INVESTIGATION OF DYNAMIC STABILITY OF ELASTIC AILERON OF WING IN SUBSONIC FLOW

Petr A. Velmisov
Department of higher mathematics
Ulyanovsk State Technical University
Russia
velmisov@ulstu.ru

Andrey V. Ankilov
Department of higher mathematics
Ulyanovsk State Technical University
Russia
ankil@ulstu.ru

Abstract
At designing structures and devices interacting with the flow of gas or liquid, it is necessary to solve the problems associated with the investigation of the stability required for their functioning and operational reliability. The definition of stability of an elastic body, taken in the article, corresponds to the Lyapunov's concept of stability of dynamical system. On the base of a proposed nonlinear mathematical model the dynamic stability of the elastic aileron taking into account the incident subsonic flow of gas or liquid (in an ideal model of a incompressible environment) is investigated. The model is described by coupled nonlinear system of differential equations for the unknown functions – the potential of the gas velocity and deformation of the elastic aileron. The sufficient conditions of the stability are obtained on the basis of the construction of functionals. The conditions impose restrictions on the free-stream velocity of the gas, the flexural stiffness of the elastic aileron, and other parameters of the mechanical system. The examples of construction of the stability regions for particular parameters of the mechanical system are presented.

Key words
Fluid-structure interaction; Stability; Dynamics; Wing aileron; Elastic plate; Subsonic flow

1 Introduction

At the design and exploitation of structures, devices, mechanisms for various applications, interacting with the flow of gas or liquid, an important problem is to ensure the reliability of their functioning and longer life. Similar problems are common to many branches of engineering. In particular, such problems arise in missilery, aircraft construction, instrumentation, at designing antenna systems, high-surface structures, and so on. The essential value in the calculation of structures that interact with the gas flow has a stability study of the deformable elements, as the impact of the flow may lead to its loss. As examples of the loss of dynamic stability can be noted: the flutter of an aircraft wing; panel flutter of plates and shells flowing around, for example, the flutter of the skin of the aircraft or missiles; stall flutter turbine blades and screws; fluctuations wires, chimneys, suspension bridges and so on.

Thus, at designing of the structures and devices interacting with the gas flow, it is necessary to solve problems related to the investigation of stability required for their functioning and operational reliability.

The stability of elastic bodies interacting with the gas flow is devoted to many theoretical and experimental studies conducted in the last decade. Among the recent studies on the dynamics, stability and flutter of the part of the aircraft, including airfoils, it should be noted the research of the scientists [Naumova, Ershov, Ivanov, 2011; Van'ko, Marchevskii, Shcheglov, 2011; Ovchinnicov, Popov, Filimonov, 2013; Plyusnin, 2014; Dimitrienko, Koryakov, Zakharov, Stroganov, 2014; Balakrishnan, 2005; Qin Zhanming and Librescu Liviu, 2002; Yatasaki Masahide, Isogai Koji, Uchida Takefumi, Yukimura Itsuma, 2004; Wu Xiao-sheng and Wu Jia-sheng, 2007; Haddadpour, 2003; Bendiksen and Seber, 2008, Florea, Hall, Dowell, 2000].

Most of the work is devoted to analytical and numerical investigations of aeroelastic oscillations the wing profile in a supersonic gas flow. For subsonic flow of the wing profile mainly used the numerical methods.

Among the works of the authors of this article on the dynamics and stability of elastic bodies interacting with the gas flow, note the monographs [Ankilov, Velmisov, 2000, 2009, 2013; Ankilov, Velmisov, Gorbokonenko, Pokladova, 2008; Velmisov, Kireev, 2011; Velmisov, Molgachev, 2012].

Taken in the work determination of stability of elastic body correspond to the Lyapunov concept of stability of dynamical systems.

2 Mathematical model of wing with elastic aileron

Let on the plane, in which take place the joint oscillations of elastic aileron and subsonic flow of an ideal gas (liquid), the segment $[a,b]$ of the axis Ox corresponds to the wing, and segment $[b,c]$ – to the aileron (fig. 1).

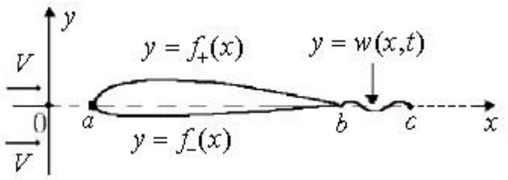

Figure 1. Wing profile.

In infinitely distant point the gas velocity is *V* and has a direction coinciding with the direction of the axis *Ox*. Assume that the deflection (strain) of the elastic aileron and the indignation homogeneous oncoming flow are small.

