IMPULSIVE CONTROL SYSTEMS WITH TRAJECTORIES OF BOUNDED P-VARIATION

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Abstract
"Rough" differential equations form a class of control-affine dynamical systems driven by input signals of a low regularity, namely, paths of bounded p-variation (BV_p), $p > 1$.
In this paper, we address impulsive rough control systems, i.e., rough differential equations driven by discontinuous BV_p-controls. For the case of scalar controls and $p \in [1, 2)$, a constructive representation of the system's states is obtained.

Key words
Rough differential equations, impulsive control, p-variation.

1 Introduction

This study is undertaken towards developing the mathematical theory of impulsive control systems with states of unbounded variation, and appeals to a relatively new and challenging branch of the modern control theory called *the rough paths theory*. This control-theoretical framework, originated in [Lyons, 1994], and further developed in [Gubinelli, 2004, Dudley and Norvaiša, 2011, Lejay, 2013, Lyons, 1998, Lyons and Qian, 2002], is, actually, the theory of control differential equations driven by paths (continuous controls) of class BV_p — introduced by N. Wiener and composed of functions having bounded p-variation, $p \geq 1$, — with deep roots in differential geometry and a rich algebraic background. Being in deterministic settings, the theory of rough paths is at the same time closely related to stochastic control and noisy differential equations by Îto and Stratonovich.

In this talk, we extend the concept of rough differential equations to the impulsive control setup by admitting discontinuous controls (and discontinuous states) of bounded p-variation, $p \in [1, 2)$.

Depending on the "order of irregularity" of input signals, one can mark out three basically different settings for control-affine dynamical systems.

1.1 The well-studied case $p = 1$: "Classical" impulsive control by signals of the Jordan's class BV

Impulsive control systems, acting over a finite control period $T = [a, b] \subset \mathbb{R}$, are commonly described by measure differential equations of the sort

$$dx = f(x)\,dt + G(x)\,dw, \;\; x(a) = x_0, \;\; t \in T, \quad (1)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^{m \times n}$ are given locally Lipschitz continuous vector and matrix functions; states $x : T \to \mathbb{R}^n$ and controls $w : T \to \mathbb{R}^m$ are (discontinuous) functions of bounded variation (BV). Differential forms dx, dw can be treated here as vector-valued Borel measures induced by respective functions. For continuous controls, a solution of (1) can be defined by Lebesgue-Stieltjes or Perron-Stieltjes integration against a given function w.

A natural way system (1) enters the scene is a trajectory compactification (relaxation) of an ordinary control-affine system

$$\dot{x} = f(x) + G(x)\,\dot{w}, \;\; x(a) = x_0, \;\; t \in T, \quad (2)$$

with inputs $w \in W^{1,1}(T, \mathbb{R}^m)$, $w(a) = 0$.[1] Such a compactification is dictated by needs of related optimal control problems, stated for system (2) under the constraint on the total "control action": $\int_T |\dot{w}|\,dt \leq M$ with a given $M > 0$. Such variational problems commonly appear to be singular in the sense [Gurman, 1997], and, generically, do not have solutions in the class of ordinary controls. A trajectory compactification, thus, requires a weaker topology (compared to the

[1] The tube of Carathéodory solutions to the Cauchy problem (2) is not generically closed in the natural topology of uniform convergence, as states can pointwise tend to discontinuous functions.

natural topology of the uniform converges of trajectories), and implies an extension of the concept of solution to the dynamical system. The topologies, for which the desired trajectory compactification can be defined constructively, are: the weak* topology of BV, the topology of pointwise convergence, and the topology of graph convergence in the Hausdorff distance. Compactifications in these topologies lead to generalized solutions of bounded variation and generalized controls of the type of vector-valued Borel measures [Arutyunov, Karamzin, and Pereira, 2014, Bressan and Rampazzo, 1988, Bressan and Rampazzo, 1994, Dykhta and Samsonyuk, 2000, Dykhta and Samsonyuk, 2015, Goncharova and Staritsyn, 2015, Goncharova and Staritsyn, 2012, Karamzin at al., 2015, Karamzin at al., 2014, Miller, 1996, Motta and Rampazzo, Miller and Rubinovich, 2003, Pereira and Silva, 2000, Sesekin and Zavalishchin, 1997, Silva and Vinter, 1996]).

