \tag{2}$$

$[\cdot, \cdot]$ staying for Lie bracket.

3 Inertia operator and Euler-Frahm equation for MRB

Readers may consult [Fedorov and Kozlov, 1995] for detailed presentation of dynamics of MRB. This is a particular case of generalized rigid body, where Lie group G is $SO(n)$ - group of orthogonal $(n \times n)$-matrices with det $= 1$, and Lie algebra $\mathfrak{g} = so(n)$ consists of skew-symmetric $(n \times n)$-matrices - *angular velocities*. We consider angular velocities in moving frame (see previous Section).

A point of MRB, whose position in the moving (attached to the body) frame is defined by vector $\rho \in \mathbb{R}^n$, possesses at moment t the velocity $\Omega_t \rho$, where $\Omega_t \in so(n)$ is angular velocity of MRB.

The kinetic energy of MRB, seen for simplicity as a set of N material points of masses $m_i,\ i = 1, \ldots, N$, is defined by formula:

$$T(\Omega) = \langle \Omega, \Omega \rangle = \frac{1}{2} \sum_{i=1}^{N} m_i \left(\Omega \rho_i \right) \cdot \left(\Omega \rho_i \right), \tag{3}$$

where $\cdot$ stays for scalar product in $\mathbb{R}^n$. Invoking some basis in $\mathbb{R}^n$ we represent the energy as

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \sum_{k=1}^{n} \left(\sum_{s=1}^{n} \Omega_{ks} \rho_s^i \right) \left(\sum_{\ell=1}^{n} \Omega_{k\ell} \rho_\ell^i \right) = \\ \frac{1}{2} \sum_{k,s=1}^{n} \left(\sum_{\ell=1}^{n} \Omega_{k\ell} \left(\sum_{i=1}^{N} m_i \rho_\ell^i \rho_s^i \right) \right) \Omega_{ks} = \\ = \frac{1}{4} \left(\Omega C + C \Omega, \Omega \right)_K,$$

where $C = \sum_{i=1}^{N} m_i [\rho^i \otimes \rho^i]$ is the sum of N rank-1 matrices $[\rho^i \otimes \rho^i]_{\ell s} = \rho_\ell^i \rho_s^i$ and $(\cdot, \cdot)_K$ stays for natural scalar product - *Killing form* - defined on $so(n)$:

$$(\Omega, \Omega')_K = -tr\,(\Omega \Omega') = 2 \sum_{j<h} \Omega_{jh} \Omega'_{jh}.$$

Note that C is *symmetric positive semidefinite matrix*.

If one identifies $so(n)$ with $so^*(n)$ by means of Killing form, then thinking of momentum M as of skew-symmetric matrix we conclude that the inertia operator is identifiable with the map

$$\mathcal{I}_C : \Omega \mapsto \frac{1}{4} \left(\Omega C + C \Omega \right) = \\ = M \in so(n)(n) \overset{\mathcal{K}}{\cong} so^*(n). \tag{4}$$

Then $\langle \Omega, \Omega \rangle = T(\Omega) = \left(\mathcal{I}_C \Omega, \Omega \right)_K$.

By direct computation

$$(\mathcal{I}_C \Omega_1, \Omega_2)_K = (\mathcal{I}_C \Omega_2, \Omega_1)_K,$$

i.e. operator $\mathcal{I}_C$ is symmetric.

Symmetric bilinear form (scalar product) corresponding to the quadratic form T, is defined as

$$\langle \Omega^1, \Omega^2 \rangle = \left(\mathcal{I}_C \Omega^1 C, \Omega^2 \right)_K. \tag{5}$$

Operator (4) maps $so(n)$ into itself and by Sylvester theorem ([Gantmacher, 1960]) is invertible if and only if the eigenvalues σ_i of C satisfy the relations $\sigma_k + \sigma_j \neq 0,\ \forall k \neq j$. This certainly would hold true for *positive definite* matrix C.