Enter designations: $u(x,t)\in C^{2,2}\left\{[b,c]\times R^{+}\right\}$ and $w(x,t)\in C^{4,2}\left\{[b,c]\times R^{+}\right\}$ – the deformations of an elastic aileron in the direction of axes of *Ox* and *Oy* respectively; $\phi(x,y,t)\in C^{2,2,1}\left\{R\times R\times R^{+}\right\}$ – the velocity potential of the disturbed flow.

The proposed mathematical model is defined by the following equations and boundary conditions: the velocity potential satisfies the Laplace equation

$$\Delta\phi\equiv\phi_{xx}+\phi_{yy}=0,\qquad (x,y)\in G=R^2\setminus[a,c],\quad (1)$$

linearized boundary conditions

$$\phi_y^{\pm}(x,0,t)=\lim_{y\to\pm0}\phi_y(x,y,t)=Vf_{\pm}'(x),\qquad x\in(a,b),\quad (2)$$

$$\phi_y^{\pm}(x,0,t)=\dot{w}(x,t)+Vw'(x,t),\qquad x\in(b,c),\quad (3)$$

condition of absence of perturbations at infinitely distant point

$$|\nabla\phi|_{\infty}^2\equiv(\phi_x^2+\phi_y^2+\phi_t^2)_{\infty}=0.\quad (4)$$

The equation of oscillations of elastic aileron have the form

$$\begin{cases}-EF\left(u'(x,t)+0.5w'^2(x,t)\right)'+M\ddot{u}(x,t)=0,\\ -EF\left[w'(x,t)\left(u'(x,t)+0.5w'^2(x,t)\right)\right]'+\\ +EJw''''(x,t)+M\ddot{w}(x,t)+\beta_0 w(x,t-\tau)+\\ +\beta_1\dot{w}(x,t)+\beta_2\dot{w}''''(x,t)=\rho(\phi_t^{+}(x,0,t)-\\ -\phi_t^{-}(x,0,t))+\rho V(\phi_x^{+}(x,0,t)-\phi_x^{-}(x,0,t)),\\ x\in(b,c).\end{cases}\quad (5)$$

The indices *x*, *y*, *t* below denote partial derivatives with respect to *x*, *y*, *t*; the bar and the point – the partial derivatives with respect to *x* and *t*, respectively; ρ – density of gas; $EJ=Eh^3/(12(1-\nu^2))$ – flexural stiffness of aileron; *h* – thickness of aileron; $M=h\rho_n$ – linear mass of aileron; $F=h/(1-\nu^2)$; E, ρ_n – elasticity modulus and the linear density of the aileron; ν – Poisson coefficient; β_2,β_1 – coefficients of internal and external damping; β_0 – stiffness coefficient of the base (compressing layer); τ – time of the delay of base reaction; $f_{\pm}(x)$ – functions determining the shape of the upper (+) and lower (–) non-deformable parts of the profile.

Using the methods of the theory of functions of a complex variable [Ankilov and Velmisov, 2013], the aerohydrodynamic loading according to (1) – (4) is possible to express through the unknown functions of deformations (*u*, *w*) of the aileron:

$$\begin{aligned}&\rho(\phi_t^{+}(x,0,t)-\phi_t^{-}(x,0,t))+\rho V(\phi_x^{+}(x,0,t)-\phi_x^{-}(x,0,t))=\\ &=-\frac{\rho}{\pi}\int_b^c[\ddot{w}(x_1,t)+V\dot{w}'(x_1,t)]K(x_1,x)dx_1-\\ &-\frac{V\rho}{\pi}\int_b^c[\dot{w}(x_1,t)+Vw'(x_1,t)]\frac{\partial K(x_1,x)}{\partial x}dx_1+\\ &+\frac{V^2\rho}{\pi}\int_a^b[f_{+}'(x_1)+f_{-}'(x_1)]G(x_1,x)dx_1,\ x\in(b,c),\end{aligned}\quad (6)$$

where

$$K(x_1,x)=2ln\left|\frac{\sqrt{(x-a)(c-x_1)}+\sqrt{(x_1-a)(c-x)}}{\sqrt{(x-a)(c-x_1)}-\sqrt{(x_1-a)(c-x)}}\right|,$$

$$G(x_1,x)=\frac{\sqrt{(x-a)(c-x)}+\sqrt{(x_1-a)(c-x_1)}}{\sqrt{(x-a)(c-x)}(x-x_1)},\ x_1\neq x.$$

We will notice that for any function $\mathrm{g}(x_1,x)\in C^{1,1}\left\{[b,c]\times[b,c]\right\}$ the improper integrals

$$\int_a^c dx\int_a^c \mathrm{g}(x_1,x)K(x_1,x)dx_1,\quad \int_a^c dx\int_a^c \mathrm{g}(x_1,x)\frac{\partial K(x_1,x)}{\partial x}dx_1,$$

$\int_a^c dx\int_a^c \mathrm{g}(x_1,x)G(x_1,x)dx_1$ are convergent.

