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VARIATIONAL RELATIVISTIC THERMOHYDRODYNAMICS

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Abstract
Working with continua at hydrodynamic level we postulate that general conservation laws are independent of irreversible phenomena. We show that this independence holds in the system's own state space and enables one to apply the reversible Hamilton's principle in order to find conservation laws for energy and momentum in imperfect systems. A general relativistic Lagrangian is constructed which yields the effect of heat flux, q, and nonequilibrium stress, τ, in the energy-momentum tensor $\mathbf{G}$ of an extended reversible fluid exhibiting thermal inertia. Extended Hamilton principle is applied that admits thermal degrees of freedom (entropy four-flux) and yields $\mathbf{G}$ as the tensor source of the gravitational field. For the instantaneous properties of this field it is inessential whether the origins of heat flux, q, and of nonequilibrium stress, τ in $\mathbf{G}$ are reversible or not, the property which makes $\mathbf{G}$ directly applicable to real dissipative fluids. Relativistic transformations for temperature and chemical potentials in moving continua composed of phonons or massive particles are discussed in the light of kinetic potentials and theory of thermodynamic transformations.

Key words
Conservation laws, relativistic Lagrangian, Hamilton principle, relativistic transformations, perfect fluids, dissipative fluids.

1 Introduction

The conservation equations for energy and momentum in thermo-hydrodynamic systems are identical for reversible and irreversible processes provided that in both cases they are considered in their general non-truncated form which does not eliminate nonequilibrium effects such as heat flux and nonequilibrium stress [Hirschfelder et al., 1963], [Sieniutycz, 1994]. This general form holds in the extended state space which is spanned not only on the traditional thermodynamic coordinates, such as densities of components and entropy, but also on the corresponding fluxes. The number of all coordinates of the extended state or the state dimensionality is a minimum number of variables necessary and sufficient to characterize the system. The energy-momentum tensor of a thermo-hydrodynamic system is thus the property of the extended state or the state described by a minimum number of thermo-hydrodynamic variables. Presence of both mechanical and thermal variables in the set of the state coordinates is essential for complete description of real thermo-hydrodynamical systems. The identity of non-truncated conservation laws in both cases (reversible and not) reflects the general property of the system in which internal contributions cancel out so that the momentum and energy balances are unaffected by irreversible sources. For example in reacting systems it is known from both statistical mechanics [Tolman, 1949], [Hirschfelder et al., 1963], [Chapman and Cowling, 1973] and thermodynamics [Bird et al, 1960], [Wisniewski, 1984], [Wisniewski et al, 1973] and [Sieniutycz,1994] that the presence of reaction rates does not change general analytical structure of energy and momentum equations. The described property is, in fact, the basis in the Onsager's theory of nonequilibrium thermodynamics [Onsager, 1931a], [Onsager, 1931b]. In the so-called perfect or adiabatic fluid the heat flux is eliminated by the adiabaticity assumption which truncates general conservation laws or rules the heat flux out as a variable; thus the state space is incomplete and the above property does not apply. The reduced number of state coordinates in the perfect fluid model causes the necessity to augment the perfect-fluid conservation laws when they are applied to real non-adiabatic systems, i.e. systems with with finite heat flux (reversible or not). The augmenting procedure, which is adviced even in the best textbooks on theoretical physics [Landau and Lifshitz, 1974a], [Landau and Lifshitz, 1987] is purely formal. Although that procedure need not result in errors in the equations themselves, it is often understood in the form of the fundamentally incorrect conclusion that any disequilibrium flux, such as heat flux or non-equilibrium stress, is the irreversible property so that the irreversibility influences the conservation laws. This incorrectness persists in many otherwise good papers on thermo-hydrodynamics. In fact, however, the general (non-truncated) energy-momentum tensor in the full state space is irreversibility-independent which is a very useful property to obtain conservation laws of real complex systems.

In the present paper this property is exploited to obtain conservation laws in relativistic thermo-hydrodynamics of irreversible gravitating fluids from a reversible while extended formulation of Hamilton's principle of least action [Sieniutycz, 1992], [Sieniutycz, 1994], [Sieniutycz, 2002a]. With the extension, the early relativistic theory of heat transfer, [Schmid, 1967b], [Schmid, 1970a], [Schmid, 1970b], and the further generalization for perfect relativistic fluids [Ray, 1972], [Ray, 1980], the Hamilton's principle can be applied to include the flux of the entropy and phenomenon of thermal inertia. For this purpose a generalized relativistic Lagrangian has been constructed [Sieniutycz, 1998c], which yields the heat flux $\mathbf{q}$ and nonequilibrium stress τ in the energy-momentum tensor of an extended reversible fluid with thermal inertia. The actual momentum of heat (thermal momentum related to the entropy flow) follows to be many orders of magnitude larger than $\mathbf{q}/c^2$ (c is the light speed) but it is consistent with the kinetic theory [Grad, 1958] and with experiments in heat conduction [Sieniutycz, 1994], [Sieniutycz, 2002a]. The resulting conservation laws exhibit the heat flux and nonequilibrium stress, thus they may be used as components of irreversible models containing explicitly irreversible equations. Such equations are, e.g., the Fourier's law of heat conduction or its non-causality-violating counterpart called the Cattaneo equation, [Cattaneo, 1958], [Vernotte, 1958], [Chester, 1963]. After its combining with the energy balance, the latter serves to obtain the hyperbolic (damped-wave) equation for temperature, i.e. Eq. (3) below or its more exact extensions.