Now we derive Euler equation for MRB. According to formulae (2) and (5)

$$\langle \mathcal{B}(\Omega^1, \Omega^2), \Omega^3 \rangle = \\ \langle [\Omega^2, \Omega^3], \Omega^1 \rangle = \left([\Omega^2, \Omega^3], C\Omega^1 + \Omega^1 C \right)_K = \\ -\text{tr} \left([\Omega^2, \Omega^3] \left(C\Omega^1 + \Omega^1 C \right) \right) = \\ -\text{tr} \left(\Omega^2 \Omega^3 \left(C\Omega^1 + \Omega^1 C \right) - \Omega^3 \Omega^2 \left(C\Omega^1 + \Omega^1 C \right) \right) = \\ -\text{tr} \left(\left(C\Omega^1 + \Omega^1 C \right) \Omega^2 \Omega^3 - \Omega^2 \left(C\Omega^1 + \Omega^1 C \right) \Omega^3 \right) = \\ = \left(\left[C\Omega^1 + \Omega^1 C, \Omega^2 \right], \Omega^3 \right)_K = \left(\left[\mathcal{I}_C \Omega^1, \Omega^2 \right], \Omega^3 \right)_K.$$

Therefore

$$\langle \mathcal{B}(\Omega^1, \Omega^2), \Omega^3 \rangle = \left(\mathcal{I}_C \mathcal{I}_C^{-1} \left[\mathcal{I}_C \Omega^1, \Omega^2 \right], \Omega^3 \right)_K = \\ = \langle \mathcal{I}_C^{-1} \left[\mathcal{I}_C \Omega^1, \Omega^2 \right], \Omega^3 \rangle,$$

and

$$\mathcal{B}(\Omega^1, \Omega^2) = \mathcal{I}_C^{-1} \left[\mathcal{I}_C \Omega^1, \Omega^2 \right]. \tag{6}$$

Euler-Frahm equation for rotation of MRB is therefore

$$\dot{\Omega} = \mathcal{I}_C^{-1}\left[\mathcal{I}_C\Omega, \Omega\right] = \mathcal{I}_C^{-1}[C, \Omega^2]. \qquad (7)$$

We will study also rotation of MRB, subject to damping. A simplest way of introducing damping into (7) results in equation

$$\dot{\Omega} = \mathcal{I}_C^{-1}[C, \Omega^2] - \nu\Omega, \ \nu \geq 0. \qquad (8)$$

Non damped rotation (7) corresponds to $\nu = 0$.

4 Controllability of rotating MRB: problem setting and main results

Controlled rotation of MRB is described by equation

$$\dot{\Omega} = \mathcal{I}_C^{-1}[C, \Omega^2] - \nu\Omega + \sum_{i=1}^{r} G^i u_i(t), \ \nu \geq 0. \qquad (9)$$

We are interested in *global controllability* of (9). This property means that for any two 'points' $\tilde{\Omega}, \hat{\Omega}$ of the state space $so(n)$ there exists a control $u(t) = (U - 1(t), \ldots, u_r(t))$ which steers (9) from $\tilde{\Omega}$ to $\hat{\Omega}$ in some time $T \geq 0$. We are interested in achieving global controllability by small number r of controls; we will prove that r can be taken ≤ 3.

Note that equation (9) is particular case of control-affine system with quadratic(+linear) *drift vector field* and *constant controlled vector fields*.

The following *genericity condition* is assumed to hold: symmetric matrix C in (9) *is positive definite and has simple eigenvalues*.

Our first result claims *global controllability of MRB by means of two controlled torques*.

Theorem 4.1. *For multidimensional rigid body with generic inertia operator $\mathcal{I}_C$ there exists a pair of directions $g^1, g^2 \in so(n)$ (depending on C), such that (9) is globally controlled by means of torques applied along g^1, g^2.* □

Remark 4.1. *In the formulation there are no a priori bounds for the magnitude of the controlled torques.* □

The proof of Theorem 4.1 is constructive and is based on computations in a special basis of $so(n)$ in which 'multiplication table' for the bilinear operator (6) takes simple form; g^1, g^2 have explicit expressions in this basis. This proof brings about certain disadvantage. The computation, on which the proof is based, does not withstand perturbations of controlled directions. Therefore one can not conclude (desirable and plausible) structural stability of global controllability property.