1.2 The case $1 < p < 2$: An extension of Stieltjes integration due to L.C. Young

For the simplest case of bounded p-variation of continuous control w, $p \in (1, 2)$, the mathematical setup behind the control theory is, principally, the same as in the above case $p = 1$. Equation (1) driven by such paths can be uniquely solved by Young's integration:

$$\int_a^t g(s)\, dw(s). \tag{3}$$

Here, w and g are assumed to have finite p- and q-variations, respectively, $p^{-1} + q^{-1} > 1$. In fact, (3) is a Stieltjes integral, which remains well defined due to a wonderful assertion by L.C. Young [Young, 1936] (see also [Dudley and Norvaiša, 1999, Norvaiša, 2015, Lejay, 2013]).

1.3 The threshold case $p = 2$ and a general setup for $p \geq 2$: Rough paths due to T. Lyons and M. Gubinelly

For $p \geq 2$, an adequate solution concept for differential equation driven by controls of the class BV_p is much more complicated, and this is the heart of the rough paths theory. As such, the term "rough path" was first introduced in [Lyons, 1994]. The basic setup here is somehow similar to the non-commutative case in impulsive control of measure-driven systems: One can design the closure of $BV \cap C$ inside $BV_p \cap C$, and points of this closure are said to be rough paths. Next, one considers a sequence $\{w_k\} \subset BV \cap C$ converging in a specific metric to a rough path $w \in BV_p \cap C$. Under certain regularity assumption of input data (say, G may be γ-Lipschitz with $\gamma > p$), one establishes the convergence of the respective states x_k of (1) to a function $x \in BV_p \cap C$, which is named a solution to (1) under the input w. In this reasoning, it occurs that the defined state $x = x[w]$, in fact, depends on the choice of an approximating sequence w_k of w; in order to perform a single-valued selection of the multivalued input-output mapping $w \mapsto x$, one should enhance control w with certain extra data. In the rough paths theory, this necessary extra information is provided by a combinatorial object called the *signature* of a path or *Chen series*.

1.4 Impulsive rough differential equation: Towards impulsive control with states of unbounded variation

Mathematical control theory for systems with trajectories of unbounded variation is, by now, a rather fragmentary framework, compared to the BV case.

A major part of studies here are confined within the simplest cases, when the vector fields, defined by the columns of the matrix function G, satisfy the well-known Frobenius commutativity condition, or its generalization called the involutivity assumption, which is also very restrictive [Bressan and Rampazzo, 1988, Dykhta and Samsonyuk, 2000, Gurman, 1997, Sesekin and Zavalishchin, 1997].

In what concerns the general setup, we should mention the recent paper [Aronna and Rampazzo, 2013], which establishes the concept of so-called L_1-limit solutions to control-affine systems, and raises the idea of impulsive control with BV_p-inputs as a potential try-out.

In the present paper, we address control dynamical systems of the form (1) with trajectories x and control inputs w being (possibly, discontinuous) functions of the Wiener's class BV_p. We call systems of type (1) *impulsive rough differential equations*. The main goals are: (i) extension of the solution concept of a rough differential equation to the case of discontinuous controls, and (ii) constructive representation of discontinuous states of bounded p-variation by a discontinuous time reparameterization.

We restrict our consideration to the case $p \in [1, 2)$.

For the ease of presentation, we operate with scalar controls and states, i.e., assume that $n = m = 1$.

Our approach is based on pointwise approximation of rough solutions to equation (1) by a sequence of regular states produced by absolutely continuous inputs w_k with uniformly bounded p-variation. In other words, we look at system (1), driven by BV_p-controls, $p \in [1, 2)$, as at a certain trajectory relaxation of ordinary control system (2).

2 Functions of bounded p-variation: Definitions, basic properties and examples

Let $p \geq 1$. Following [Wiener, 1924], the total p-variation of a function $g : T \to \mathbb{R}^k$ on an interval T is the quantity $V_p(g; T)$, defined by

$$V_p(g; T) \doteq \left(\sup_{\pi} \sum_{i=1}^{N} \big|\big|g(t_i) - g(t_{i-1})\big|\big|^p \right)^{1/p},$$

where sup is taken over all finite partitions $\pi = \{t_0, t_1, \ldots, t_N\}$ of T, $a = t_0 < t_1 < \ldots < t_N = b$. The value $V_p(g;T)$ can be infinite. If $V_p(g;T) < \infty$, we say that g is a function of bounded p-variation. The set of functions $T \to \mathbb{R}^k$ of bounded p-variation is denoted by $BV_p(T, \mathbb{R}^k)$. It is a Banach space with the norm $||g||_{BV_p} \doteq ||g||_{L_\infty} + V_p(g;T)$.

