The dynamics of a ring of coupled Phase-Locked loops

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Abstract

The dynamics of two and three coupled Phase-Locked loops is considered. In the bifurcation diagrams the domains of different dynamic regimes are extracted.

1. A Phase-Locked loop (PLL) is one of the typical systems of synchronization. It is a closed loop frequency system; its functioning is based on the detection of phase difference between the input and output signals of the controlled oscillators. PLLs have high accuracy, reliability, noise resistance, controllability, capability to provide high power and frequency. These properties of PLLs make them highly promising for data communication by means of chaotic signals too. The dynamical chaos may be occurred either in the single PLLs with the high order filter or in the ensemble of the PLLs with the low order filters. At the present time the ring of PLLs haven't been considered in the scientific literature; as a result this paper is concerned with research of this rings.

2. In this part the dynamics of a ring of two phase-locked loops is considered. The equation for such a system is written as follows:

$$\begin{aligned}\frac{d\varphi}{d\tau} &= y, \quad \frac{dy}{d\tau} = z,\\ \varepsilon_1\varepsilon_2\frac{dz}{d\tau} &= \gamma - [(1-\kappa_1)b + 1 - \kappa_2]\sin\varphi - \\ &- (\varepsilon_1+\varepsilon_2)z - \\ &- \{1 + [\varepsilon_1(1-\kappa_1)b + \varepsilon_2(1-\kappa_2)]\cos\varphi\}y.\end{aligned} \qquad (1)$$

Here, φ - current phase difference between the first and the second partial generators, $b, \varepsilon_{1,2}$ - parameters of the partial systems, $\kappa_{1,2}$ - parameters of the connections. System (1) has been defined on the 3D phase cylinder $U{=}\{\varphi(\text{mod}2\pi), y, z\}$, which makes possible the existence of different types of attractors.

For the simplest case of the zero order filters, when $\varepsilon_{1,2} \ll 0$, equation (1) can be written as follows:

$$\frac{d\varphi}{d\tau} = \gamma - \sin\varphi[(1-\kappa_1)b + 1 - \kappa_2] \qquad (2)$$

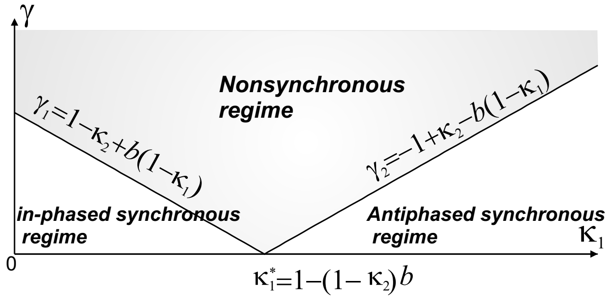


Figure 1: Domains of the existence of syncronous and nonsynchronous regimes of system (2).

Dynamical system (2) has two equilibrium states: $O_1(\varphi_1^* = \arcsin(\gamma/(1+b-b\kappa_1-\kappa_2)))$ and $O_2(\varphi_2^* = \pi - \varphi_1^*)$. If O_1 is a stable equilibrium state, system (2) works in the in-phased synchronous regime; if O_2 is a stable equilibrium state, system (2) works in the antiphased synchronous regime. The results of the investigations of system (2) are presented in fig.1.

The dynamics of system (1) is considered in two cases: when the in-phased synchronous regime is realized and when the antiphased synchronous regime is realized.The analysis of model (1) shows that for ring of two PLLs with a first-order filter the following dynamic modes are typical:

– *synchronization of the partial generators*, i.e., the frequencies of the generators become equal, and the phase difference between them takes on a constant value. Stable equilibrium states with co-ordinates $O_1(\varphi_1^* = \arcsin(\gamma/(1+b-b\kappa_1-\kappa_2)))$ or $O_2(\varphi_2^* = \pi - \varphi_1^*)$ correspond to this mode in the phase space U;

– *quasi-synchronization*, when the phase difference between generators fluctuates around some average value. Regular attractors L_0 [Fig. 2(b)] and L_0^2 [Fig. 2(d)] correspond to this mode in the phase space U;