We pass now to formulation of some structurally stable criteria starting with controllability of *non damped* MRB. In this case, given *recurrence* of drift (uncontrolled) Euler-Frahm dynamics, *bracket generating property* suffices for guaranteeing global controllability. This property means that (evaluations at each point of) iterated Lie brackets of drift and controlled vector fields span $so(n)$. Given high dimension of $so(n)$ and fixed and small number of controlled vector fields, proving bracket generating property is a nontrivial task. Still we are able to establish this property and derive from it *global controllability by means of single control applied along generic direction.*

Theorem 4.2. *Let $r = 1$, and $\nu = 0$ in (9). For generic inertia operator $\mathcal{I}_C$ the system*

$$\dot{\Omega} = \mathcal{I}_C^{-1}[C, \Omega^2] + gu(t), \qquad (10)$$

is globally controllable for all g from some open dense subset of $so(n)$. □

Remark 4.2. *The controllability property hods if one bounds control by $|u| \leq b$ for any $b > 0$.* □

We now pass to the damped case. Our method requires one of the controlled directions to correspond to *stationary rotation* of MRB.

Definition 4.1. *Principal axis of MRB, is a matrix $\hat{g} \in so(n)$ such that*

$$\mathcal{I}_C\hat{g} = \alpha\hat{g}, \ \alpha \in \mathbb{R}.$$

Stationary direction for MRB is a matrix $\hat{g}$ for which $[\mathcal{I}_C\hat{g}, \hat{g}] = [C, \hat{g}^2] = 0$; equivalently $\hat{g}$ is an equilibrium point of (7).

Remark 4.3. *Obviously all principal axes are stationary directions. The contrary is true for $n = 3$ but not for $n > 3$.* □

The results obtained for the damped case differ for *odd* and *even* n.

Theorem 4.3. *Let $\nu \geq 0$, n be odd, $r = 2$ in (9), and the inertia operator $\mathcal{I}_C$ be generic. For a stationary direction g^1 and generic $g^2 \in so(n)$ (element of a dense open subset of $so(n)$) the system (9) is globally controllable.* □

An additional symmetry in the case of even n, obliges us to involve additional controlled direction for achieving global controllability.

Theorem 4.4. *Let $\nu \geq 0$, n be even in (9), $r = 3$ and the inertia operator $\mathcal{I}_C$ be generic. For a stationary direction $g^1 \in so(n)$ and generic pair (g^2, g^3) of directions (element of a dense open subset of $so(n) \times so(n)$) the system (9) is globally controllable.* □

Let us mention preprint [Deryabin, 2007], in which author proved that non damped multidimensional rigid body is controllable by using of a pair of controlled 'flywheels'. This is different kind of "internal-force controls" modeled by bilinear control system on Lie group.

The next Sections contain geometric control ideas which underly proofs of the above formulated controllability criteria. In particular we will sketch proofs of Theorems 4.1 and 4.2.

5 Controllability of control-affine systems by Lie extensions

First result - Bonnard-Lobry theorem - is well known and widely used. We formulate it for control-affine system on smooth manifold M

$$\dot{q} = f(x) + \sum_{i=1}^{r} g^i(x)v_i(t), \tag{11}$$

with drift vector field f being complete and possessing *recurrence property*. This property means that all points are *nonwandering* for the flow e^{tf}: for each neighborhood W of each point x and each $T > 0$ there exists $T' > T$ such that $e^{T'f}(W) \cap W \neq \emptyset$.