Some literature references cited in the present paper serve to mention particular concepts and formal techniques, and do not exhaust even in a small part very large research material obtained for relativistic continua. For a coherent review of macroscopic approaches to relativistic thermomechanics of continuous media a book is recommended [Baranov and Kolpascikov, 1974]. Use of composite variational principle is based on a mathematical theory which also shows a way of derivation of the conservation laws [Caviglia, 1988]. Elementary calculus of variations is used here to derive the Lagrange equations of motion for the relativistic fluid [Gelfand and Fomin, 1963]. With these results conservation laws and matter tensor can be found, and other problems, such as stability and causality in dissipative fluids, may be considered [Hiscock and Lindblom, 1983]. Diffusion processes can be treated in curved phase space [Graham, 1977]. Even dissipative phenomena can be included provided that the Riemannian geometry is replaced by the Finslerian geometry [Ingarden, 1996]. Applications are broad and may lead to variational principles for stellar structures [Kennedy and Bludman, 1997]; their text reformulates the four basic equations of stellar structure as two alternate pairs of variational principles. Applying the geometry of spaces with the Jacobi metric, a *generalized* Maupertuis principle has been formulated for systems with the Lagrangian and an indefinite form of the kinetic energy [Szydlowski et al., 1996]. The generalization was applied to the theory of gravity and cosmology where the metric determined by the kinetic energy form has a Lorentz signature. The theorem was proved concerning the behavior of trajectories in a neighborhood of the boundary of the region admissible for motion. This region is not a smooth manifold but turns out to be a differential space of constant differential dimension. This fact allows us to use geometric methods analogous to those elaborated for smooth manifolds. It was shown that singularities of the Jacobi metric are not dangerous for the motion; its trajectories are smooth in the sense of the theory of differential spaces. An analysis of the classical derivation of the continuity equation in general relativity was performed [Sandoval-Villalbazo and Garcia-Colin, 1999]. The consequences of the equation obtained were discussed and compared with similar results related with the conservation of particles (baryons and leptons). Emphasis was made on the fact that different transport equations may be derived from the possible choices of the mass balance equation in non-equilibrium formalisms. Nevertheless, the continuity equation agrees with the notion of particle conservation. Heat conduction in relativistic extended thermodynamics is treated [Pavon et al, 1980] along with the equilibrium and nonequilibrium fluctuations in relativistic fluids [Pavon et al, 1983]. Extended relativistic thermodynamics (ERT) is developed [Pavon et al., 1992]. However their paper deals with applications to astrophysics and to cosmology, rather than with the general theory. The reader interested in causal relativistic theories is refereed to papers by [Israel, 1976] and [Hiscock and Linblom, 1983]. The second moments of fluctuations are determined from the Einstein probability formula involving the second differential of the nonequilibrium entropy. These moments are subsequently used to determine the transport coefficients of heat conductivity, bulk viscosity and shear viscosity of a radiative fluid. By exploiting correlation formulae, second order coefficients of transport are determined. Nonequilibrium fluctuation theory assuming the validity of the Einstein formula close to equilibrium, is used to determine the nonequilibrium corrections of the bulk viscous pressure. An analysis of survival of the protogalaxies in an expanding universe completes the presentation of astrophysical applications of ERT. Cosmological applications are discussed next. It is shown that nonequilibrium effects and transport properties (such as, e.g., bulk viscosities) play a certain role in the cosmic evolution of the universe. In that context, the following issues are outlined: cosmological evolution (described by a Bianchi-type model with the bulk viscosity), production of entropy in the leptonic period, the inflationary universe and FRW cosmology. Some cosmological aspects of the second law are also discussed.

In conventional radiative hydrodynamics, dissipative fluxes are proportional to the gradients of the temperature, velocity, etc., with proportionality constants inferred from the linearized radiative transfer equation. The dynamical evolution is determined by the conservation of matter, energy, and momentum. As stated by [Schweitzer, 1988], conventional theory breaks down if the geometry under consideration is photospheric, i.e., if emission/absorption processes are weak compared to scattering and photons can "random walk" over large distances without being absorbed. The fundamental dynamical equations of motion include the relativistic transfer equation and form the starting point for establishing extended versions of radiative hydrodynamics. In [Schweitzer, 1988], the standard treatment of the transfer equation is replaced by a hierarchy of relativistic moment equations, and the dissipative fluxes are treated as dynamical variables in their own right. The general moment equations involve source terms which are a priori undetermined. If the radiation is almost thermal the source terms can be expressed with Rosseland means, see, e.g., Wikipedia: Opacity (optics), also as the information about the degree to which light is not allowed to travel through. In this case, the moment equations reproduce the structure equations postulated by [Israel, 1976] for general dissipative continuous media. In the limit of slow motion, the moment equations are studied separately and interpreted as hyperbolic diffusion equations.

Also, a treatment was presented [Sieniutycz, 1990], which, while nonrelativistic, deals with a macroscopic extension of the de Broglie microthermodynamics [de Broglie, 1964], so as to preserve the usual thermostatistical effects ignored in the de Broglie theory. The extension includes basic macroscopic concepts, the usual statistical temperature T and the chemical potential μ in a generalized macroscopic theory of fluids. The derivative $d = \partial E/\partial s$ of the energy density E with respect to the entropy density ρ_s, is found as the sum of the intrinsic relativistic temperature of de Broglie θc^2 (a counterpart of the c^2 term in the relativistic chemical potential $g = c^2 + \mu$), and the usual statistical T, so that $d = \theta c^2 + T$ and $\theta = m/k_B$. Nonequlibrium multicomponent fluids are investigated in the context of the Hamiltonian action and the resulting conservation laws, under a "nonrelativistic" approximation of low transport velocities and for the c^2 terms ignored. The sourceless constraints for mass and entropy ("reversibility") are sufficient to derive the standard form of conservation laws for the energy and momentum, improved by the presence of a generalized energy of diffusion. Thermal inertia effects appear naturally due to the finite value of θ. The resulting thermohydrodynamics exhibits many new features both for the static and dynamic aspects of the theory. In particular, both the thermal equilibrium and the heat flow are affected by the gravitational field. The entropy flux is associated with a momentum (thermal momentum), and causes phenomena such as heat flow, selfdiffusion, and nonequilibrium stresses, which are usually regarded as irreversible. General Gibbs equations are obtained which take into account the role of the transport velocities or momenta and the macroscopic averages of the quantum phases of the species. A unification is achieved for the mechanical and thermal Legendre transformations within the field hydro-thermodynamics. It is shown that for inherently irreversible cases the thermodynamic formalism must involve quantum phases explicitly and that the entropy source is incorporated in the Gibbs equation as the derivative of the energy with respect to the thermal phase.

Further developments of the above approach are formulated in the relativistic context. [Sieniutycz, 1992], [Sieniutycz, 1998], [Sieniutycz, 1994], and [Sieniutycz, 2002a] has constructed a general relativistic Lagrangian which yields the effect of heat flux **q** and nonequilibrium stress τ in the energy-momentum tensor of an extended reversible fluid exhibiting thermal inertia. The actual momentum of heat (thermal momentum related to the entropy flow) follows to be many orders of magnitude larger than $\mathbf{q}/c^2$ (c is the light speed) but it is consistent with the predictions of the kinetic theory [Grad, 1958] and with the experiments in heat conduction. On the other hand, the net momentum of heat remains $\mathbf{q}/c^2$, in agreement with the standard relativistic result, this net momentum being the result of incomplete compensation of the actual thermal momentum and the momentum associated with self-diffusion of particles. Whenever inertial effects prevail, the classical densities of mass and entropy, ρ and ρ_s, cease to be natural variables of energy density E in the Callen's sense [Callen, 1988]. This fact necessitates the use of what may be called the thermal potential T^-, a new quantity replacing the classical temperature T. Changes in thermodynamic formalism require the replacement of T by T^-. The admission of a freely varied four-flux of entropy in an extended Hamilton principle implies all nonequilibrium corrections (**q** and τ) to the energy-momentum tensor, thus making it possible to investigate the effect of nonequilibrium phenomena on the properties of associated gravitational fields. For the instantaneous properties of this field it is inessential whether the origins of heat flux **q** and of nonequilibrium stress τ in the energy-momentum tensor **G** are reversible or not, which is the property that makes **G** directly applicable to real dissipative fluids. Selected aspects of these relativistic formulations are contained in the present text below.

