SYNCHRONIZATION AND CONTROL IN ENSEMBLES OF PERIODIC AND CHAOTIC NEURONAL ELEMENTS WITH TIME DEPENDENT COUPLING

Vladimir N. Belykh and Evgeniya V. Pankratova[*]

[*] *Mathematics Department, Volga State Academy,*
5, Nesterov street, Nizhny Novgorod, 603600, Russia.
belykh@unn.ac.ru, pankratova@aqua.sci-nnov.ru

Abstract: The study of complete synchronization in networks of periodic and chaotic neurodynamical elements with different coupling configurations is performed. Using the connection graph stability method we obtain the sufficient conditions for achievement of synchronous behavior of all elements involved in these ensembles. The theoretical predictions we compare with the numerical results obtained for the networks composed of the classical Hodgkin-Huxley neuronal elements. The problem how to control the synchronization of networks growing in time is discussed. *Copyright © 2007 IFAC*

Keywords: Neural networks, synchronization, stability criteria, chaotic behavior

1. INTRODUCTION

The stability of a synchronous state in large ensembles of coupled oscillators is one of the most intensively studied problem arising in different fields of science. This topic is of significant interest in the context of electronic circuits, chemical and biological systems, and secure communication (for particular examples see (Pikovsky *et al.*, 2001)).

In the present work we study the complete synchronization in the context of neural networks. The dynamics of individual element of the network is described by the classical Hodgkin-Huxley equations (Hodgkin and Huxley, 1952). Several types of possible topology for the network are examined. Among the objectives of the study of such networks is to get a better understanding of ba-

sic mechanisms of sensory processing, motor control, memory and higher information-processing functions of the brain. From experimental works of Swadlow (1992) on mammalian neocortex it is known that the delays on the synaptic connections could be small enough. In this case a synfire activity of the neurons in a network should be considered (Izhikevich, 2006). Only such type of activity can provide further transmission of information along the network. The point is that, the synchronously generated spikes arrive to the target at the same time, thereby evoking potent postsynaptic responses. If the neurons fire asynchronously their spikes arrive to the postsynaptic target at different times evoking only weak or no response. In this context, the stability of the synfire activity is of great importance.

In order to determine the stability, various criteria can be used (Pecora and Carroll, 1998; Pogromsky and Nijmeijer, 2001; Wu and Chua, 1996). In this work the results of theoretical prediction obtained within the framework of recently developed con-

nection graph stability method (Belykh *et al.*, 2004) are presented. These results are compared with the data of numerical calculations. Some aspects of the synchronization in multilayered neuronal networks that are suggestive of sensory-motor systems are also touched upon.

2. COMPLETE SYNCHRONIZATION: THE STATE OF THE PROBLEM

Let us consider a network of n coupled identical oscillators:

$$\dot{x}_i = F(x_i) + \sum_{j=1}^{n} \varepsilon_{ij}(t)\mathcal{P}x_j, \quad i = 1, ..., n \quad (1)$$

Here $x_i = (x_i^1, x_i^2, ..., x_i^d)$ is the d-vector containing the coordinates of the i-th oscillator, $F(x_i)$ is a nonlinear vector function defining the dynamics of the individual element. The non-zero elements of the $(d \times d)$ matrix $\mathcal{P} = diag(p_1, p_2, ..., p_d)$, where $p_h = 1$ for $h = 1, 2, ..., s$ and $p_h = 0$ for $h = s+1, ..., d$ determine which variables couple the individual systems.

The matrix $\mathcal{G} = \{\varepsilon_{ij}(t)\}$ is an $(n \times n)$ symmetric matrix with non-negative off-diagonal elements. The diagonal elements of the connectivity matrix are chosen from a necessary condition for the existence of the synchronous solution of the system (1), namely, the invariance of hyperplane $M = \{x_1(t) = x_2(t) = ... = x_n(t)\}$. This means that diagonal elements of the matrix $\mathcal{G}$ are assumed to be equal $\varepsilon_{ii} = -\sum_{j=1;j\neq i}^{n} \varepsilon_{ij}, \ i = 1, 2, ..., n$. The global asymptotical stability of the invariant manifold M corresponds to the completely synchronous state of the network. In this case any trajectory of the system (1) unrestrictedly converges to any attractor on M.