Let us recall some basic properties of BV_p-functions [Chistyakov and Galkin, 1998]:

- For any $g \in BV_p(T, \mathbb{R}^k)$, the set of discontinuity points of g is at most countable, and, for all points $a \le s < t \le b$, there exist one-sided limits

$$g(t-) \doteq \lim_{\tau \to t-} g(\tau), \quad g(s+) \doteq \lim_{\tau \to s+} g(\tau).$$

- $g \in BV_p(T, \mathbb{R}^k)$ iff there exists a bounded non-decreasing function $\varphi : T \to \mathbb{R}$, and a Hölder continuous function $h : \varphi(T) \to \mathbb{R}^k$ of exponent $\gamma = 1/p$ with the Hölder constant $H(g) \le 1$, such that $g = h \circ \varphi$.
- A generalization of the Helly's selection principle (a compactness theorem for functions of bounded p-variation): Let K be a compact subset of $\mathbb{R}^k$. Let $\mathcal{F}$ be an infinite family of functions $T \to K$ of uniformly bounded p-variation, that is, $\sup\limits_{g \in \mathcal{F}} V_p(g;T) < \infty$. Then there exists a sequence $\{g_k\} \subseteq \mathcal{F}$ converging pointwise on T to a function $g \in BV_p(T, \mathbb{R}^k)$.
- Any function $g \in BV_p(T, \mathbb{R}^k)$ admits a unique representation $g = g_c + g_d$, where g_c is a continuous function called the continuous component of g, and g_d is the sum of jumps of g.

We also cite a basic result for rough differential equations with BV_P-controls, $p \in [1, 2)$, [Lejay, 2013]. Let $F = (F_1, F_2, \ldots, F_k)$ be a matrix function $\mathbb{R}^n \to \mathbb{R}^{n \times k}$. Consider a control equation

$$x(t) = x_0 + \sum_{i=1}^{k} \int_a^t F_i\big(x(t)\big) dw_i(t), \;\; t \in T, \quad (4)$$

where w is a continuous function of bounded p-variation with $p \in [1, 2)$.

Theorem 2.1. [Lejay, 2013]. *Let F be α-Hölder continuous with $\alpha > p - 1$. Then there exists a continuous function x of bounded p-variation being a solution to* (4)*. Furthermore, assume that F is bounded and continuous, and its derivative is bounded and α-Hölder continuous with $\alpha > p - 1$. Then the solution x is unique.*

3 Solution concept for impulsive rough differential equations. Representation of states of bounded p-variation, $p \in [1, 2)$, by a discrete-continuous integral equation

In what follows, we adopt the following hypotheses:
(H_1) The functions f and G are locally Lipschitz continuous, f is of sublinear growth, and G is bounded on $\mathbb{R}$, i.e, for any compact $Q \subset \mathbb{R}$, there exist constants $L_{f,G} = L_{f,G}(Q)$ such that, for all $x_1, x_2 \in Q$, it holds

$$\begin{aligned} |f(x_1) - f(x_2)| &\le L_f\, |x_1 - x_2|, \\ |G(x_1) - G(x_2)| &\le L_G\, |x_1 - x_2|, \end{aligned} \quad (5)$$

furthermore, there exist constants $c_f,\ c_G > 0$ such that

$$|f(x)| \le c_f\,(1 + |x|),\ |G(x)| \le c_G \quad \forall\, x \in \mathbb{R}. \quad (6)$$

(H_2) The derivative G_x is bounded on $\mathbb{R}$, and satisfies the Hölder condition with exponent $\alpha > p - 1$.

Given $p \ge 1$, consider solutions of system (2) produced by control inputs w with bounded total p-variation $V_p(w;T)$. Let us show that any sequence of such solutions contains a subsequence converging pointwise to a function of bounded p-variation.

Consider a control sequence $\{w_k\} \subset W^{1,1}(T, \mathbb{R})$ with uniformly bounded p-variations, that is, there exists $M > 0$ such that

$$V_p(w_k;T) \le M$$

for all $k \ge 1$. According to the Helly's selection principle — passing, if necessary, to a subsequence — we can assume that $\{w_k\}$ is pointwise converging to a function $w \in BV_p(T)$ with $w(a) = 0$.

Let $\{x_k\}$ be a sequence of Carathéodory solutions to (2), generated by $\{w_k\}$. Pointwise limits of $\{x_k\}$ are said to be **generalized solutions** of (2).