– *regular or chaotic beats*, when the phase difference

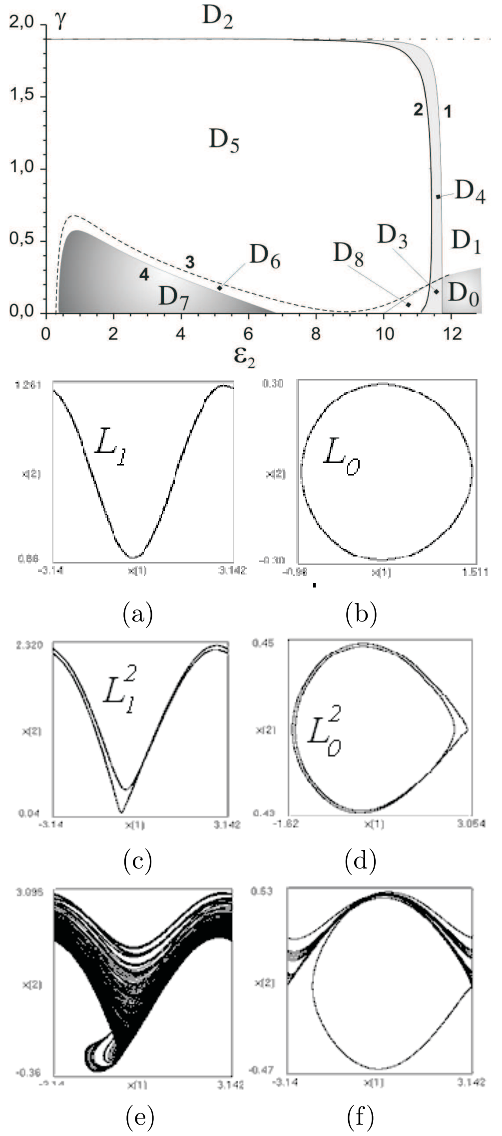


Figure 2: The bifurcation diagram and phase portraits of system (1) for $b = 1.5, \kappa_1 = 3.2, \kappa_2 = -4.2, \varepsilon_1 = 15$.

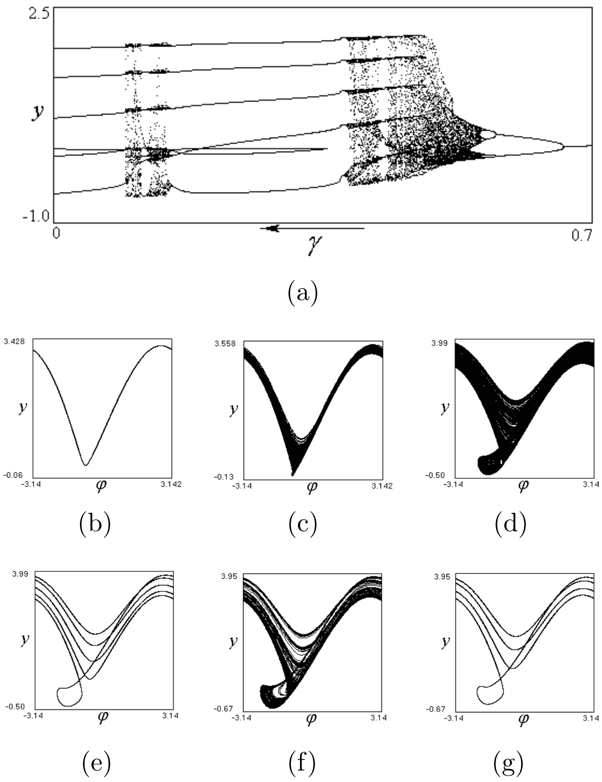
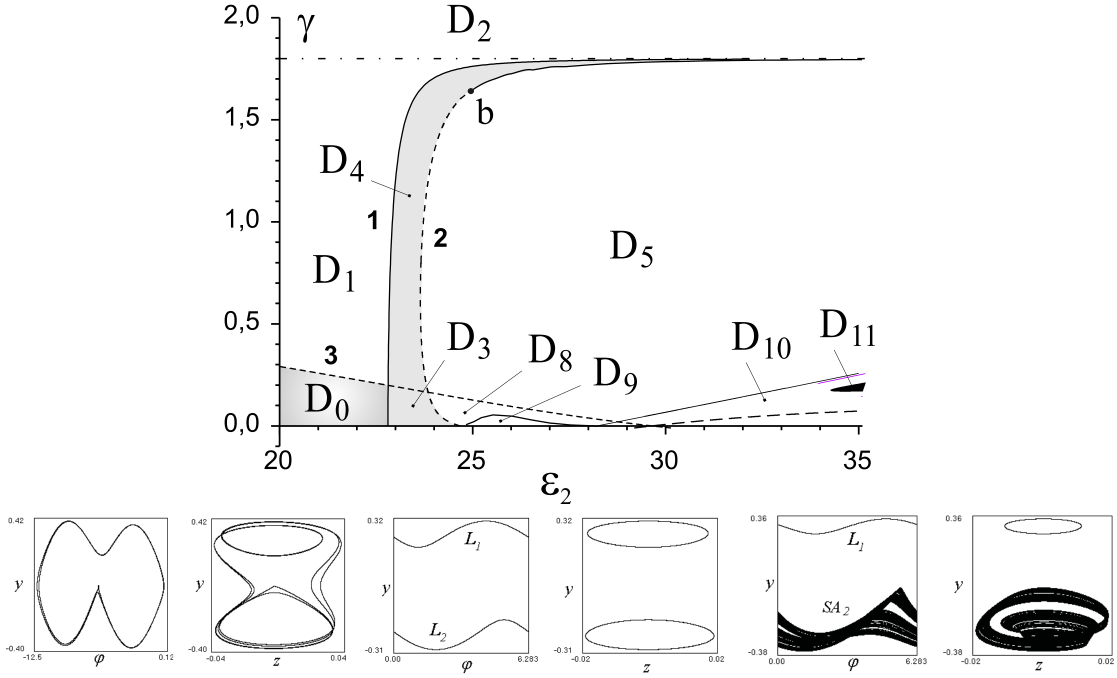