2 Action, Lagrangian and Thermo-Hydrodynamic Potentials

We restrict in this section to the Lorenzian frames of special relativity, for which we outline the role of Lagrangians (or related Hamiltonians) in the development of the theory of thermo-hydrodynamic transformations and the related Gibbs equations describing various thermodynamic potentials. This information is purposely condensed for the sake of brevity. For more details the reader is referred to previous publications [Sieniutycz, 1990], [Sieniutycz, 1992a], [Sieniutycz, 1994], [Sieniutycz, 2002a].

The basis of the variational theory of a reversible fluid in Eulerian representation of fluid's motion is the field action, *A,* based on the four-dimensional integrand over pressure *P*. Due to the relativistic invariance of four-volume element d*V*d*t* the invariance of *A* is consistent with the invariance of *P*. For example, in the Lorenzian frames

$$A = \int P dV dt = \int P^0 dV^0 dt^0 = A^0, \qquad (1)$$

where the superscript 0 refers to rest-frame quantities. (For the simplicity of notation single integrals are symbollically used in place of multiple four-dimensional and theree-dimensional integrals.) As the four-dimensional volume element, d*V*d*t,* is the relativistic invariant, or d*V*d*t*= dV^0dt^0, Eq. (1) implies the invariance of *P* in the light of the invariance of *A*. Moreover, in view of the invariance of *P* and selfconsistency of thermodynamic transformations the three-dimensional integral of Mathieu thermodynamic potential (the ratio of the grand potential and temperature *T*)

$$\Omega = \int \frac{PdV}{T} = \int \frac{P^0 dV^0}{T^0} = \Omega^0 \qquad (2)$$

is Lorentz invariant only if the temperature transforms in the same way as the volume *V*, i.e. moving body appears cooler. This means that the relativistic temperature transformation satisfies the Einstein-Planck transformation formulae rather than Ott's transformations. Indeed, a comparison of Eqs. (1) and (2) proves the frequency-like transformation for the relativistic temperature. This agrees with the interpretation in which temperature is a measure of the frequency of thermal agitation. More details on this issue can be found in Sec. 5, where transformations of a generalized temperature refer to systems with thermal inertia.

Associated with the entropy density ρ_s is the total entropy flux $\mathbf{J}_s$ whose diffusive part is defined operationally as ratio of density of heat flux **q** and *T* in the frame of fluid particles. In an arbitrary frame the ratio $\mathbf{u}_s$= $\mathbf{J}_s/\rho_s$ defines the velocity of entropy transfer which is composed on the sum of hydrodynamic velocity **u** and diffusive part **q**/$T\rho_s$. The quantities ρ_s and $\mathbf{J}_s$ and constitute a relativistic four-vector. The ratio $\mathbf{J}_s/\rho_s$ describes the absolute velocity of entropy flow, d$\mathbf{x}_s$/d*t,* or the change of the entropy coordinate $\mathbf{x}_s$ (Lagrangian thermal coordinate) in time *t.* The idea of the Lagrangian thermal coordinate is associated with the Lagrangian formulation of the Maxwell-Cattaneo hydrodynamics [Grmela and Teichman, 1983]. The above consideration means that besides of hydrodynamic concept of moving fluid particles we also use the parallel concept of moving thermal elements associated with quantal heat particles or quasiparticles of the thermal process, [Markus and Gambar, 2005]; [Vazquez et al., 2009] These quasiparticles are relatives of photons or quasiparticles of quantized acoustic fields.

Various hypotheses for thermal mass and thermal inertia are known [Veinik, 1966], [Sieniutycz and Berry, 1997]. Quantum theories such as de Broglie microthermodynamics [De Broglie, 1964], [De Broglie, 1970] offer a reinterpretation of wave mechanics and may accommodate EIT effects [Sieniutycz, 1990], [Sieniutycz, 1992]. An approach towards constructing the Hamiltonian formulation as a basis of quantized thermal process [Markus, 2005] has turned out to be quite successful. The approach is based on a generalized Hamilton-Jacobi equation for dissipative heat flow [Markus and Gambar, 2004], [Markus and Gambar, 2005], [Vázquez et al., 2009] whose physical implications differ substantially from those ignoring dissipation [Schmid, 1967a].

Only in reversible processes the identity $\mathbf{u}_s$ = **u** is satisfied. In the case of irreversible processes both velocities $\mathbf{u}_s$ and **u** differ. The simplest example is the heat transfer in a resting medium where only $\mathbf{u}_s$ is non-vanishing (the consequence of a finite **q**), whereas the hydrodynamic velocity **u** equals zero. The velocity of the entropy transfer $\mathbf{u}_s$ is a concept more suitable and more exact than the concept of the velocity of energy transfer because the energy density and energy flux are not scalars but components of a four-tensor.

The kinetic potential density *L* is an important quantity in the hydrodynamic theory of heat conducting fluid. It is essential that in the realm of thermodynamic transformations *L* need to be considered as a function of densities and velocities, whereas the energy density *E* (which is the Legendre transform of *L* with respect to velocities) is a function of densities and momenta, $\mathbf{p}=\partial L/\partial \mathbf{u}$, $\mathbf{p_s}=\partial L/\partial \mathbf{u}_s$, etc. On the other hand, negative partial derivatives of *L* with respect to densities of entropy and particles are respectively the temperature *T* and chemical potential μ.

A simplest example is the kinetic potential of black photons who represent the perfectly disordered radiation. In this case the contribution of the rest mass to *L* vanishes due to vanishing mass of the photon, and only density of internal energy defined by the Stefan-Boltzmann law contributes to *L*. Moreover, in this case, a truncated kinetic potential holds

$$L = -\rho_e^0(\rho_s^0, \rho_n^0) = -\rho_e^0(\rho_s\sqrt{1-\mathbf{u}_s^2/c^2})$$

which proves that densities ρ_e^0 and L are independent of ρ_n^0. Therefore, for black photons, partial derivative $\mu_n = -\partial L/\partial\rho_n = 0$, which is the classical result in the thermodynamics of radiation. Moreover, partial derivative

$$T = -\partial L/\partial\rho_s = T^0\sqrt{1-u_s^2/c^2)}$$

defines the temperature of the black photon system that moves with macroscopic velocity **u**s= **J**s $/\rho_s$. The same formulas can be derived by considering the energy density *E* in terms of corresponding momentum densities.