Assume the wing profile is symmetric, i.e. $f_{+}(x)=-f_{-}(x)$ (this also takes place for the keel of the aircraft with an elastic rudder (fig. 2)).

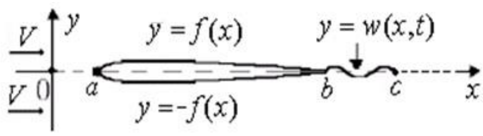

Figure 2. Profile of aircraft keel.

In this case, according to (5), (6) we will have the homogeneous system:

$$\begin{cases}-EF\left(u'(x,t)+0.5w'^2(x,t)\right)'+M\ddot{u}(x,t)=0,\\ -EF\left[w'(x,t)\left(u'(x,t)+0.5w'^2(x,t)\right)\right]'+EJw''''(x,t)+\\ +M\ddot{w}(x,t)+\beta_0 w(x,t-\tau)+\beta_1\dot{w}(x,t)+\beta_2\dot{w}''''(x,t)=\\ =-\frac{\rho}{\pi}\int_b^c[\ddot{w}(x_1,t)+V\dot{w}'(x_1,t)]K(x_1,x)dx_1-\\ -\frac{V\rho}{\pi}\int_b^c[\dot{w}(x_1,t)+Vw'(x_1,t)]\frac{\partial K(x_1,x)}{\partial x}dx_1,\quad x\in(b,c).\end{cases}\quad (7)$$

The boundary conditions at the ends of the aileron in $x=b$ and $x=c$ have the form:

$$w(b,t)=0,\quad w''(b,t)=\alpha w'(b,t),\quad u(b,t)=0,$$
$$w''(c,t)=0,\quad w'''(c,t)=0,\quad u'(c,t)+0.5w'^2(c,t)=0, \tag{8}$$

that corresponds to the elastic fastening of the left end and free right end. Number α – coefficient of rigidity of the elastic connection between the wing and aileron.

3 Investigation of stability of elastic aileron

We will obtain the sufficient conditions for the stability of solutions of integro-differential equations (7) with respect to perturbations of the initial conditions.

Introduce a functional

$$\Phi(t)=\int_b^c\Big\{M(\dot u^2+\dot w^2)+EJw''^2+EF\left(u'+0.5w'^2\right)^2+$$
$$+\beta_0 w^2+\beta_0\int_{t-\tau}^{t}dt_1\int_{t_1}^{t}\dot w^2(x,s)ds\Big\}dx+\alpha EJw'^2(b,t)+ \tag{9}$$
$$+I(t)+J(t),$$

where

$$I(t)=\frac{\rho}{\pi}\int_b^c dx\int_b^c \dot w(x,t)\dot w(x_1,t)K(x_1,x)dx_1,$$
$$J(t)=-\frac{\rho V^2}{\pi}\int_b^c dx\int_b^c w'(x,t)w'(x_1,t)K(x_1,x)dx_1.$$

Lets find the derivative of Φ by t. In view of equality $w(x,t-\tau)=w(x,t)-\int_{t-\tau}^{t}\dot w(x,s)ds,$ for functions $w(x,t)$ and $u(x,t)$, that are solutions of the equations (7), the expression for $\dot\Phi(t)$ takes the form:

$$\dot\Phi(t)=2\int_b^c\Big\{EF\dot u\left(u'+0.5w'^2\right)'+EF\dot w\left[w'\left(u'+0.5w'^2\right)\right]'-$$
$$-EJ\dot w w''''-\beta_0\dot w w+\beta_0\dot w\int_{t-\tau}^{t}\dot w(x,s)ds-\beta_1\dot w^2-$$
$$-\beta_2\dot w\dot w''''-\frac{\rho}{\pi}\dot w(x,t)\int_b^c(\ddot w(x_1,t)+V\dot w'(x_1,t))K(x_1,x)dx_1-$$
$$-\frac{V\rho}{\pi}\dot w(x,t)\int_b^c(\dot w(x_1,t)+Vw'(x_1,t))\frac{\partial K(x_1,x)}{\partial x}dx_1+EJw''\dot w''+$$
$$+EF\dot u'\left(u'+0.5w'^2\right)+EFw'\dot w'\left(u'+0.5w'^2\right)+$$
$$+\beta_0 w\dot w+\frac{\beta_0\tau}{2}\dot w^2(x,t)-\frac{\beta_0}{2}\int_{t-\tau}^{t}\dot w^2(x,s)ds\Big\}dx+ \tag{10}$$
$$+2\alpha EJw'(b,t)\dot w'(b,t)+\dot I(t)+\dot J(t).$$