The connectivity matrix $\mathcal{G}$ defines a graph with n vertices and m edges. The number of edges m equals the number of non-zero above diagonal elements ε_{ij}. The i-th vertex of the graph corresponds to the i-th oscillator of the network. Therefore, if l-th and k-th oscillators of the ensemble are coupled, i.e. $\varepsilon_{lk} = \varepsilon_{kl} > 0$, then the corresponding graph has the edge linking l-th and k-th vertices. Between these vertices there is a path with the unit length P_{lk}. For the general case the length of the path $z(P_{ij})$ equals to the number of edges involved in P_{ij}, that links i-th and j-th vertices in accordance with the connectivity matrix $\mathcal{G}$.

The main statement of the connection graph stability method is that for the definite conditions (see (Belykh *et al.*, 2004)) synchronization manifold $M = \{x_1(t) = x_2(t) = ... = x_n(t)\}$ is globally asymptotically stable if the following inequality holds:

$$\varepsilon_k(t) > \varepsilon_k^* = \frac{a}{n}b_k(n, m) \quad (2)$$

where $b_k(n, m) = \sum_{j>i;k\in P_{ij}}^{n} z(P_{ij})$ is the sum of the lengths of all chosen paths P_{ij} which pass through a given edge k that belongs to the coupling configuration. The parameter a is a constant related to the dynamical properties of the individual dynamical systems.

In general, the dynamics of the elements in networks can be described by an arbitrary model. In computational neuroscience there are a lot of mathematical models illustrating the richness and complexity of spiking behavior of individual neurons. These models are defined at a different level of abstraction and trying to simulate different aspects of neural systems. The choice of a certain model depends on the type of the problem. This could be, for example, some conductance-based models such as Morris-Lecar describing oscillations in barnacle giant muscle fiber, or Wilson model for cortical neurons, etc. This could be some phenomenological neuronal models such as FitzHugh-Rinzel or Hindmarsh-Rose model, etc. Therefore, in the following theoretical approach the synchronization threshold of the form

$$\widetilde{\varepsilon_k^*} = \frac{\varepsilon_k^*}{a} = \frac{b_k(n, m)}{n}, \quad (3)$$

will be considered. According to (3), the variety of the sums $b_k(n, m)$ gives the variety of synchronization thresholds $\widetilde{\varepsilon_k^*}$, that are sufficient to achieve globally stable synchronization in system (1).

3. TWO STAR-COUPLED NETWORKS CONNECTED BY THE CHAIN

In this section let us consider a network composed of n elements, whose topology is illustrated in Fig. 1. From neurophysiological point of view this type of the structure corresponds to a couple of diffusively connected pacemaker neuronal cells. For convenience, we introduce the following notations: m_c is the number of elements in the chain linking the central nodes of the stars; m_{st}^l and m_{st}^s are the numbers of elements for the most loaded star and the star with low concentration of load, respectively. Thus, the total number of cells in the network is $n = m_{st}^l + m_{st}^s + m_c$.
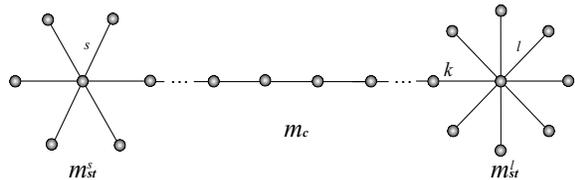
Fig. 1. The coupling structure of an ensemble with two stars connected by the chain.