Here we give merely an introduction to the important problem of thermodynamic potentials of thermo-hydrodynamic systems [Sieniutycz, 1990], [Sieniutycz, 1992], [Sieniutycz, 1994]. Functions or functionals that can be obtained from *L* or *E* through Legendre transformations are called the thermo-hydrodynamic potentials (THP). Their dependence on velocities or momenta, and their general properties make them the extensions of classical mechanical Lagrangians and usual thermostatic potentials. Their perfect differentials describe various Gibbs equations for the thermo-hydrodynamic system.

One may distinguish three types of the THP's. Those obtained from the kinetic potential *L* by the thermodynamic Legendre transformation of thermodynamic variables (densities, intensities) are called the THP's of the Lagrangian type, or Lagrangians. These THP's always contain velocities. Those obtained from the kinetic potential *L* (or other Lagrangian) by the mechanical Legendre transformation involving all velocities are called the THP's of the energy type, because they always contain the momenta, the natural variables of the energy. For example, the pressure function $P(\mathbf{u}, \mathbf{u}_s, \rho, \rho_s)$ is the Lagrangian type THP, whereas the potential $\Omega(\mathbf{p}, \mathbf{p}_s, \rho, \rho_s)$, which coincides with the negative of *P* only in the static case when the role of kinetic terms is inessential, is a THP of the energy type. In the static situation all Lagrangians become the usual thermodynamic potentials with negative sign, and all the energy type THP's as well as Lagrangian type THP's, become the usual THP's.

The third group is composed of mixed structures, containing both velocities and momenta (and some thermodynamic variables). They are called Routhians, *per analogiam* with such mixed structures in classical mechanics. From the viewpoint of their sign they may behave both like Lagrangians and like energy type THP's. The Routhian form is especially useful if some but not all of the extensive (coordinate-like) variables do not appear in the Lagrangian.

Further details and the corresponding examples of some THP's are given in next sections where we consider systems containing massive particles and exhibiting thermal inertia.

3 Basic Information on Thermal Inertia

By applying equations of Cattaneo type instead of the Fourier's law, one takes into account the phenomenon of thermal inertia, which causes a gradual change of the heat flux under a rapid change of the temperature gradient [Chester, 1963]. This phenomenon has attracted the researchers attention as the way of solving the well-known paradox of infinite propagation of thermal disturbances contained in the classical Fourier's model of the heat conduction. In resting systems with non-vanishing thermal inertia an extended Fourier law holds which links the heat flux operator $\mathbf{Q} \equiv \mathbf{q} + \partial\mathbf{q}/\partial t$ with the negative product of the thermal conductivity and temperature gradient, $-\lambda\nabla T$, [Chester, 1963]. The operator contains the thermal relaxation time τ, whose order of magnitude is that of the mean time between molecular collisions. In a stationary rigid continuum of constant specific heat capacity *C*, thermal conductivity λ and density ρ, the combination of the extended Fourier law with the conservation law for the thermal energy leads to the hyperbolic equation for the temperature

$$\nabla^2 T - \frac{\partial^2 T}{c_0^2\partial t^2} - \frac{\partial T}{\tau c_0^2 \partial t} = 0 \qquad (3)$$

where

$$c_0 = \sqrt{\lambda/\rho C\tau}$$

is the propagation speed of thermal disturbances. Thermal waves of finite speed implied by this equation, known collectively as second sound, were first detected in solid helium 3He [Ackerman et al., 1996], and then in high purity dielectric crystals of sodium fluoride, NaF, [Jackson et al., 1970]. Then, it was shown soon that the classical irreversible thermodynamics is insufficient to describe these phenomena correctly. Only extended irreversible theories, [Jou et al., 1989], [Jou et al., 2001], which take into the contribution of the heat flux to the (disequilibrium) entropy, are appropriate. Such contributions are obtained in nonequilibrium statistical mechanics (both relativistic and nonrelativistic), in particular in Grad's moment approach to the solution of Boltzmann kinetic equation, [Grad,1958]. Along with usual thermal conductivities λ, (nonvanishing) thermal relaxation times τ and the corresponding finite values of propagation speeds, c_0, can be estimated from the Grad's solutions, both nonrelativistic and relativistic, as shown in many recent works on the subject [Jou et al., 2001].

It follows from the results of Grad's analysis [Sieniutycz and Berry, 1989], [Sieniutycz and Berry, 1997] that, in the entropy representation, the thermal waves with finite speeds are associated with a local deviation internal energy from its equilibrium,

$$\Delta\rho_e = \frac{1}{2}\rho_e^{-1}c_0^{-2}\mathbf{q}^2$$

In terms of entropy density ρ_s and entropy flux $\mathbf{j}_s$=$\mathbf{q}$/T, the deviation is:

$$\Delta\rho_e = \frac{1}{2}\rho_s^{-1}Tc_0^{-2}\mathbf{j}_s^2$$

(In nonequilibrium media, which are by nature inhomogeneous, the actual local value of the entropy density ρ_s is taken instead the constant ρ_{s0} of linearized models.) One can thus introduce and then work with the coefficient θ=Tc_0^{-2}, which is constant since c_0^2 is of the order of the thermal speed, $(c_0)^2 \cong k_B T/(am)$, [Chester, 1963]; [Jou et al., 2001], where k_B is the Boltzmann's universal constant and a is close to 1. (The constant a equals 1 for an ideal gas.) The constant $\theta = am/k_B$ depends only on the mass of the particle, m. Nowadays diverse hypotheses exist which predict values of the numerical coefficient a at the quotient m/k_B in θ. These are discussed elsewhere [Sieniutycz and Berry, 1989], [Sieniutycz and Berry, 1997]. θ is, in fact, the thermal inertia per unit of the entropy [Sieniutycz and Berry, 1997]. Occasionally the density $\rho_{s\prime} = \rho_s\theta$, that is the entropy density in mass units, is used [Sieniutycz and Berry, 1997]. The important conclusion stemming from the constancy of θ is that the statistical mechanical evaluation of the disequilibrium entropy in the form