Integrating by parts, taking into account (8), obtain

$$\int_b^c \dot w w''''dx=\alpha\dot w'(b,t)w'(b,t)+\int_b^c\dot w''w''dx,$$
$$\int_b^c \dot w\dot w''''dx=\alpha\dot w'^2(b,t)+\int_b^c\dot w''^2dx,$$
$$\int_b^c\dot u\left(u'+0.5w'^2\right)'dx=-\int_b^c\dot u'\left(u'+0.5w'^2\right)dx,$$
$$\int_b^c\dot w\left[w'\left(u'+0.5w'^2\right)\right]'dx=-\int_b^c\dot w'w'\left(u'+0.5w'^2\right)dx.$$

Changing the order of integration, effect the integration by parts

$$\int_b^c dx\int_b^c\dot w(x,t)\dot w(x_1,t)\frac{\partial K(x_1,x)}{\partial x}dx_1=\int_b^c dx_1\int_b^c\dot w(x,t)\times$$
$$\times\dot w(x_1,t)\frac{\partial K(x_1,x)}{\partial x}dx=\int_b^c\dot w(x,t)\dot w(x_1,t)K(x_1,x)\,|_b^c\,dx_1-$$
$$-\int_b^c dx_1\int_b^c\dot w'(x,t)\dot w(x_1,t)K(x_1,x)dx=$$
$$=-\int_b^c dx\int_b^c\dot w'(x_1,t)\dot w(x,t)K(x_1,x)dx_1,$$

where in the last equality the integration variables x and x_1 are changed places, considering that $K(x_1,x)=K(x,x_1)$.

Similarly we find

$$\int_b^c dx\int_b^c\dot w(x,t)w'(x_1,t)\frac{\partial K(x_1,x)}{\partial x}dx_1=$$
$$=-\int_b^c dx\int_b^c\dot w'(x,t)w'(x_1,t)K(x_1,x)dx_1.$$

Substituting these relations in (10), we obtain

$$\dot\Phi(t)=\int_b^c\Big\{2\beta_0\dot w\int_{t-\tau}^{t}\dot w(x,s)ds+\beta_0\tau\dot w^2(x,t)-$$
$$-\beta_0\int_{t-\tau}^{t}\dot w^2(x,s)ds-2\beta_1\dot w^2-2\beta_2\dot w''^2\Big\}dx-2\beta_2\alpha\times$$
$$\times\dot w'^2(b,t)-\frac{2\rho}{\pi}\int_b^c\Big\{\dot w(x,t)\int_b^c\ddot w(x_1,t)K(x_1,x)dx_1\Big\}dx+ \tag{11}$$
$$+\frac{2\rho V^2}{\pi}\int_b^c\Big\{\dot w'(x,t)\int_b^c w'(x_1,t)K(x_1,x)dx_1\Big\}dx+\dot I+\dot J.$$

Using the inequality $2ab\le a^2+b^2$, we obtain $2\dot w(x,t)\dot w(x,s)\le\le\dot w^2(x,t)+\dot w^2(x,s)$. Substituting this estimate in (11), finally find

$$\dot\Phi(t)\le\int_b^c\left\{2\beta_0\tau\dot w^2(x,t)-2\beta_1\dot w^2-2\beta_2\dot w''^2\right\}dx-$$
$$-2\beta_2\alpha\dot w'^2(b,t)-\frac{2\rho}{\pi}\int_b^c\Big\{\dot w(x,t)\int_b^c\ddot w(x_1,t)K(x_1,x)dx_1\Big\}dx+ \tag{12}$$
$$+\frac{2\rho V^2}{\pi}\int_b^c\Big\{\dot w'(x,t)\int_b^c w'(x_1,t)K(x_1,x)dx_1\Big\}dx+\dot I+\dot J.$$

Transform the integral $\dot I(t)$:

$$\dot I(t)=\frac{d}{dt}\frac{\rho}{\pi}\int_b^c dx\int_b^c\dot w(x,t)\dot w(x_1,t)K(x_1,x)dx_1=$$
$$=\frac{\rho}{\pi}\int_b^c dx\int_b^c\ddot w(x,t)\dot w(x_1,t)K(x_1,x)dx_1+$$

$$+\frac{\rho}{\pi}\int_b^c dx\int_b^c \dot{w}(x,t)\ddot{w}(x_1,t)K(x_1,x)dx_1.$$

Since the $K(x_1,x)=K(x,x_1)$, then, changing at first the order of integration, and then the variables x_1 and x by places, we will have:

$$\frac{\rho}{\pi}\int_b^c dx\int_b^c \dot{w}(x_1,t)\ddot{w}(x,t)K(x_1,x)dx_1 =$$

$$=\frac{\rho}{\pi}\int_b^c dx\int_b^c \dot{w}(x,t)\ddot{w}(x_1,t)K(x_1,x)dx_1.$$