3.1 Theoretical approach

Statement 1: *For an ensemble of two stars connected by the chain, the synchronization thresholds for the edges belonging to the chain are defined as:*

$$\widetilde{\varepsilon}^*_k = -\frac{k^2}{2} +$$
$$\frac{m_c(m_c + 2m^s_{st} + 2) + 5m^s_{st} - m^l_{st}}{2(m_c + m^l_{st} + m^s_{st})}k +$$
$$\frac{[m_c(m_c + 2m^s_{st} + 3) + 6m^s_{st}](m^l_{st} - 1)}{2(m_c + m^l_{st} + m^s_{st})}. \tag{4}$$

Here parameter k, is the number of the edge in the chain starting from the central node of the most loaded star (m^l_{st}), $k = 1, 2, ..., (m_c + 1)$. *The synchronization thresholds for the edges of the stars* $(\zeta_s$ *or* $\zeta_l)$ *are the following:*

$$\widetilde{\varepsilon}^*_{m^{l,s}_{st}} = \frac{(m_c + 3)(m_c + 2m^{s,l}_{st}) + 4m^{l,s}_{st} - 8}{2(m_c + m^l_{st} + m^s_{st})}. \tag{5}$$

The values of the coupling strengths for the edges belonging to the chain are larger than for the edges of the stars. Moreover, the equality (4) allows to predict the number of the most heavily loaded edge:

$$k = \frac{m_c^2 + 2m_c + 2m_c m^s_{st} + 5m^s_{st} - m^l_{st}}{2(m_c + m^l_{st} + m^s_{st})}. \tag{6}$$

Number of this edge depends on both number of elements in the chain linking the stars and number of elements in the stars. Fig. 2 shows the distribution of synchronization thresholds $\widetilde{\varepsilon}^*_k$ for symmetric and asymmetric structures of a network when the number of elements in the chain connecting the stars is fixed $(m_c = const)$. From this figure we see that the increase of the number of elements in one of the stars causes
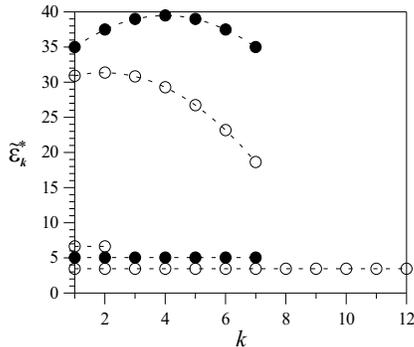
Fig. 2. The distribution of $\widetilde{\varepsilon}^*_k$ for the coupling schemes with the fixed number of elements in the chain linking the stars $(m_c = 6)$. The results for asymmetric case are shown by symbols "○" $(m^l_{st} = 13, m^s_{st} = 3)$, for the case when the number of elements in both stars are equal (symmetric case with $m^l_{st} = m^s_{st} = 8)$ are shown by symbols "●".
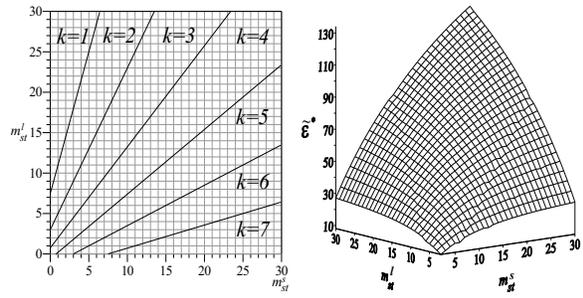
Fig. 3. (a) The ranges with the numbers of the most heavily loaded edges for the two star-coupled networks connected by the chain with $m_c = 6$. (b) The surface of maximal values for synchronization thresholds.

the thresholds shift for the edge requiring the largest coupling strength. This shift is directed to the star with the larger concentration of load on the central node. In this case the values of $\widetilde{\varepsilon}^*_k$ become smaller. Thus, the disturbance of the network symmetry decreases the synchronization threshold.

Figure 3(a) presents the diagram where the ranges with the numbers of the edges having the maximum traffic load on it (the so-called *weakest links* (Belykh *et al.*, 2005)) are shown ($m_c = 6$). On the basis of this diagram the surface of maximal values for $\widetilde{\varepsilon}^* = \max_k \widetilde{\varepsilon}^*_k$ was obtained, Fig. 3(b). This surface defines the values of $\widetilde{\varepsilon}^*$, that are sufficient for complete synchronization in ensembles with various numbers of elements in stars. Namely this value of $\widetilde{\varepsilon}^*$ determines the synchronization threshold for the networks with homogeneously coupled elements, when all coupling strengths are equal $(\varepsilon_k = \varepsilon$ for all $k)$. For this particular case the following statement allows to predict the lower bound for ε.