$$\Delta\rho_s = \frac{1}{2}\rho_s^{-1}\theta\mathbf{j}_s^2,$$

or in the equivalent form

$$\Delta\rho_s = \frac{1}{2}\rho_s\theta\mathbf{v}_s^2$$

($\mathbf{v}_s$ =≡ $\mathbf{j}_s/\rho_s$ is the rest-frame-velocity of entropy transfer), is consistent with the simple two-phase model of the medium in which the massive entropy and bare matter are two independent massive entities, and the kinetic energy of each entity has the classical structure of the usual kinetic energy. The inertial entropy and bare matter may thus be regarded as two phases which flow separately one of each other in general disequilibrium situation. The classical heat conduction process in solids in which the solid skeleton does not move ($\mathbf{u}_m$=0) and the entropy still flows due to a nonvanishing heat flux is the suitable example. In essence, there are a few basic contemporary hypotheses of thermal inertia: those of Grad-Boltzmann, de Broglie and Veinik [Sieniutycz and Berry, 1997]. While Grad 's results follow from the kinetic theory [Veinik, 1966] and [Veinik, 1973], has formed his ideas on the electrochemical grounds; they stem from his experimental investigations of simultaneous transfer of heat and electricity in a thermodynamic couple. On the surface, the model based on the Grad-Boltzmann theory seems to exclude the linearity of the kinetic energy of heat with respect to θ. However, this may only be an effect of the barycentric reference frame for heat flow contained in that theory: the inertial heat is defined there in the frame of the vanishing momentum, $\mathbf{\Gamma}$**=** $\mathbf{0}$, not in the frame of bare particles. While the barycentric frame causes no difficulties for the inertialess heat, in processes with pronouced thermal inertia it necessarily involves a particle drift, in the direction opposite to the heat flow. This involves extra kinetic energy of the particle motion which causes apparent values of θ, say θ', that are state dependent. Yet, the inertia of pure heat flux, defined as the energy flux in the frame of particles (where the particle drift vanishes), is linear with respect to θ. (When the frame is different the boundary conditions for heat flux are changed into those which may disagree with an experiment.) In fact, this consideration proves that Eckart's particle frame is the most natural frame to describe the heat transfer in the way consistent with real experiments. For an observer at rest in this frame, the flux of particles is zero, and the heat flux represents the flux of energy relative to the particle stream [Landau and Lifshitz,1997]. Thus the particle frame is both suitable and essential in relativistic descriptions. When the frame is properly adjusted, it appears that the constancy of inertial coefficient θ in the relativistic Lagrangian is approximately satisfied also for models stemming from the Grad-Boltzmann theory. Therefore all analysis here use a constant value of θ, the thermal inertia per unit of entropy. But all this enhances our basic interpretation of the heat flux as the effect of the inertial entropy flow in the particle frame.

4 Matter Tensor of General Relativistic Theory

In this paper we deal with general relativistic systems in motion, in which, as shown by Israel, second order terms in the entropy four-flux are sufficient to overcome the paradox of infinite propagation speeds of thermal signals. Israel's theory involves the energy-momentum tensor G^{ik} as the quantity generalizing mass in both a covariant form of the Gibbs equation and a generalized Gibbs-Duhem formula containing the four-velocity, [Israel, 1976]. It follows that the corrections to the classical entropy and entropy flux must be of, at least, second order in dissipative fluxes to make the theory compatible with standard thermodynamics in the quasistatic limit.

The basic ingredients of the standard relativistic theory of a one-component fluid are the symmetric energy-momentum tensor G^{ik}, the particle four-flux J^i, and the entropy flux S^i (*i, k* = 1,.., 4). Both $G^{ik}$and J^i are sourceless, i.e.

$$G^{ik}_{j} = 0, J^{i}_{;i} = 0 \tag{4}$$

The entropy four-flux, S^i, of an irreversible process obeys the second law constraint: the positive entropy production $S^{i}_{;i} \geq 0$, where the equality sign refers to a limiting reversible process to which the Hamiltons principle can be applied. (This is done in Sec. 6 below.) Equation (4) contains the energy and momentum balances and the balance of matter. The signature convention is (+++-). The energy-momentum tensor of Eckart's theory has the following structure

$$G^{ik} = c^{-2}\ \left(E^0 U^i U^k + q^i U^k + q^k U^i\right)\ + \tau^{ik} + P h^{ik} \tag{5}$$

where U^i is the particle four-velocity and h^{ik} is the projection tensor [Eckart, 1940]. The heat flux vector $\mathbf{q}^i$ and disequilibrium stress tensor τ^{ik} satisfy

$$q^i U_i = 0; U_i \tau^{ik} = 0.$$

Equations (4) and (5) refer to Eckart's relativistic scheme. Yet, in the framework of Eckart's theory thermal inertia is possible only as relativistic phenomenon. Moreover, Eq. (5) is not derived therein; it is rather only postulated on the basis of general covariance principles. This is not a case of the present approach. Our inclusion of thermal inertia is accompanied with the recognition of the importance of a free entropy flow (independent of the flow of the particles) in the Hamilton's principle. On this basis we shall derive Eq. (5) and some of its important associates from the stationary action approach. The resulting theory, which is in the spirit of an extended thermodynamics, is an extension of Ray's variational construction of the energy-momentum tensor G^{ik} in the general relativity of adiabatic fluids, [Ray, 1972], [Ray, 1980]. We shall show that not only Eq. (5) for G^{ik} can be derived in a direct way but also that definitions of heat flux $\mathbf{q}$ and disequilibrium stress τ can be furnished in G^{ik}.

5 Thermal Mass and Modified Temperatures

The hypothesis of the thermal mass incorporates the assumption that a part of the observed rest mass of a macroscopic body is of purely thermal origin, meaning that it should be attributed to entropy rather than to particles. Consistently, the kinetic potential (unconstrained Lagrangian) *L* is used which is based on the split of the background relativistic energy $\rho^0 c^2$ into the "bare matter" part,ρ^0_mc^2, and the "thermal part", $\rho^0_s c^2$, such that the their sum remains equal to $\rho^0 c^2$. The zero superscript refers to the rest frame quantities. The particle of the traditional theory, where the entropy is inertialess, is "clothed in the thermal mass robe" while that of the modified theory is a bare ("unclothed") particle [De Broglie, 1964], [De Broglie, 1970], [Sieniutycz, 1994], [Sieniutycz and Berry, 1997]. This reorientation changes neither internal energy ρ_e of the medium nor the observed density of mass ρ, (and many other important thermodynamic quantities, as the pressure *P*) but it allows to share the inertial responsibilities between the entropy and matter in a more balanced way. Suggested by the Grad-Boltzmann theory, the mass-entropy equivalence constant, θ, is applied, which is the amount of the thermal mass per unit of the entropy. For the internal consistency of the description, the explicit use of thermal mass requires, however, modifications in definitions of thermodynamic quantities, the most important being the redefinition of the temperature. This redefinition may be regarded as the consequence of the passage from conventional variables (ρ_s, ρ) to "canonical variables" (ρ_s, ρ_m) in the Gibbs equation for the differential of the relativistic internal internal energy.