Figure 3: The bifurcation diagram (a) of system (1) for $\varepsilon_2 = 1$ and examples of the attractors for $\gamma = 0.1(b), 0.52(c), 0.39(d), 0.35(f), 0.135(f), 0.7(g)$.

between generators grows without restriction. Rotatory or oscillatory-rotatory (with phase difference advance φ more than 2π) attractors that may be either regular [Fig. 2(a),(c)] or chaotic [Fig. 2(e),(f)] correspond to this mode in phase space.

3. The bifurcation diagram $[\gamma, \varepsilon_2]$ in fig.2 illustrates possible modes of the behavior of system (1) in the in-phased synchronous case. System (1) has a single attractor in the parameter domain D0 - an equilibrium state $O_1(\varphi_1^* = \arcsin(\gamma/(1+b-b\kappa_1-\kappa_2)))$. The attractor O_1 corresponds to the global synchronous regime of the coupled PLLs. There is one more attractor in the parameter domain D1 - a rotatory limit cycle L_1 [Fig.2]. The attractor L_1 corresponds to the nonsynchronous regime. System (1) has no equilibrium states in the parameter domain D2, so the nonsynchronous regime is globally stable here. For the values of the parameters from domain D3, a single oscillatory attractor L_0 [Fig.2(b)] exists in the phase space of system (1). Line 1 corresponds to changing the stability of the equilibrium state O_1. The parameter domain D4 is a bistability domain because of the existence of two attractors: the rotatory limit cycle L_1 and the oscillatory limit cycle L_0. The limit cycle L_0 is doubled in the domain D5. As a result, the limit cycle L_0^2 takes place here . The attractor L_1 may be either regular [Fig.2(a)] or chaotic one [Fig.2(e)]. An attractor chaotisation results from the sequence of period doubling bifurcations. The evolution of the attractor L_1 is shown in the bifurcation diagram (fig.3). The domain of the existence of chaotic attractors is typically intermitted by the "windows" of multi-turn limit cycles, which are transformed into the chaotic attractors again with a slight change in γ.

4. The bifurcation diagram for the antiphased synchronous regime is presented in [Fig.4(a)]. In this case there are some new effects beside the in-phased synchronous regime. First, in the parameter domain



(a) (b) (c)

Figure 4: The bifurcation diagram and phase portraits of system (1) for $b = 1.5, \kappa_1 = 3.2, \kappa_2 = -0.5, \varepsilon_1 = 15$

D9 the complex oscillatory limit cycle is occurred. Changing of the amplitude of this cycle is larger than 2π [Fig.4(a)]. Second, in the parameter domains D10 and D11 the existence of two limit cycles is exposed, and one of them may be chaotic [Fig.4(c)].

5. In the present part the dynamics of a ring of three PLLs with zero-order filters in the control loop is considered. The mathematical equation for such an ensemble is written as follows:

$$
\begin{aligned}
\frac{d\varphi_1}{d\tau} &= \gamma_1 - (1 - \kappa_1)\sin\varphi_1 - \kappa_2 \sin\varphi_2 - \\
&\qquad - b_3 \sin(\varphi_1 + \varphi_2) \qquad (3)\\
\frac{d\varphi_2}{d\tau} &= \gamma_2 - (b_2 - \kappa_2)\sin\varphi_2 + \sin\varphi_1 + \\
&\qquad + \kappa_3 \sin(\varphi_1 + \varphi_2)
\end{aligned}
$$

Here, φ_1 - current phase difference between the first and the second partial generators, φ_2 - current phase difference between the second and the third partial generators $b_{2,3}$ - parameters of the partial systems, $\kappa_{1,2,3}$ - parameters of the connections, $\gamma_{1,2}$ - is a deviation of the frequency between neighbouring generators. System (3) has been defined on the phase torus $U^*=\{\varphi_1(\mathrm{mod}2\pi), \varphi_2(\mathrm{mod}2\pi)\}$, so quasi-syncronous regime will be defined by oscillatory and rotatory limit cycles.