Statement 1.1: *The synchronization threshold for homogeneously coupled network composed of two stars connected by the chain takes the form*

$$\widetilde{\varepsilon}^* = \frac{(m_c + 3)(m^l_{st} - 1)(m_c + 2m^s_{st})}{2(m_c + m^l_{st} + m^s_{st})} +$$
$$\frac{[m^l_{st} - 5m^s_{st} - m_c(m_c + 2m^s_{st} + 2)]^2}{8(m_c + m^l_{st} + m^s_{st})^2}. \tag{7}$$

Note, that synchronization threshold for the symmetric structure of the network $(m^s_{st} = m^l_{st})$ will always be larger than for asymmetric one $(m^s_{st} \neq m^l_{st})$.

3.2 Numerical calculations for ensembles of mutually coupled Hodgkin-Huxley neuronal models

In the following we examine the synchronization properties for networks, whose elements are the classical Hodgkin-Huxley systems subjected to a sinusoidal drive:

$$C_m \dot{v}_i = -I_{ion} + I_{ext}(t) + \sum_{k=1}^{n} \varepsilon_{ik}(t) v_k,$$
$$\dot{m}_i = \alpha_m(v_i)(1 - m_i) - \beta_m(v_i) m_i, \qquad (8)$$
$$\dot{h}_i = \alpha_h(v_i)(1 - h_i) - \beta_h(v_i) h_i,$$
$$\dot{n}_i = \alpha_n(v_i)(1 - n_i) - \beta_n(v_i) n_i.$$

Here C_m is the capacitance of the membrane; $v_i = v_i^m - v_i^{eq}$ is the deviation of the membrane potential v_i^m from its equilibrium value v_i^{eq}; $I_{ion}(v_i, m_i, h_i, n_i) = G_{Na} m_i^3 h_i (v_i - v_{Na}) + G_K n_i^4 (v_i - v_K) - G_L(v_i - v_L)$ is the sum of the individual ionic currents through the membrane of i-th cell. The parameters G_{Na}, G_K, and G_L are the maximal conductances for the sodium, potassium and leakage channels, respectively, and v_{Na}, v_K, v_L are the corresponding reversal potentials. The last three kinetic equations in (8) describe the dynamics of the gating variables. Here, m_i and h_i are responsible for the activation and the inactivation of the Na^+-current, respectively, and n_i controls the K^+-current activation. The rate functions $\alpha(v_i)$ and $\beta(v_i)$ are interpreted as mean transition rates of ionic channels from the closed to the open state, and vice versa. In numerical simulations we employ the same set of parameter values as in (Pankratova *et al.*, 2006). We assume that the total synaptic current received by the neuron is $I_{ext}(t) = A \sin(2\pi f t)$. Note, that sinusoidal input is suprathreshold. This means that, subjected to this periodic drive, the system produces a generation of periodic or chaotic train of spikes. Particularly, the drive in some range of frequencies around $f = 123.5\ Hz$ causes a chaotic response of the cell.

In the following numerical analysis, to elicit the synchronous regime in ensembles of Hodgkin-Huxley elements, we compute the *average relative synchronization error* (Zhou and Kurths, 2003):

$$\delta = \frac{\overline{< |v_i - v_j| >_n}^t}{\sigma_v}. \qquad (9)$$

Here v_i is the membrane potential of i-th cell, σ_v is the standard deviation of $v(t)$ over time. Note, that the numerator in (9) involves both averaging over differences $|v_i - v_j|$, $i > j$, $\{i, j\} = 1, 2, ..., n$ (n is the number of elements in an ensemble) for the fixed time point, and the time averaging.

We consider three scenarios determining the increase of the number of elements n in a network composed of two stars connected by the chain. For the synchronization threshold the law of dependence on n in such ensembles is predicted by the use of formula (7).