In accord with the above considerations, the hydrodynamic velocity $\mathbf{u}$ and the total mass density ρ satisfy the relations

$$\rho\mathbf{u} = \rho_m \mathbf{u}_m + \theta\rho_s\mathbf{u}_s; \rho = \rho_m + \theta\rho_s \tag{6}$$

Using these relationships we shall now consider the differential of the relativistic internal energy density in terms of the densities of the entropy and bare mass (ρ_s and ρ_m) called the "canonical densities". The related (relativistic) intensity variables are subscripted by asterisks. Yet, since these intensities coincide with the usual temperature *T* and chemical potential μ only when θ equals zero and the effect of usual *T* is decreased, we designate them with $^{-0}$superscript. The index 0 refers to the rest fame whereas the negative sign points out their basic property according to which they transform according to Planck-Einstein when passing to moving frames ("cooler intensities"; [Sieniutycz, 1998c]. In the rest frame and in terms of canonical densities

$$dE^0 = \frac{\partial E^0}{\partial \rho^0_s} d\rho^0_s + \frac{\partial E^0}{\partial \rho^0_m} d\rho^0_m \equiv T^{-0}_* d\rho^0_s +\ \mu^{-0}_* d\rho^0_m. \tag{7}$$

The energy differential can next be expressed in terms of traditional densities ρ^0_s and ρ^0

$$dE^0 = T_*^{-0} d\rho_s^0 + \mu_*^{-0} d(\rho^0 - \theta\rho_s^0) = \quad (8)$$

$$(T_*^{-0} - \theta\mu_*^{-0}) d\rho_s^0 + \mu_*^{-0} d\rho^0.$$

Comparison of this result with the traditional form Gibbs equation which operates with the traditional densities ρ_s^0 and ρ^0

$$dE^0 = T_*^{-0} d\rho_s^0 + \mu_*^0 d\rho^0 = T^0 d\rho_s^0 + (\mu^0 + c^2) d\rho^0 \quad (9)$$

yields the desired connection between the *relativistic* intensities

$$\mu_*^{-0} = \mu_*^0 \equiv \mu^0 + c^2; \quad (10)$$

$$T_*^{-0} = T^0 + \theta\mu_*^0 \equiv T^0 + \theta(\mu^0 + c^2).$$

The asterisk-free intensities are their nonrelativstic counterparts. The canonical intensities (with minus superscript) represent then the usual relativistic chemical potential and its thermal analogue,T_*^{-0}, which we call the thermal potential. The ratio T_*^{-0}/θ has units of the chemical potential and, as it represents the modified temperature T_*^{-0} in the energy units, it may be regarded as the chemical potential of the thermal matter. Equation (10) implies that the rest frame nonrelativistic canonical chemical potential coincides with the classical one, whereas the nonrelativistic thermal potential T^- differs from the classical temperature T due to the contribution of the nonrelativistic chemical potential. As discussed elsewhere [Sieniutycz, 1998c] the transformations contained in Eq. (10) set the appropriate rest-frame inputs (non-relativistic T^{-0} and μ^{-0}) to the relativistic transformations of thermodynamic intensities in moving systems. Importantly, it is the relativistic canonical intensities, not their truncated nonrelativistic counterparts, that obey the well-known Planck-Einstein formula for the relativistic temperature transformation. At disequilibrium these relativistic intensities incorporate two different velocities of thermal and bare matter, $\mathbf{u}_s$ and $\mathbf{u}_m$. For example, in the the Lorentzian frames of special relativity

$$T_*^- = T_*^{-0}\sqrt{(1 - u_s^2/c^2}; \mu_*^- = \mu_*^{-0}\sqrt{(1 - u_m^2/c^2} \quad (11)$$

When $\theta = 0$ and for small velocities, the velocity $\mathbf{u}_m$ approaches the hydrodynamic velocity $\mathbf{u}$, $T = T^0$, and only the chemical potential transforms;

$$\mu = \mu^0 - \mathbf{u}^2/2$$

This result is well-known in the theory of hydrodynamic fluctuations [Keizer, 1987]. Thus, in the framework of the present theory, temperature satisfies the Einstein-Planck tranformations. This result might be confronted with that of [De Parga *et al.*, 2005] who derived the scheme of thermodynamic relativistic transformations inspired by the Planck–Einstein theory, but changing the relativistic transformation of energy; the change which ensures the form invariance of thermodynamics. The key point to obtain these results was their use of finite time thermodynamics (FTT), to demonstrate and creatively exploit the relativistic invariance of thermal efficiency.

6 Tensor of Matter Including Heat and Viscous Stress

We shall determine the matter tensor **G** for a general relativistic fluid with thermal inertia. The standard kinetic potential of the general theory can be expressed in terms of the canonical densities,

$$L = -\rho^0 c^2 - \rho^0 e(\rho^0, \rho_s^0) = \quad (12)$$

$$-(\theta\rho_s^0 + \rho_m^0)[c^2 + e(\rho_m^0, \rho_s^0)].$$

The corresponding Lagrangian density in its invariant form adjoins with the help of Lagrange multipliers relevant constraints containing the four-velocity vectors U_s^i and U_m^i(or corresponding fluxes), each obeying the same formulae as the four-vector of velocity $U^i \equiv (U^\alpha, U^4)$,

$$U^i U^i = \mathbf{U}.\mathbf{U} - U^4 U^4 = -c^2, \quad (13)$$

where $i = 1, .., 4$ and $\alpha = 1, .., 3$. In this formalism any four-flux vector can be written as

$$(\mathbf{J}, J^4) = (\rho\mathbf{u}, \rho c) = \left(\rho^0 U^\alpha, \rho^0 U^4\right) = \rho^0 U^i. \quad (14)$$

With subscripts *s* and *m* these equations apply both to thermal and bare matter. Our approach here extends to media with heat the variational formalism first formulated by Ray for perfect or adiabatic fluids [Ray, 1972], [Ray, 1980]. There are also generalizations of

Ray's approach to spinning fluids [Ray and Smalley, 1982], [Ray and Smalley, 1983] whose non-adiabatic extensions we shall relegate to another paper. The four-velocity constraint

$$g_{ik}U^iU^i + c^2 = 0 \tag{15}$$

contains in the case of gravitational fields the metric tensor g_{ik}. The signature convention is (+++-). Inclusion of the both degrees of freedom (flows of the entropy and matter) and absorbing constraints by Lagrange multipliers yields the Lagrangian density

$$\Lambda = \frac{c^3}{16\pi\kappa'}(-g)^{1/2}R$$

$$c^{-1}(-g)^{1/2}[(\theta\rho_s^0 + \rho_m^0)c^2 + \rho_e^0(\rho_s^0, \rho_m^0)]$$

$$+(-g)^{1/2}\gamma_s(g_{ik}U_s^iU_s^k + c^2)$$

$$+(-g)^{1/2}\gamma_m(g_{ik}U_m^iU_m^k + c^2) \tag{16}$$

$$+(-g)^{1/2}\eta_s(\rho_s^0U_s^i)_{;i}$$

$$+(-g)^{1/2}\phi_m(\rho_m^0U_m^i)_{;i}$$

$$+(-g)^{1/2}\lambda_sX_{s,i}U_s^i$$

$$+(-g)^{1/2}\lambda_mX_{m,i}U_m^i.$$

where κ' is the gravitational constant. *R* is the Riemannian curvature scalar which determines the Lagrangian of the gravitational field; its implicit form is here sufficient. The first line of the above equation represents the kinetic potential *L* of the gravitational field and matter. The second line contains the four-velocity constraint (15) applied for both thermal and bare matter. The continuity equations of the reversible evolution are adjoined to *L* in the third line of this equation. Its last line contains the corresponding identity constraints [Ray, 1972].

