# LOCALIZATION OF COMPACT INVARIANT SETS OF THE COUPLED LASER SYSTEM

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The localization problem of compact invariant sets of a nonlinear system of ordinary differential equations is one of attractive problems due to interest to the long-time behaviour of the system. In this paper we examine a localization of compact invariant sets of the 5-dimensional coupled laser system which exhibits chaos for some values of parameters, see in [Saha *et al,* 2003]. Here by a localization we mean a description of a set containing all compact invariant sets of the system under consideration in terms of equalities and inequalities defined in the state space. Our approach is described in [Krishchenko, 1997] for periodic orbits and in [Krishchenko & Starkov, 2006a; 2006b] for compact invariant sets. It is based on using the first order extremum conditions.

Below we examine the linearly coupled single mode Nd:YAG laser system:: $\dot{x} = \beta xy - \beta\alpha_1 x - kz\cos v$, $\dot{y} = \gamma p_1 - \gamma y - \gamma yx^2$, $\dot{z} = \beta uz - \beta\alpha_2 z - kx\cos v$, $\dot{u} = \gamma p_2 - \gamma u - \gamma uz^2$, $\dot{v} = \triangle\omega + k\frac{x^2+z^2}{xz}\sin v\dot{v} = \triangle\omega + k\frac{x^2+z^2}{xz}\sin v$, denoted by $\sum$. Here $x$ and $z$ are amplitudes of laser-1 and laser-2; $y$ and $u$ are gains of laser-1 and laser-2; $v = \varphi_2 - \varphi_1$, where $\varphi_1$ and $\varphi_2$ are phases of laser-1 and laser-2. The parameter $\beta = \tau_c^{-1}$, where $\tau_c$ is the cavity round trip time. The parameter $\gamma = \tau_{fl}^{-1}$, where $\tau_{fl}$ is the fluorescence time of the upper lasing level of the $Nd^{3+}$ ion. Parameters $p_1$ and $p_2$ are the pump coefficients, $\alpha_1$ and $\alpha_2$ are the cavity loss coefficients, $\omega_1$ and $\omega_2$ are the detunings of the laser from a common cavity mode, $k$ is a coupling parameter. All parameters excepting $k; \triangle\omega := \omega_1 - \omega_2$ are positive. In addition, we suppose that $k\triangle\omega \neq 0$.

Below we introduce a few sets $K_j, j = 0, 1, ..., 4$, in the state space $\mathbf{R}^5$ of the system $\sum$. Let $K_1 = \{0 \leq y \leq p_1\}; K_2 = \{0 \leq u \leq p_2\}$. Also, we define numbers

$$\begin{aligned} R_1 &: = \gamma\sqrt{\frac{(p_1^2+p_2^2)(4\alpha_1\beta\gamma-\beta^2)}{2(4\alpha_1\beta\gamma-\beta^2)(4\alpha_2\beta\gamma-\beta^2)-32\gamma^2k^2}}; \\ R_2 &: = \gamma\sqrt{\frac{p_1^2+p_2^2}{2(4\alpha_1\beta\gamma-\beta^2)}} + \frac{2k\gamma^2}{(4\alpha_1\beta\gamma-\beta^2)}\sqrt{\frac{(p_1^2+p_2^2)(4\alpha_1\beta\gamma-\beta^2)}{2(4\alpha_1\beta\gamma-\beta^2)(4\alpha_2\beta\gamma-\beta^2)-32\gamma^2k^2}} \\ h_* &: = R_1^2 + R_2^2 + p_1^2 + p_2^2 \end{aligned}$$

and come to our main result.

**Theorem 1** *Suppose that inequalities*

$$4\alpha_1\gamma > \beta; \beta^2(2\alpha_1 - \frac{\beta}{2\gamma})(2\alpha_2 - \frac{\beta}{2\gamma}) > 4k^2$$

*hold. Then each compact invariant set of the system* $\sum$ *is contained in the set* $K_0 := K_1 \cap K_2$ $\cap K_3$, *with* $K_3 := \{x^2 + y^2 + z^2 + u^2 \leq h_*\}$.

In addition, we have

**Theorem 2** *Let* $K_4 = \{x^2 - z^2 \geq 0; \alpha_2 > p_2 + \alpha_1\}$ *and* $K_4 = \{x^2 - z^2 \leq 0; \alpha_1 > p_1 + \alpha_2\}$. *We state that all compact invariant sets are located in the set* $K_0 \cap K_4$.

**Theorem 3** *1. Let* $k > 0$. *Then there are not any periodic orbits and homoclinic orbits which are contained in the set* $\{2m\pi < v < (2m+1)\pi, m = 0, \pm1, \pm2, ...\} \cap \{xz > 0\}$ *or in the set* $\{(2m+1)\pi < v < (2m+2)\pi, m = 0, \pm1, \pm2, ...\} \cap \{xz < 0\}$.

*2. Let* $k < 0$. *Then there are not any periodic orbits and homoclinic orbits which are contained in the set* $\{2m\pi < v < (2m+1)\pi, m = 0, \pm1, \pm2, ...\} \cap \{xz < 0\}$ *or in the set* $\{(2m+1)\pi < v < (2m+2)\pi, m = 0, \pm1, \pm2, ...\} \cap \{xz > 0\}$.

Finally, we describe briefly how to improve localization results presented above by using specially constructed parabolic cylinders and some other polynomial surfaces. Also, we give a number of localization results for the system $\sum$ in the case $\triangle\omega = 0$. All details will be included into the journal version of this work.

**References**

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