- *Symmetrical structure with $m_c = const$ ($m_c = 8$). The numbers of elements in both stars are equal ($m_{st}^l = m_{st}^s \equiv m_{st}$) and increased. Thus, the total number of cells in*
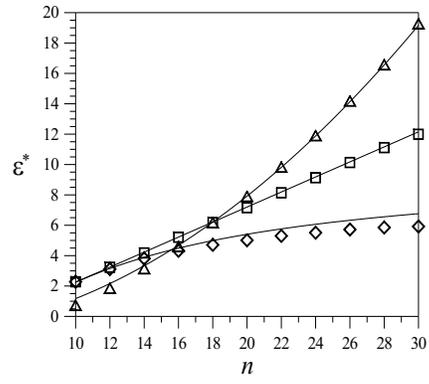
Fig. 4. The dependence of synchronization threshold ε^* ($a = 0.18$) on number of elements n for two symmetrical cases ($m_c = const$ is shown by "□"; $m_{st}^l = m_{st}^s = const$ is shown by "△") and extremely asymmetrical structure (with degeneracy of one of the stars; is shown by "◇"). The theoretical prediction is shown by curves, the numerical results are shown by symbols.

such network is $n = 8 + 2 m_{st}$. Substituting $m_{st} = 0.5(n - 8)$ into (7), we obtain:

$$\widetilde{\varepsilon^*} = \frac{11n}{4} - 15. \qquad (10)$$

- *Symmetrical structure with $m_{st}^l = m_{st}^s \equiv m_{st} = const$ ($m_{st} = 5$). The number of elements in the chain m_c is increased. Substitution of $m_c = n - 10$ into (7) yields:*

$$\widetilde{\varepsilon^*} = \frac{n^2}{8} - 6. \qquad (11)$$

- *Asymmetrical structure with $m_c = const$ and $m_{st}^s = const$ ($m_c = 8$, $m_{st}^s = 1$). The enlargement of the network due to the increase of m_{st}^l occurs. Similarly to the previous cases from (7) we obtain:*

$$\widetilde{\varepsilon^*} = \frac{(110 - 21n)^2}{8n^2}. \qquad (12)$$

As it is seen from Fig. 4, the theoretical estimation based on connection graph stability method ($\varepsilon^* = a\widetilde{\varepsilon^*}$), gives good agreement with the numerical data.

4. CONTROL OF SYNCHRONIZATION OF TIME-DEPENDENT MULTILAYERED NEURAL NETWORKS

A multilayered network describes a set of neural subdivisions each of which is a single layer of the network. This kind of the structure might be considered as a possible model for sensory-motor system. The last layer, thus, represents sensory processing and the first layer represents motor control. For the network considered here, the following assumptions will be done:
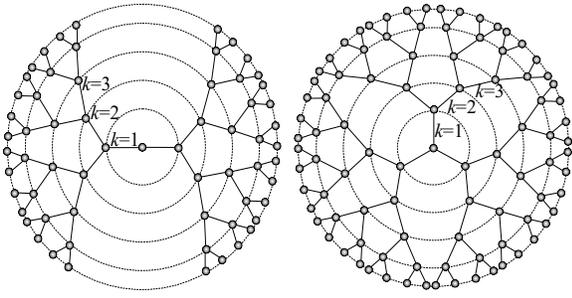
Fig. 5. Two types of multilayered structure of a network. The total number of elements involved is defined by the following power laws: $n = 2 \cdot 2^m - 1$ for the structure (a), and $n = 3 \cdot 2^m - 2$ for the structure (b). Here, m is the total number of layers in such a network.

  - there are no synapses between the elements of the same layer. All interactions are arranged in a hierarchical fashion, i.e. the information is transmitted between the neighboring layers only;
  - each element is affected by only a few other elements from the neighboring layers. In particular, we consider the case when each neuron of the k-th layer sends the signal to a couple of the neurons of the $(k+1)$-th layer.

A schematic illustration of two possible types of such a network are presented in Fig. 5. For both coupling schemes the interactions with the larger number of elements belonging to the following layer of hierarchy are established. In neurobiology these structures of the networks with a divergent type of the connections are widely used for simulation of various neuronal networks (Bloom *et al.*, 1985).

We now consider the network shown in Fig. 5(b) in more detail. In this structure the degree of all vertices for corresponding graph equals to three.