The energy-momentum tensor is obtained by varying the action with respect to g_{ik}, ρ_s^0, ρ_m^0, U_s^i, U_m^i, γ_s, γ_m, ϕ, η, and λ which yield a set of equations of motion. The extremum conditions with respect to the densities are

$$\eta_{,i}\,U_s^i = -c^{-1}(\theta c^2 + T^{-0});\, \phi_{,i}\,U_m^i = -c^{-1}(c^2 + \mu^{-0}), \tag{17}$$

whereas those with respect to the velocity components are

$$\gamma_s g_{ik}U_s^i = \rho_s^0\eta_{,k} - \lambda_sX_{s,k}; \tag{18}$$

$$\gamma_m g_{ik}U_m^i = \rho_m^0\phi_{,k} - \lambda_mX_{m,k}.$$

When the above conditions are combined with the velocity constraints for U_s and U_m, we obtain

$$\gamma_s = (2c^3)^{-1}\rho_s^0(\theta c^2 + T^{-0}); \tag{19}$$

$$\gamma_m = (2c^3)^{-1}\rho_m^0(c^2 + \mu^{-0}).$$

Thus the multipliers of four-velocities are additive components of the relativistic enthalpy density of the fluid.

Einstein's equations are contained in the extremum conditions of the action with respect to the components of the metric tensor

$$\frac{\partial\Lambda}{\partial g_{ik}} - \left(\frac{\partial\Lambda}{\partial g_{ik,r}}\right)_{,r} + \left(\frac{\partial\Lambda}{\partial g_{ik,r,l}}\right)_{,l,r} = 0. \tag{20}$$

For the Lagrangian (16) these equations are obtained in the usual form

$$E^{ik} = (8\pi\kappa'/c^4)G^{ik}, \tag{21}$$

where E^{ik} is the Einstein's tensor of the gravitational field. The energy-momentum tensor or the matter tensor, G^{ik}, is the source of this field. The matter tensor G^{ik} is affected by the heat flow, thermal inertia and the nonequilibrium stress. Here it is obtained in the form

$$G^{ik} = c^{-2}\rho_s^0(\theta c^2 + T^{-0})U_s^i U_s^k$$

$$+c^{-2}\rho_m^0(c^2 + \mu^{-0})U_m^i U_m^k \quad (22)$$

$$+g^{ik}[-(\theta\rho_s^0+\rho_m^0)c^2-\rho_e^0+\rho_s^0(\theta c^2+T^{-0})+\rho_m^0(c^2+\mu^{-0})].$$

where the conditions (19) for the Lagrangian multipliers have been used. Since the expression in the last line is exactly the pressure scalar, *P*, we find,

$$G^{ik} = c^{-2}\rho_s^0(\theta c^2 + T^{-0})U_s^i U_s^k \quad (23)$$

$$+c^{-2}\rho_m^0(c^2 + \mu^{-0})U_m^k U_m^k + g^{ik}P.$$

This equation takes into account effects of heat and nonequilibrium stress (the total viscous stress in the case of a purely dissipative fluid) through relative four-velocities of the entropy and bare matter with respect to the hydrodynamic four-velocity. It allows one to investigate the effect of the heat flow on the solution of the Einstein's equations. The rest mass (or relativistic energy without the statistical term) furnishes the definition of the hydrodynamic velocity U^i

$$\rho_s^0\theta U_s^i + \rho_m^0 U_m^i - \rho^0 U^i = 0 \quad (24)$$

(equivalent to the set represented by Eq. (6). With this, Eq. (23) can be cast into the extended form of **G** of Eq. (5)

$$G^{ik} = c^{-2}\left(E^0 U^i U^k + q^i U^k + q^k U^i\right) \quad (25)$$

$$+\tau^{ik} + h^{ik}P,$$

which uses the projection tensor $h^{ik} = g^{ik} + c^{-2}U^iU^k$. An equivalent form of the matter tensor G^{ik} is

$$G^{ik} = c^{-2}[(\rho^0c^2 + \rho^0e^0)U^iU^k + q^iU^k + q^kU^i] \quad (26)$$

$$+\tau^{ik} + h^{ik}P.$$

In these equations q^i is the four-vector of heat expressed as

$$q^i = \rho_s^0(\theta c^2 + T^{-0})U_s^i + \rho_m^0(c^2 + \mu^{-0})U_m^i \quad (27)$$

$$-\rho^0(c^2 + h^{-0})U^i$$

and τ^{ik} is the four-tensor of nonequilibrium stresses expressed as

$$\tau^{ik} = c^{-2}[\rho_s(\theta c^2 + T^{-0})v_s^i v_s^k \quad (28)$$

$$+\rho_m^0(c^2 + \mu^{-0})v_m^i v_m^k.$$

Here $v_s^i = U_s^i - U^i$ is the relative four-velocity. The heat flux is defined as the difference between the actual energy flux and the energy flux of a corresponding perfect fluid. Eq. (26) is known; [Ray, 1972]; [Sieniutycz and Berry, 1997], but the expressions (27) and (28) are relatively unknown. Note that in terms of the traditional temperature T^0 the relativistic heat flux equals $q^i = T^0\rho_s^0(U_s^i - U^i) = T^0\rho_s^0 v_s^i$. Indeed, with Eq. (24)

$$q^i = T^{-0}\rho_s^0 U_s^i + \mu^{-0}\rho_m^0 U_m^i - h^{-0}\rho^0 U$$

$$= T^{-0}\rho_s^0(U_s^i - U^i) + \mu^{-0}\rho_m^0(U_m^i - U^i) \quad (29)$$

$$= (T^{-0} - \theta\mu^{-0})\rho_s^0(U_s^i - U^i) = T^0\rho_s^0(U_s^i - U^i) = T^0 j_s^i.$$

Equations (28) and (32) may be regarded as macroscopic definitions of the heat flux and nonequilibrium stress tensor in barycentric frames. Unlike adiabatic fluid models, the superconductor model with thermal inertia preserves both heat and nonequilibrium stress, which is the substantial improvement. For a special case when U^i_s= U^i_m=U^i, the tensor G^{ik} simplifies to the form

$$G^{ik} = c^{-2}\rho^0(c^2 + h^0)U^iU^k + g^{ik}P. \quad (30)$$

which describes an adiabatic, perfect relativistic fluid.