For $\dot{I}(t)$ thus we obtain the following expression:

$$\dot{I}(t)=\frac{2\rho}{\pi}\int_b^c dx\int_b^c \dot{w}(x,t)\ddot{w}(x_1,t)K(x_1,x)dx_1. \qquad (13)$$

Similar transformations for $\dot{I}(t)$, we find an expression for $\dot{J}(t)$

$$\dot{J}(t)=-\frac{2\rho V^2}{\pi}\int_b^c dx\int_b^c \dot{w}'(x,t)w'(x_1,t)K(x_1,x)dx_1. \qquad (14)$$

Substituting (13) and (14) to the right side of (12), we will have

$$\dot{\Phi}(t)\le 2\int_b^c\left\{\beta_0\tau\dot{w}^2(x,t)-\beta_1\dot{w}^2-\beta_2\dot{w}''^2\right\}dx- \\ -2\beta_2\alpha\dot{w}'^2(b,t). \qquad (15)$$

The boundary value problem for the equation $\psi^{IV}(x)=\mu\psi(x),\ x\in[b,c]$ with the boundary conditions (8) is considered. This problem is self-adjoint and completely defined on condition

$$\alpha\ge 0. \qquad (16)$$

Indeed, integrating by parts, can easily verify that

$$\int_b^c u(x)v^{IV}(x)dx=\int_b^c v(x)u^{IV}(x)dx,\quad \int_b^c u(x)u^{IV}(x)dx>0,$$

for any functions $u(x)$ and $v(x)$, which satisfy the considered boundary conditions and have on $[b,c]$ the continuous derivatives of the fourth order. For functions $\dot{w}(x,t)$ we write Rayleigh's inequality [Kollatc, 1968]:

$$\int_b^c \dot{w}(x,t)\dot{w}^{IV}(x,t)dx\ge\mu_1\int_b^c \dot{w}(x,t)\dot{w}(x,t)dx,$$

where μ_1 – the smallest eigenvalue of the considered boundary value problem. Integrating by parts, the inequality is represented in the form:

$$\int_b^c \dot{w}''^2(x,t)dx+\alpha\dot{w}'^2(b,t)\ge\mu_1\int_b^c \dot{w}^2(x,t)dx. \qquad (17)$$

Thus, taking into account (17), the inequality (15) takes the form

$$\dot{\Phi}(t)\le-\frac{2}{\mu_1}\int_b^c(\beta_1+\mu_1\beta_2-\beta_0\tau)\dot{w}''^2dx. \qquad (18)$$

Let condition

$$\beta_0\tau-\beta_1-\mu_1\beta_2\le 0, \qquad (19)$$

then $\dot{\Phi}(t)\le 0$. Integrating from 0 to *t*, we obtain:

$$\Phi(t)\le\Phi(0). \qquad (20)$$

For an assessment (20) we will use the proved in [Ankilov and Velmisov, 2009] following theorem.

Theorem 1. *Suppose that: 1) the function $f(x)$ is continuous on the interval $x\in[a,c]$; 2) the function $K(x_1,x,c)$ is defined and continuous by x and x_1 in a region $x\in[a,c],\ x_1\in[a,c]$ (except, perhaps, line $x=x_1$) and integrable in this area; 3) the function $K(x_1,x,c)$ is continuously differentiable with respect to c and the equality $\frac{\partial K}{\partial c}=\omega(x,c)\cdot\omega(\tau,c)$ is verified; 4) for any $\alpha\in(a,c],\quad x,\tau\in(a,\alpha)$ the equality $K(\alpha,x,\alpha)=K(x_1,\alpha,\alpha)=0$ is verified; 5) $\lim_{c\to a}\int_a^c dx\int_a^c K(x_1,x,c)dx_1=0$, then repeated (proper or improper) integral is nonnegative* $\int_a^c dx\int_a^c f(x)\times$

$$\times f(x_1)K(x_1,x,c)dx_1=\int_a^c\left(\int_a^\alpha f(x)\omega(x,\alpha)dx\right)^2 d\alpha\ge 0.$$

We introduce the notation

$$f(x)=\begin{cases}0,\ x\in[a,b];\\ \dot{w}(x,t_0),\ x\in[b,c].\end{cases}$$

The kernel

$$K(x_1,x,c)=\ln\left|\frac{\sqrt{(x-a)(c-x_1)}+\sqrt{(x_1-a)(c-x)}}{\sqrt{(x-a)(c-x_1)}-\sqrt{(x_1-a)(c-x)}}\right|,$$

where as a parameter the c is taken, satisfies the conditions of theorem 1:

1) $$\frac{\partial K}{\partial c}=\frac{\sqrt{(x-a)(x_1-a)}}{(c-a)\sqrt{(c-x)(c-x_1)}}=\frac{\sqrt{x-a}}{\sqrt{(c-a)(c-x)}}\times \\ \times\frac{\sqrt{x_1-a}}{\sqrt{(c-a)(c-x_1)}}=\omega(x,c)\cdot\omega(x_1,c);$$

2) $K(c,x,c)=\ln\left|\frac{\sqrt{c-x}}{\sqrt{c-x}}\right|=0,\ K(x_1,c,c)=\ln\left|\frac{\sqrt{c-x_1}}{\sqrt{c-x_1}}\right|=0,$ $\forall x,x_1\in(a,c)$;

3) as $K(x_1,x,c)\in[0,+\infty)$ in the field $x\in[b,c],\ x_1\in[b,c]$, and the function is integrable in this area, then by the mean value theorem exist numbers $\theta_1\ne\theta_2,\ 0<\theta_1,\theta_2<1$ such that

$$\lim_{c\to a}\int_a^c dx\int_a^c ln\left|\frac{\sqrt{(x-a)(c-x_1)}+\sqrt{(x_1-a)(c-x)}}{\sqrt{(x-a)(c-x_1)}-\sqrt{(x_1-a)(c-x)}}\right|dx_1=$$

$$=\lim_{c\to a}(c-a)^2 ln\left|\frac{\sqrt{\theta_2(1-\theta_1)}+\sqrt{\theta_1(1-\theta_2)}}{\sqrt{\theta_2(1-\theta_1)}-\sqrt{\theta_1(1-\theta_2)}}\right|=0.$$

Hence, by theorem 1, the improper integral is non-negative

$$\int_a^c dx \int_a^c f(x) f(x_1) K(x_1, x, c) dx_1 =$$

$$= \int_a^c \left(\int_a^\alpha \frac{\sqrt{x-a}\, f(x)}{\sqrt{(\alpha-a)(\alpha-x)}} dx \right)^2 d\alpha \geq 0.$$

Substituting the function $f(x)$, obtain

$$\int_b^c dx \int_b^c \dot{w}(x, t_0) \dot{w}(x_1, t_0) K(x_1, x) dx_1 \geq 0.$$

Since the integral is non-negative for any value $t = t_0$, then

$$\int_b^c dx \int_b^c \dot{w}(x,t) \dot{w}(x_1,t) K(x_1, x) dx_1 \geq 0 . \quad (21)$$

Similarly,

$$\int_b^c dx \int_b^c w'(x,t) w'(x_1,t) K(x_1, x) dx_1 \geq 0 . \quad (22)$$

The study of the functional is continued. Taking into account the expression (9) and inequalities (21), (22), the right and left sides of (20) are estimated as follows:

$$\Phi(t) \geq \int_b^c EJw''^2 dx - \frac{\rho V^2}{\pi} \int_b^c dx \int_b^c w'(x,t) \times$$
$$\times w'(x_1,t) K(x_1,x) dx_1 + \alpha EJw'^2(b,t), \quad (23)$$

$$\Phi(0) \leq \int_b^c \left\{ M(\dot{u}_0^2 + \dot{w}_0^2) + EJw_0''^2 + EF\left(u_0' + \frac{1}{2} w_0'^2\right)^2 + \right.$$
$$\left. + \beta_0 w_0{}^2 \right\} dx + \frac{\rho}{\pi} \int_b^c dx \int_b^c \dot{w}(x,0) \dot{w}(x_1,0) K(x_1,x) dx_1 + \quad (24)$$
$$+ \alpha EJw'^2(b,0),$$

where the notations $\dot{w}_0 = \dot{w}(x,0),\ u_0' = u'(x,0),$ $w_0 = w(x,0),\ \dot{u}_0 = \dot{u}(x,0),\ w_0' = w'(x,0),\ w_0'' = w''(x,0)$ are introduced.

Using the obvious inequalities $2ab \leq a^2 + b^2$, $-2ab \geq -(a^2 + b^2)$, symmetric and non-negative kernel $K(x_1, x)$, boundary conditions (8), we obtain:

$$\int_b^c dx \int_b^c \dot{w}(x,0) \dot{w}(x_1,0) K(x_1,x) dx_1 \leq \int_b^c dx \int_b^c \dot{w}^2(x,0) \times$$
$$\times K(x_1,x) dx_1 \leq \int_b^c K_0 \dot{w}^2(x,0) dx,\ K_0 = \sup_{x\in(b,c)} \int_b^c K(x_1,x) dx_1.$$

Similarly,

$$\int_b^c dx \int_b^c w'(x,t) w'(x_1,t) K(x_1,x) dx_1 \leq \int_b^c K_0 w'^2(x,t) dx.$$