<u>Statement 2:</u> *For a network with multilayered structure, whose graph has all vertices of the third degree, Fig. 5(b), the synchronization threshold for an edge belonging to the k-th layer ($k = 1, 2, ..., m$), is defined by the equality:*

$$\widetilde{\varepsilon}_k^* = \frac{2^{2(m-k+1)}[(3m-5)2^k - 2(m-k) + 4]}{3 \cdot 2^m - 2} + \frac{2^{(m-k)}[(13 - 3m - 3k)2^k - 6] - 3}{3 \cdot 2^m - 2}. \tag{13}$$

The dependences of the synchronization thresholds on the current number of layers k and on the total number of layers m are presented in Fig.6.

The statement 2 allows as to consider the problem of synchronization control of growing in time networks. To better illustrate the requirements of the growing networks preserving complete synchronization we introduce two characteristics. The first is the *summary coupling strength of a layer*:
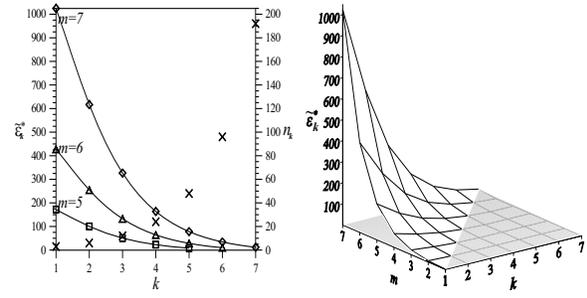
Fig. 6. (a) The decrease of the synchronization thresholds with the increase of k for the structure shown in Fig. 5(b), composed of m =5, 6, and 7 layers. The number of elements within the k-th layer is pointed by symbol " $\times$ ". (b) The surface of the bounds for the coupling strengths $\widetilde{\varepsilon}_k^*$ for the same structure.

$$\widetilde{\varepsilon}_l^* = n_k \widetilde{\varepsilon}_k^*, \tag{14}$$

where n_k is the number of elements in k-th layer.

In order to control the saving of the synchronous state with the growing number of layers in the network, we should increase the summary coupling strength for each layer in a certain manner. This increase may have an arbitrary dependence on time. The main goal of such an increase is to transfer values of $\widetilde{\varepsilon}_l^*$ to the curve for the larger m, see Fig.7(a). It is seen from the figure that the use of the summary coupling strength reduces the multilayered structure to the well-studied chain configuration, whose k-th element is a shrunk (integral) image of all elements from k-th layer. Therefore, as it was shown for the chain, the weakest link is the closest to the center of this chain. Namely this link requires the maximal increase of $\widetilde{\varepsilon}_l^*$ with the growing number of layers in network.

The second description is the *total coupling store*:

$$B = \sum_{k=1}^{m} \widetilde{\varepsilon}_l^*. \tag{15}$$

This characteristics allows to predict the increase of the costs for the total coupling store sufficient for the enlargement of the network without loss of stability of the synchronous state. This is a pri-
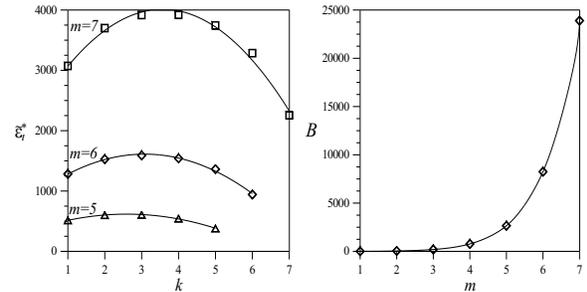
Fig. 7. (a) The summary coupling strength vs the number of layer k. (b) The increase of the total coupling store with the growing number of layers in network.
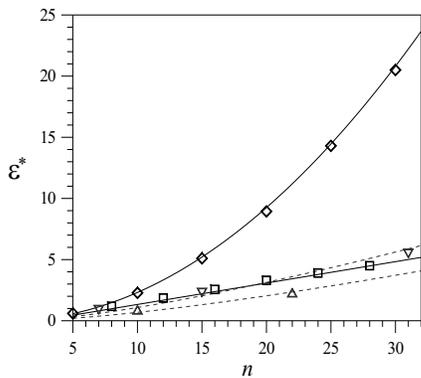
Fig. 8. Synchronization thresholds ε^* versus n for various homogeneously coupled networks. The theoretical estimates are shown by the curves, the numerical data are shown by symbols: $\diamond$ - the chain, $\square$ - the two star-coupled structure (symmetrical case with $m_c = 2$), ∇ - the multilayered structure shown in Fig. 5(a), $\triangle$ - the network shown in Fig. 5(b).

mary estimate. The correct distribution over the layers of network is defined by characteristics (14). For the coupling structure shown in Fig. 5(b), the increase of B with the increase of total number of layers m is presented in Fig. 7(b). It is seen, that the saving of the synchronous state in multilayered structure with the growing number of layers is "expensive".