Theorem 5.1 ([Bonnard, 1981; Lobry, 1974]). *Let f be complete and recurrent and system of vector fields $\{f, g^1, \ldots, g^r\}$ be bracket generating. Then system* (11) *is globally controllable.* □

To verify recurrence property of the vector field at the right-hand of (7) we observe that kinetic energy $E = \langle \Omega, \Omega \rangle$ is preserved by the flow of (7). Any energy level $E = c$ is compact and by Liouville theorem ([Abraham and Marsden,1978]) there exists an invariant measure on it. Therefore all points of each level are nonwandering for the uncontrolled dynamics.

For establishing controllability of damped MRB, where recurrence property is missing, it does not suffice to evaluate *all* Lie brackets. One has to select some specific Lie brackets which 'contribute' to controllability. This can be formalized in terms of *Lie extensions*.

Lie extensions amount to finding vector field X which are compatible with control system in the sense that closures of attainable sets of the control system are invariant for X. There are various methods of choosing iterative Lie brackets which result in compatible vector fields. If one is able to prove global controllability of the system *extended* by adding some compatible vector fields, then controllability of the original system can be concluded by standard argument ([Jurdjevic, 1997; Sarychev, 2006]).

Key Lie extension, we employ, is described by the following Proposition, formulated for two-input control-affine system

$$\dot{x} = f(x) + g^1(x)\hat{v}_1 + g^2(x)\hat{v}_2. \tag{12}$$

Proposition 5.1. *Let*

$$[g^1, g^2] = 0, \ [g^1, [g^1, f]] = 0. \tag{13}$$

Then the system

$$\dot{x} = f(x) + g^1(x)\tilde{v}_1 + g^2(x)\tilde{v}_2 + [g^2, [g^1, f]]v_{12}, \tag{14}$$

is Lie extension of (12). □

Remark 5.1. *The conclusion of the Proposition means that vector fields $\pm[g^2, [g^1, f]]$ and $f + [g^2, [g^1, f]]w$, $w \in \mathbb{R}$ are compatible with (12). The Lie bracket $[g^2, [g^1, f]]$ is called extending controlled vector field.* □

One can employ Proposition 5.1 repeatedly. For example, assuming that g^1 and $g^{12} = [g^2, [g^1, f]]$ commute, we can extend (14) once more 'upgrading' it to 4-input system

$$\dot{x} = f(x) + g^1(x)\tilde{v}_1 + g^2(x)\tilde{v}_2 + \\ +[g^2, [g^1, f]]v_{01} + [[g^1, [g^2, f]], [g^1, f]]v_{001},$$

with $[[g^1, [g^2, f]], [g^1, f]]$ being another *extending controlled vector field*.

If one arrives after a series of extensions to a system with full-dimensional input (dimension of the extended input coinciding with the dimension of the state) then global controllability of the extended and original systems can be easily concluded (see [Agrachev and Sarychev, 2005]).

6 Controllability of damped Euler-Frahm equation via Lie extensions; proof of Theorem 4.1

We will apply Proposition 5.1 repeatedly for proving global controllability of (9) with two controls.

At each iteration the first one of the assumptions (13) will be trivially satisfied as long as original and extending controlled vector fields will be constant and hence commuting.

Being drift vector field f in (9) quadratic, and (original or extending) controlled vector field g^1 *constant*, double Lie bracket $[g^1, [g^1, f]]$ is constant vector field with value $[\mathcal{I}_C g^1, g^1] = [C, g^1 g^1]$. The second relation (13) would hold if and only if g^1 is *stationary direction* for MRB. In fact we will manage to choose at each step of our proof g^1 to be principal axis of MRB. We start with describing the set of principal axes of MRB.

6.1 Diagonalization of inertia operator and principal axes of MRB

Let symmetric matrix C in (4) be represented as $C = \mathrm{Ad}\, SD = SDS^{-1}$ with S being orthogonal and $D = \mathbf{diag}\{I_1, \ldots, I_n\}$, $I_1 < I_2 < \cdots < I_n$.

Introduce matrices $\Theta^{rs} = \mathbf{1}_{rs} - \mathbf{1}_{sr} \in so(n)$, where $r < s$ and $\mathbf{1}_{rs}$ stays for matrix with (the only) unit element at row r and column s.