Taking into account these estimates, the inequalities (23) and (24) take the form

$$\Phi(0) \leq \int_b^c \left\{ \left(M + \frac{\rho K_0}{\pi} \right) \dot{w}_0^2 + M\dot{u}_0^2 + EJw_0''^2 + \right.$$
$$\left. + EF\left(u_0' + \frac{1}{2} w_0'^2\right)^2 + \beta_0 w_0{}^2 \right\} dx + \alpha EJw'^2(b,0). \quad (25)$$

$$\Phi(t) \geq \int_b^c \left\{ EJw''^2 - \frac{\rho V^2 K_0}{\pi} w'^2 \right\} dx + \alpha EJw'^2(b,t), \quad (26)$$

Applying the Cauchy-Bunyakovski inequality, have

$$\int_b^c w''^2(x,t) dx \geq \frac{2}{(c-b)^2} \int_b^c \left(w'(x,t) - w'(b,t) \right)^2 dx, \quad (27)$$

$$w^2(x,t) \leq (c-b) \int_b^c w'^2(x,t) dx. \quad (28)$$

Using (27), from (26) we obtain

$$\Phi(t) \geq \int_b^c \left\{ \frac{2\pi EJ - \rho K_0 V^2 (c-b)^2}{\pi (c-b)^2} w'^2(x,t) - \right.$$
$$\left. - \frac{4EJ}{(c-b)^2} w'(x,t) w'(b,t) + \frac{(2+\alpha(c-b))EJ}{(c-b)^2} w'^2(b,t) \right\} dx. \quad (29)$$

According to the Sylvester criterion, the quadratic form with regard to $w'(x,t),\ w'(b,t)$ is positive definiteness, if next condition is verified:

$$V^2 < \frac{2\pi\alpha EJ}{(c-b)\rho K_0 (2+\alpha(c-b))}. \quad (30)$$

Then, according to (28), (29), we obtain

$$\Phi(t) \geq w^2(x,t) \times$$
$$\times \frac{\left(2EJ\pi - \rho K_0 V^2 (c-b)^2\right)(2+\alpha(c-b)) - 4EJ\pi}{\pi (c-b)^3 (2+\alpha(c-b))}. \quad (31)$$

Thus, from (20), (25) and (31) we obtain the inequality

$$w^2(x,t) \leq \frac{\pi (c-b)^3 (2+\alpha(c-b))}{\left(2EJ\pi - \rho K_0 V^2 (c-b)^2\right)(2+\alpha(c-b)) - 4EJ\pi} \times$$
$$\times \left(\int_b^c \left\{ \left(M + \frac{\rho K_0}{\pi} \right) \dot{w}_0^2 + M\dot{u}_0^2 + EJw_0''^2 + \right. \right.$$
$$\left. \left. + EF\left(u_0' + \frac{1}{2} w_0'^2\right)^2 + \beta_0 w_0{}^2 \right\} dx + \alpha EJw'^2(b,0) \right),$$

from which follows the next theorem.

Theorem 2. *Let the conditions (16), (19), (30). Then the solution $w(x,t)$ of the system of equations (7) is stable with respect to perturbations of the initial data $\dot{w}_0, w_0, w_0', w_0'', \dot{u}_0, u_0'$, if the function $w(x,t)$ satisfies the boundary conditions (8).*

4 Example of mechanical system

We consider the example of a mechanical system. Assume that the wing is located in the air flow ($\rho = 1$), and the aileron is made from aluminum ($E = 7 \cdot 10^{10}$, $\rho_n = 8480$). Other parameters of the mechanical system: $a = 0$; $b = 3$; $c = 4$; $\nu = 0.31$; $\beta_0 = 4$; $\beta_1 = 0.4$; $\beta_2 = 0.4$; $\alpha = 0.1$ (all values are given in the SI system).

For inequality (30) the stability region (gray area) on the plane «aileron thickness *h* – flow velocity *V*» is constructed (fig. 3).

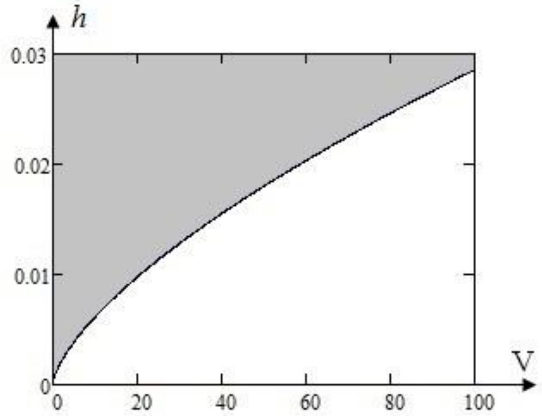

Figure 3. The stability region in the plane (*h*, *V*).