The following lemma provides the existence of generalized solutions of (2).

Lemma 3.1. *Let $\{w_k\} \subset W^{1,1}(T, \mathbb{R})$ be a control sequence such that*

$$\sup_{k \ge 1} V_p(w_k;T) < \infty \quad (7)$$

and $\{x_k\}$ be the sequence of the corresponding solutions to differential equation (2)*. Then,*

i) *$\{x_k\}$ is uniformly bounded, and there exists a constant $K > 0$ such that*

$$V_p(x_k;T) \le K \qquad \forall\, k \ge 1; \quad (8)$$

ii) *There exist a function $x \in BV_p(T, \mathbb{R})$ with $V_p(x;T) \le K$ and a subsequence $\{x_{k_j}\} \subseteq \{x_k\}$ such that $x_{k_j}(t) \to x(t)$ for all $t \in T$.*

The proof readily follows from applying the so-called nonlinear Goh's transform [Dykhta and Samsonyuk, 2000] to (2).

Given $p \in [1,2)$, let $\mathcal{W}_p = \mathcal{W}_p(T)$ denote the set of functions $w \in BV_p(T,\mathbb{R})$, which are right continuous on $(a,b]$ and satisfy $w(a) = 0$.[2]

Let $w \in \mathcal{W}_p$. On the interval T, consider the following discrete-continuous integral equation: $x(a) = x_0$,

$$x(t) = x_0 + \int_a^t f\big(x(\varsigma)\big)\,d\varsigma + \int_a^t G\big(x(\varsigma)\big)\,dw_c(\varsigma)$$

$$+ \sum_{s\le t,\ s\in S_d(w)} \big(z_s(1) - x(s-)\big),\ \ t\in(a,b], \qquad (9)$$

Here,

$$S_d(w) = \{s \in T \mid [w(s)] \doteq w(s) - w(s-) \neq 0\}$$

denotes the set of jump points of w. The integral with respect the continuous part $w_c \in BV_p(T,\mathbb{R})$ of control w in the right-hand side of (9) is understood in the Young's sense, and the functions z_s, $s \in S_d(w)$, are defined as solutions on $[0,1]$ of the ordinary differential equations

$$\frac{dz_s(\tau)}{d\tau} = G\big(z_s(\tau)\big)[w(s)], \quad z_s(0) = x(s-). \qquad (10)$$

By a solution to impulsive rough differential equation (2) under a control input $w \in \mathcal{W}_p$ we mean a right continuous on $(a,b]$ function $x \in BV_p(T,\mathbb{R})$ satisfying discrete-continuous system (9), (10), i.e., it turns (9) into an identity.

Theorem 3.1. *Given $p \in [1,2)$, assume that hypotheses (H_1) and (H_2) are satisfied. Then the following assertions hold true.*

i) **(The existence of a solution)**: *For any $w \in \mathcal{W}_p$, there exists a unique solution $x = x[w;x_0] \in BV_p(T,\mathbb{R})$ of* (9).
ii) **(Approximation by ordinary control processes)**: *For any $w \in \mathcal{W}_p$, there exists a sequence $\{w_k\} \subset W^{1,1}(T,\mathbb{R})$ of control inputs of system* (2) *such that*
   - *there exist positive constants M_w and M_x independent of k and such that $V_p(w_k;T) \le M_w$ and $V_p(x_k;T) \le M_x$, where $x_k \doteq x[w_k;x_0]$;*
   - *x_k converges to x at continuity points and at $t = T$.*

The proof is based on the nonlinear Goh's transform (described just above), and a special discontinuous time change [Samsonyuk and Staritsyn, 2017] generalizing the so-called space-time reparameteization [Miller, 1996, Miller and Rubinovich, 2003, Motta and Rampazzo, Sesekin and Zavalishchin, 1997] to the case of states with bounded p-variation. Note that, by the discontinuous time change, equation (1) is transformed to an auxiliary rough differential equation with controls and states being continuous functions of class $BV_p(\mathbb{R})$.

4 Conclusion

The paper raises a pretty new and challenging issue of mathematical control theory: impulsive control of dynamical systems driven by signals of unbounded variation, i.e., of a lower regularity than the familiar class of impulsive controls represented by Borel measures.

At the present step, we are confined within the simplest case $p \in [1,2)$. A further extension of the impulsive control framework to systems acted by BV_p-controls with $p \ge 2$ has to heavily rely on the apparatus of the theory of rough paths, briefly discussed in Introduction.

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