The dynamical system (3) has no equilibrium states in the parameter domain D_2, so the nonsynchronous regime is globally stable here. Curve 1 corresponds to the birth of two equilibrium states: O_1 - stable or unstable equilibrium state and saddle O_2. Curve 4 corresponds to the loss of stability of equilibrium state O_1, so in the parameter domain $D^S_{O_1}$ a stable oscillatory limit cycle exists and in the parameter domain $D^U_{O_1}$ an unstable oscillatory limit cycle exists. Curve 2 and 3 corresponds to the birth of two more equilibrium states: stable or unstable equilibrium state O_3, saddle O_4 (curve 2) and stable or unstable equilibrium state O_5, saddle O_6 (curve 3). Changing of stability of equilibrium state O_3 results to the soft onset of the one more stable oscillatory limit cycle, which takes place in the parameter domain $D^S_{O_3}$. Similary, unstable limit cycle exists in the parameter domain $D^U_{O_5}$. Curve 5 is a border of the existence of a rotatory limit cycle. In the parameter domain $D^S_{L_{1,0}}$ there is a stable rotatory limit cycle with the φ_1 coordinate rotation, and in the parameter domain $D^S_{L_{0,-1}}$ there is a stable rotatory limit cycle with the φ_2 coordinate rotation. They corresponds to the quasi-synchronous regime, as we said. In the other parameter domains rotatory limit cycles with different number of rotation exist. They correspond to the regular beats regime.

6. In this article we have studied small rings of PLLs with low-order filter in the control loops. We

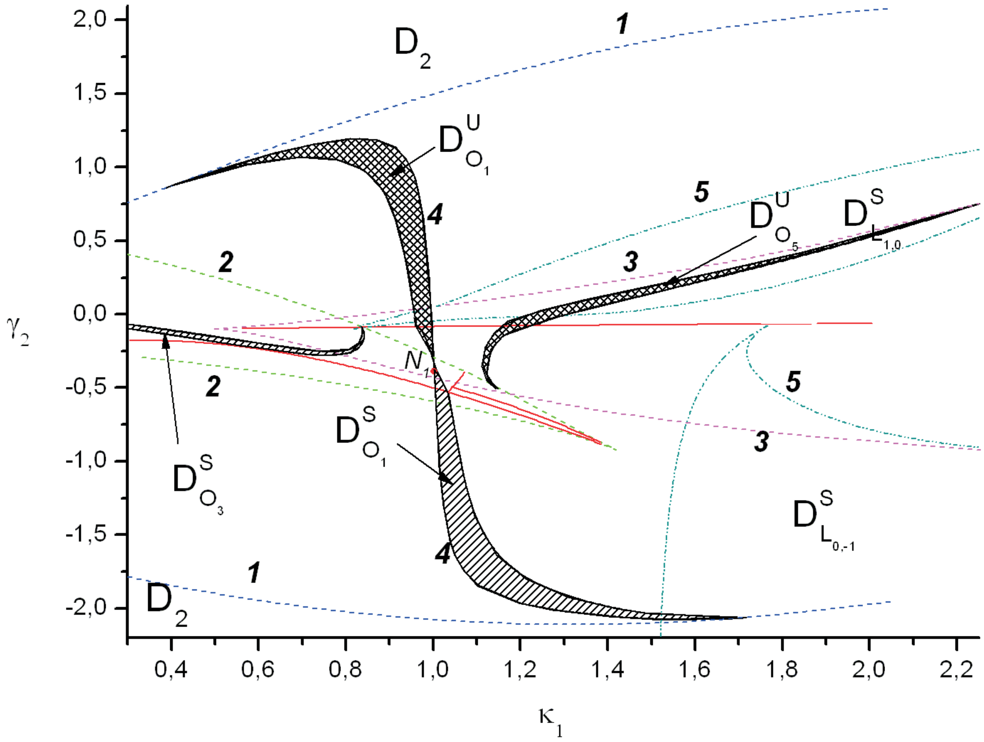


Figure 5: The bifurcation diagram of system (3) for $\gamma_1 = 0.5; \kappa_2 = 0.9; \kappa_3 = 1.1$

have received the following results: first, the collective dynamics of PLLs is described by the following variety of the dynamical regimes - synchronous regime, quasi-synchronous regime , chaotic or regular beats and complex regular regimes. Second, in the ring of two PLLs two synchronous regimes exist: in-phased synchronous regime and antiphased synchronous regime. They are defined by stability of the equilibrium states O_1 and O_2. It has been established that even in the ring of three same PLLs with zero-order filters in the control loops quasi-synchronous, various bistability and beats regimes may be appeared.

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