5 Conclusion

Based on the proposed mathematical model of flow around a wing with elastic aileron to subsonic flow of a liquid or gas (in the model of an ideal incompressible environment) obtain sufficient conditions of the dynamic stability of the aileron. The conditions impose restrictions on the velocity of the gas, the flexural stiffness of the elastic aileron and other parameters of the mechanical system. The case of elastic fastening one end and the free other end of the elastic aileron is considered. The region of stability on the plane of the two parameters (*h*, *V*) is built for specific examples of mechanical systems.

Acknowledgements

The assignment is done within the framework of the State Job № 2014/232 of the Russian Ministry of Education and Science and is supported by grants № 15-01-08599, № 15-41-02455-r-volga-a of Russian Foundation for Basic Research.

References

Ankilov, A.V., Velmisov, P.A. (2000). Stability of viscoelastic elements of walls of flowing channels. Ulyanovsk State Technical University. Ulyanovsk.

Ankilov, A.V., Velmisov, P.A., Gorbokonenko, V.D., Pokladova, Yu.V. (2008). Mathematical modeling of mechanical system «the pipeline – pressure sensor». Ulyanovsk State Technical University. Ulyanovsk.

Ankilov, A.V., Velmisov, P.A. (2009). Dynamics and stability of elastic plates at aerohydrodynamic influence. Ulyanovsk State Technical University. Ulyanovsk.

Ankilov, A.V., Velmisov, P.A. (2013). Mathematical modeling in problems of dynamic stability of deformable elements of constructions at aerohydrodynamic influence. Ulyanovsk State Technical University. Ulyanovsk.

Balakrishnan, A.V. (2005). Toward a mathematical theory of aeroelasticity. IFIP, № 166, pp. 1-24.

Bendiksen, O.O., Seber, G. (2008). Fluid-structure interactions with both structural and fluid nonlinearities. J. Sound and Vibr., V. 315, № 3, pp. 664-684.

Dimitrienko, Yu.I., Koryakov, M.N., Zakharov, A.A., Stroganov, A.S. (2014). Computational modeling of conjugated gas dynamic and thermo mechanical processes in composite structures of high speed aircraft. Mathematical modeling and numerical methods, V. 2, №3-3 (3), pp. 3-24.

Florea Razvan, Hall Kenneth C., Dowell Earl H. (2000). Analysis of eigenvalues and reduced order model of nonstationary transonic potential flow of profiles to define the boundaries of flutter. J. Aircraft, №3 (37), pp. 454-462.

Kollatc, L. (1968). Problems on eigenvalues. Science. Moscow.

Naumova, N.V., Ershov, V.A., Ivanov, D.N. (2011). Deformation of elastic spherical shell, fixed at the equator, in the flow of a viscous incompressible fluid // Bulletin of St. Petersburg university, series 1, issue 3, pp. 124-130.

Ovchinnicov, V.V., Popov, V.M., Filimonov, S.V. (2013). Application of the extended hypothesis of harmonicity for the calculation of flutter characteristics of the aircraft. Scientific Bulletin of the Moscow State Technical University of Civil Aviation, № 195, pp. 93-100.

Plyusnin, A.V. (2014). Boundary-element-method modelling of inside and outside non-stationary interaction of aircraft body and liquid. Mathematical modeling and numerical methods, V. 2, №2-2 (2), pp. 77-100.

Qin Zhanming, Librescu Liviu (2002). Management of flutter of wing at subsonic flight. [Conference on «Smart Structures and Materials: Smart Structures and Integrated Systems», San Diego, Calif., 18-21 March, 2002] Proc. SPIE. 4701, pp. 281-292.

Soltani, N., Esfahanian, V., Haddadpour, H. (2003). Analytical prediction of panel flutter using unsteady potential flow. Journal of Aircraft, V. 40, № 4, pp. 805-807.

Van'ko, V.I., Marchevskii, I.K., Shcheglov, G.A. (2011). Numerical and Analytical Method for Studying the Stability of Airfoil Equilibrium Positions in Flow. Bulletin of Bauman Moscow State Technical University. Series: Natural sciences. № 5, pp. 3-10.

Velmisov, P.A., Kireev, S.V. (2011). Mathematical modeling in problems of static instability of elastic elements of constructions at aerohydrodynamic influence. Ulyanovsk State Technical University. Ulyanovsk.

Velmisov, P.A. Molgachev, A.A. (2012). Mathematical modeling in problems of dynamic stability of viscoelastic elements of flowing channels. Ulyanovsk State Technical University. Ulyanovsk.

Wu Xiao-sheng and Wu Jia-sheng. (2007). Numerical method for the calculation of the swept wing flutter. Trans Beijing Inst. Technol., №5 (27), pp. 385-389.

Yatasaki Masahide, Isogai Koji, Uchida Takefumi, Yukimura Itsuma. (2004). Shock-stall-flutter of a two-dimensional airfoil. AIAA Journal, №2 (42), pp. 215-219.