Since the largest value of the coupling strength $\widetilde{\varepsilon^*}$ is required for the first layer, that is the closest to the central node of the network, this value determines the synchronization threshold for homogeneously coupled elements of the ensemble. For this case the following statement holds:

<u>Statement 2.1:</u> *The synchronization threshold for homogeneously coupled network, whose graph has the central node of the second degree, Fig. 5(a), is defined as:*

$$\widetilde{\varepsilon^*} = (m-1)2^m + 1. \qquad (16)$$

For graph with the central node of the third degree, Fig. 5(b), the threshold is:

$$\widetilde{\varepsilon^*} = [(m-1)2^m + 1]\frac{4 \cdot 2^m - 3}{3 \cdot 2^m - 2}. \qquad (17)$$

The corresponding dependences for the bounds of coupling strength on the total number of elements n are shown in Fig. 8.

5. CONCLUSIONS

In the present work the complete synchronization in application to the neuronal networks of mutually coupled Hodgkin-Huxley systems in a chaotic regime have been studied. The complete synchronization for various coupling schemes of

the network has been examined. In particular, we have considered the so-called multilayered neuronal networks that are suggestive of sensory-motor systems. Within the framework of recently developed connection graph stability method, the sufficient conditions on the coupling strengths were obtained. These conditions allow to increase the number of elements in network keeping the synchronized state stable. The numerical calculations with a high degree of precision have been predicted by this estimation.

REFERENCES

Belykh, V.N., I.V. Belykh and M. Hasler (2004). Connection graph stability method for synchronized coupled chaotic systems. *Physica D* **195**, 159-187.

Belykh, I., M. Hasler, M. Lauret and H. Nijmeijer (2005). Synchronization and graph topology. *Int. J. Bif. and Chaos* **15**, 3423-3433.

Bloom, F.E., A. Lazerson and L. Hofstadter (1985). *Brain, Mind, and Behavior.* W. H. Freeman and Company, New York.

Hodgkin, A.L. and A.F. Huxley (1952). A quantative description of membrane current and its application conduction and excitation in nerve. *J. Physiology* **117**, 500-544.

Izhikevich, E.M. (2006). Polychronization: Computation with spikes. *Neural Computation* **18**, 245-282.

Pankratova, E.V., V.N. Belykh and E. Mosekilde (2006). Role of the driving frequency in a randomly perturbed Hodgkin-Huxley neuron with suprathreshold forcing. *Europ. Phys. J. B* 00401.

Pecora, L.M. and T.L. Carroll (1998). Master stability functions for synchronized coupled limit-cycle and chaotic systems. *Phys. Rev. Lett.* **80**, 2109-2112.

Pikovsky, A., M. Rosenblum and J. Kurths (2001). *Synchronization: A Universal Concept in Nonlinear Science* (Cambridge University Press, Cambridge).

Pogromsky, A.Yu. and H. Nijmeijer (2001). Cooperative oscillatory behavior of mutually coupled dynamical systems. *IEEE Trans. Circuits Syst. I* **48**, 152-162.

Swadlow, H.A. (1992). Monitoring the excitability of neocortical efferent neurons to direct activation by extracellular current pulses. *J. Neurophysiol.* **68**, 605-619.

Wu, C.W. and L.O. Chua (1996). On a conjecture regarding the synchronization in an array of linearly coupled dynamical systems. *IEEE Trans. Circuits Syst. I* **43**, 161-165.

Zhou, C. and J. Kurths (2003). Noise-induced synchronization and coherence resonance of a Hodgkin-Huxley model of thermally sensitive neurons. *Chaos* **13**, 401.