Lemma 6.1. *Matrices $\Omega^{rs} = Ad\, S\Theta^{rs}$ are eigenvectors of the operators $(ad\, C)$ and $\mathcal{I}_C$ with eigenvalues $I_r - I_s$ and $I_r + I_s$ respectively.* □

The proof goes by direct computation.

Corollary 6.1. *Matrices $\Omega^{rs} = Ad\, S\Theta^{rs}$ are eigenvectors of the operator $\mathcal{I}_C^{-1} \circ ad\, C$ with the eigenvalues $\frac{I_r - I_s}{I_r + I_s}$. These matrices form set of principal axes of the MRB.* □

Remark 6.1. *The set of stationary directions of n-dimensional rigid body is much richer, when $n \geq 4$, on the contrast to the 3-dimensional case where these two sets coincide.* □

6.2 Computing Lie extension for Euler-Frahm equation

For drift vector field f from (9) and two constant controlled vector fields $g^1, g^2 \in so(n)$ the extending controlled vector field $g^{12} = [g^2, [f, g^1]]$ is constant; its value is $\mathcal{B}(g^1, g^2)$, with $\mathcal{B}$ defined by (6). The correspondence

$$(g^1, g^2) \mapsto \mathcal{B}(g^1, g^2) = \mathcal{I}^{-1}[C, g^1 g^2 + g^2 g^1]. \quad (15)$$

is bilinear operation on $so(n)$.

Lemma 6.2. *If g^1, g^2 are (extending or original) constant controlled vector fields (directions), then (15) defines extending controlled direction.* □

'Multiplication table' for $\mathcal{B}$ with respect to the basis Ω^{rs} from the previous Subsection, is described by the following

Lemma 6.3. *For $r < s$:*

$$\mathcal{B}(\Omega^{rs}, \Omega^{k\ell}) = 0, \textit{if } r, s, k, \ell \textit{ are distinct}$$
$$\mathcal{B}(\Omega^{rs}, \Omega^{rs}) = 0$$
$$\mathcal{B}(\Omega^{rs}, \Omega^{r\ell}) = \frac{I_\ell - I_s}{I_s + I_\ell}\Omega^{s\ell}. \ \square$$

Proof. It suffices to verify these computations for $S = I$, matrix C coinciding with $D = \mathrm{diag}\{I_1, \ldots, I_n\}$ and $\Omega^{rs} = \Theta^{rs}$.

Using this multiplication table one can provide constructive proof of Theorem 4.1.

6.3 Proof of Theorem 4.1

Let $C = \mathrm{Ad}\, SD,\ g^1 = \mathrm{Ad}\, S\Omega^{12}$ and

$$g^2 = \mathrm{Ad}\, S\left(\Omega^{23} + \Omega^{34} + \cdots + \Omega^{n-1,n}\right). \quad (16)$$

Remark 6.2. *Here g^1 is principal axis, while g^2 is neither principal axis nor stationary direction.* □

By Lemma 6.3

$$g^3 = \mathcal{B}(g^2, g^1) = \mathcal{B}(\Omega^{12}, \Omega^{23}) = \frac{I_2 - I_3}{I_2 + I_3}\Omega^{13}, \quad (17)$$

is principal axis. Calculating subsequently

$$g^i = \mathcal{B}(g^{i-1}, g^2),\ i > 3,$$

we see that g^i coincide up to a nonzero multiplier with $\Omega^{1,i}$, and are principal axes.

According to Lemma 6.2 all matrices $\Omega^{1i}, i = 2, \ldots, n$ are extending control directions.

By Lemma 6.3

$$\mathcal{B}(\Omega^{1i}, \Omega^{1k}) = ((I_k - I_i)/(I_k + I_i))\Omega^{ik},$$

and again by Lemma 6.2 *all* matrices Ω^{ik} are extending controlled directions.