7 Conclusions and Final Remarks

We have derived (not just assumed or guessed) relativistic definitions of heat and nonequilibrium stress, Eqs. (28) and (29), from the extended Hamilton principle allowing thermal degrees of freedom. Flows of the matter and inertial entropy have similar effect on the matter tensor, and the split of the total mass into the thermal mass and bare mass does not change observable effects at the thermal equilibrium. Nonequilibrium descriptions are benefited by the concept of the thermal mass, where both the heat flow **q** and the nonequilibrium stress τ emerge as effects of the entropy flow (thermal mass flow) in the fluid frame.

The substantial virtue of the approach based on the Lagrangian of a superconducting fluid is that it does not truncate terms in the tensor **G**; the obtained energy flux contains both the heat flux, **q**, and the nonequilibrium flux of momentum, τ. In the standard model of adiabatic (perfect) fluid these quantities are absent. Thus nontruncated conservation laws are found, usable even for dissipative fluids, since they contain disequilibrium fluxes **q** and τ.

Clearly, the superconducting model does not admit any dissipative mechanisms for fluxes. In fact, the assumed zero entropy production admits that fluxes can only be related to purely reversible effects, such as "ballistic" nondissipative heat transfer or elastic transport of momentum. This reversibility is the typical limitation of all classical action-type approaches. Yet, for the derived structure of **G**, conservation laws and gravitational metrics g_{ik}, is immaterial whether the fluxes **q** and τ evolve reversibly or not. As long as models are accepted which do not eliminate entropy flows (the case of the present model and not that of adiabatic fluid), general conservation laws are obtained with fluxes **q** and τ from the present two-phase (reversible) model of a superconducting fluid. This condition is the only requirement necessary to describe gravitational metrics in the general relativity, where the relativistic tensor **G** is the unique source of the gravitational field, Eq. (21).

Therefore, our conclusion should remove a frequent misunderstanding concerning the role of dissipative effects in the relativistic theory of gravitation. Here the effect of dissipation on gravitational fields is shown to be indirect at most: as a possible phenomenon causing definite flows, **q** and τ, which could otherwise be attributed to some reversible causes. This is similar to effects of electric currents which cause magnetic fields regardless whether they are reversible (caused by the motion of the conductor) or irreversible (caused by the conductivity electrons).

In view of its link with the de Broglie thermodynamics [de Broglie, 1964], the present paper may be seen as an approach that bridges the mainstream conservative ideas of relativistic thermodynamics, [Tolman, 1949], with the recent leading edge exterior works calling for exploration of alternatives to a Bigbang universe [Amoroso *et al.*, 2002]. Of certain interest in this realm is a book [Grössing, 2000] that discusses the contemporary conflict between relativity and quantum mechanics, and regards it as the apparent conflict. Grössing proposes a resolution of dilemma based on de Broglie's causal interpretation [De Broglie, 1964], [De Broglie, 1970] as elaborated by David Bohm. He claims that a "medium" or "aether" may be introduced in a manner consistent with both relativity and quantum theory, which allows the two theories to be unified by the identification of circularly causal processes at their core. He also describes several experiments confirming his predictions. However, the reviewer of the present paper raises doubts on the validity of some ideas in the Grössing's book, which, in his view, has a limited link with cybernetics.

As an example of a healthy link between the cybernetics, control and optimal control, series of papers by H. Rosenbrock treating, in particular, quantum mechanics via control theory, are recommended [Rosenbrock, 1986]; [Rosenbrock, 1995]; [Rosenbrock, 1997]; [Rosenbrock, 1999]. Indeed, starting with the well known fact that the Hamilton-Jacobi equation can be derived from Hamilton's variational principle by the methods of control theory, [Rosenbrock, 1986] shows that a suitable random disturbance added to the formulation of Hamilton's principle leads to Schrödinger's equation, and to some other results in quantum theory. Later, [Rosenbrock, 1995], extends in several directions his earlier development of quantum mechanics from a stochastic variational principle. Extensions are given to relativistic systems, to Dirac's equation, and to elementary quantum field theory. Rosenbrock's aim is to show that results in the standard theory can be obtained from an extended form of Hamilton's principle, which has the advantage of conciseness and a relatively close relation to the classical formulations. Rosenbrock's wave function appears as a modified form of the optimal cost function, and the photon is identified with a singularity in the electromagnetic field. Interference is explained by optimization of an expected value, the ensemble over which the expectation is taken being dependent upon the information available. A note [Rosenbrock, 1997] corrects an error in a previous paper [Rosenbrock, 1995] which is set out to extend a variational principle for quantum mechanics to special relativity. A next Rosenbrock's paper [Rosenbrock, 1999] gives further applications of the generalized Hamilton's principle to quantum mechanics. Namely, the wave function does no meet all the requirements in this treatment to be the state, and the way in which it can be made to do so is described and applied to the two-slit experiment. In the present journal a paper may be noted on the state estimation for a class of nonlinear dynamic systems through the Hamilton-Jacobi-Bellman technique [Filippova, 2013]. It would be interesting to confront the stochastic results of Rosenbrock with the theories postulating statistical origin of quantum mechanics [Kaniadakis, 2002].

Also, another approach towards constructing the Hamiltonian formulation as a basis of quantized thermal processes [Markus, 2005] has turned out to be quite successful. It involves a generalized Hamilton-Jacobi equation for dissipative heat processes [Markus and Gambar, 2004] as the fundamental component of the variational theory developed by these authors. The basic results are quasiparticles in thermal processes [Markus and Gambar, 2005], [Vazquez *et al.*, 2009], siblings of phonons or quasiparticles of quantized acoustic fields.

Other independent approaches are applied in the context of unified theory. The Evans wave equation of an unified theory can be derived from an appropriate Lagrangian density and action in general relativity, identifying the origin of the Planck constant h in general relativity [Evans, 2004]. The action so defined is shown to be the origin both of the Planck constant h and of the Evans spin field B. The classical Fermat principle of least time, and the classical Hamilton principle of least action, are expressed in terms of a tetrad multiplied by a phase factor $\exp(iS/h)$, where S is the action in general relativity. Wave (or quantum) mechanics emerges from these classical principles of general relativity for all matter and radiation fields, giving a unified theory of quantum mechanics based on differential geometry and general relativity. The phase factor $\exp(iS/h)$ is an eigenfunction of the Evans wave equation and is the origin in general relativity and geometry of topological phase effects in physics, including the Aharonov-Bohm class of effects, the Berry phase, the Sagnac effect, related interferometric effects, and all physical optical effects through the Evans spin field B and the Stokes theorem in differential geometry, [Evans, 2004]. The Planck constant h is thus identified as the least amount possible of action or angular momentum or spin in the universe. This is also the origin of the fundamental Evans spin field B which is observed in any physical optical effect. It originates in torsion, spin and the second (or spin) Casimir invariant of the Einstein group. Mass originates in the first Casimir invariant of the Einstein group. These two invariants define every particle [Evans, 2004].

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