Thus we have arrived to the system

$$\dot{\Omega} = \mathcal{I}^{-1}[C, \Omega^2] - \nu\Omega + \sum_{i<j} \Omega^{ij} u_{ij}(t),$$

with full-dimensional input. This system is globally controllable and the original system (9) is globally controllable as well ([Agrachev and Sarychev, 2005]).

Remark 6.3. *If one perturbs in generic way the vector field g^2 in (16) then computation of extending controlled direction by Lemma 6.2 does not result in a stationary direction on contrast to (17). Thus we would not be able to iterate application of the Proposition 5.1. Hence the provided construction would not allow to conclude structural stability of global controllability property.* □

7 Non damped MRB: bracket generating property and Hautus criterion

As we explained in Section 5 global controllability of (10) would follow from 'bracket generating property' of the couple of vector fields: $f(\Omega) = \mathcal{I}_C^{-1}[C, \Omega^2],\ g(\Omega) \equiv g$, on $so(n)$. While for $n = 3$ the generating brackets can be constructed explicitly, for high n and generic g the problem can hardly be attacked via direct computation.

To cope with the problem we look at the Lie bracket $[f, g]$, which is Ω-linear vector field:

$$[f, g](\Omega) = \mathcal{I}_C^{-1}[C, (\Omega g + g\Omega)] = F(g)\Omega, \quad (18)$$

where $F(g)$ is Jacobian operator of $[F, g]$, acting on $\Omega \in so(n)$..

Vector fields, defined iteratively as

$$G - 1 = g,\ g_{i+1} = [g_i, [f, g]] = F(g)g_i,\ i \geq 1,$$

are all constant; $g_i = (F(g))^i g$. If for $N = n(n-1)/2$

$$\text{span}_{\mathbb{R}} \left\{g, F(g)g, \dots (F(g))^{N-1} g\right\} = so(n), \quad (19)$$

then bracket generating property would immediately follow. The relation (19) is *Kalman condition for controllability* of the pair $(F(g), g)$.

Unfortunately direct computation of iterated applications in (19) for generic $g \in so(n)$ is also hardly realizable. Instead we invoke another controllability condition - so called *Hautus controllability test* ([Sontag, 1998]).

Proposition 7.1. *The pair $F(g), g$ is controllable if and only if for $N = n(n-1)/2$:*

$$rank(F(g) - zI|g) = N, \ \forall z \in \mathbb{C}. \ \square \quad (20)$$

Corollary 7.1. *The pair of vector fields $f(g), g$ is bracket generating if (20) holds.* $\square$

Therefore proof of Theorem 4.2 will be deduced from

Proposition 7.2. *For generic inertia operator $\mathcal{I}_C$ the condition (20) holds for g from an open dense subset of* $so(n)$. $\square$

Let us interpret condition (20) as *non intersection* of the family $z \mapsto \Phi(z; g) = (F(g) - zI, g), z \in \mathbb{C}$ of complex $(N \times (N+1))$-matrices with stratified manifold $\mathcal{M}_d$ of $(N \times (N+1))$-matrices of rank $< N$.

Note that codimension of $\mathcal{M}_d$ equals

$$(N - (N-1))((N+1) - (N-1)) = 2.$$

Transversality of the family $z \mapsto \Phi(z; g)$ to $\mathcal{M}_d$ would imply lack of intersection.

We will conclude such transversality for generic $g \in so(n)$ from the following

Proposition 7.3. *For generic inertia operator $\mathcal{I}_C$ the map $(z, g) \mapsto \Phi(z; g)$ is transversal to the stratified manifold $\mathcal{M}_d$.* $\square$

For the lack of space we postpone proof of the last Proposition to a later publication.

It rests to to observe that if Proposition 7.3 holds, then according to [Golubitsky and Guillemin, 1973, Ch. 2, §4] the set of $g \in so(n)$ for which 'individual' maps $z \mapsto \Phi(z; g)$ are transversal to $\mathcal{M}_d$ is open and dense. For any such g bracket generating property and Theorem 4.2 is now derived from Corollary 7.1.

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