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REFINED ASYMPTOTICS FOR SINGULARLY PERTURBED REACHABLE SETS

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Abstract
We study the limit behavior of the reachable sets to singularly perturbed linear dynamic systems with time dependent coef£cients under geometric constraints on control. The system data are assumed to be Lipschitz continuous functions of time. The fast component of the phase vector is governed by a strictly stable linear system. It is shown that the reachable sets converge as the small parameter ε of singular perturbation tends to zero, and the rate of convergence is $O(\varepsilon \log 1/\varepsilon)$. Under an extra assumption pertaining to singularities of the boundaries of sets of admissible controls, we £nd the coef£cient of $\varepsilon \log 1/\varepsilon$ in the asymptotic expansion for the support function of the reachable set.

Key words
Singularly perturbed linear dynamic control systems, reachable sets, asymptotics.

1 Problem Statement

The paper concerns the study of the asymptotic behavior of the reachable sets to singularly perturbed linear control systems. The motions described by such systems evolve in the two different time scales: "slow" and "fast" times, and, in a natural way, split into the two dynamics. Note that the system coef£cients vary slowly with respect to fast-time scale.

The purpose of this work is to £nd an asymptotic estimate for the reachable set to singularly perturbed linear systems, when the small parameter of singular perturbations tends to zero.

Consider the following singularly perturbed linear dynamic system under geometric constraint on control

$$\begin{aligned} \dot{x} &= Ax + By + Fu, \\ \varepsilon\dot{y} &= Cx + Dy + Gu,\ u \in U, \end{aligned} \tag{1}$$

with a given initial state $x(0) = 0$, $y(0) = 0$, and $t \in [0, T]$, where $T > 0$ is £xed. Here, $\varepsilon > 0$ is a small parameter of singular perturbation, the phase vector $z = (x, y)$ consists of "slow" $x \in \mathbf{R}^n$, and "fast" $y \in \mathbf{R}^m$ components. An admissible control is by definition a measurable function $u(\cdot)$ such that $u(t) \in U$ for almost all $t \in [0, T]$, where $U \subset \mathbf{R}^k$ is a nonvoid convex compact.

We assume the matrix functions $A, \dots, G$ and the convex compact U are Lipschitz continuous with respect to t. The latter means that the support function $h = H_U$ of the set U is Lipschitz continuous in t. The matrix D is assumed to be asymptotically stable, i.e. $\operatorname{Re} \operatorname{Spec} D < 0$, for any t.

Denote by $\mathcal{D}_\varepsilon(T)$ the reachable set to system (1) at time T, i.e. the set of ends at time T of all admissible trajectories of system (1). We will study the limit behavior of the set $\mathcal{D}_\varepsilon(T)$ as $\varepsilon \to 0$.

Under the same assumptions, this issue was addressed in [Dontchev and Slavov, 1988; Dontchev and Veliov, 1983], where it was shown that the sets $\mathcal{D}_\varepsilon(T)$ have a limit $\mathcal{D}_0(T)$ with respect to the Hausdorff metric as $\varepsilon \to 0$, and the rate of convergence of the reachable sets is $O(\varepsilon^\alpha)$, where $0 < \alpha < 1$ is arbitrary.

In this paper we succeeded to get an essential re£nement of the results [Dontchev and Slavov, 1988; Dontchev and Veliov, 1983]. We proved that the said rate of convergence of the reachable sets is, in fact, $O(\varepsilon \log 1/\varepsilon)$. In suitable coordinates the "slow" and "fast" state components split, and the limit set $\mathcal{D}_0(T)$ is the direct product of the limit reachable sets of the "slow" and "fast" subsystems. Each factor in the product is a (topological) manifold with boundary, while the product is a manifold with corners. Thus, in the £rst approximation, slow and fast controlled motions are independent. Under an extra assumption that the support function h of the set U is C^1-smooth outside the origin, we found the exact coef£cient of $\varepsilon \log 1/\varepsilon$ in the asymptotic expansion for the support function of the prelimit reachable set. Thus, the estimate $O(\varepsilon \log 1/\varepsilon)$ for the rate of convergence is sharp. The correction term in the asymptotics turns out to be negative. Geometrically, this looks like "rounding off" the corners of the limit reachable set $\mathcal{D}_0(T)$.

The assumption on Lipschitz continuity of the system coef£cients is essential. There are examples where the system parameters are Hölder continuous with exponent $0 < \alpha < 1$, while the rate of convergence of the reachable sets is $\Omega(\varepsilon^\alpha)$.

2 Splitting Dynamic System

Following [Dontchev and Slavov, 1988; Kokotovich, 1984] we can simplify the original problem by using gauge transformations. If $z = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathfrak{A} = \begin{pmatrix} A & B \\ C/\varepsilon & D/\varepsilon \end{pmatrix}$, and $\mathfrak{B} = \begin{pmatrix} F \\ G/\varepsilon \end{pmatrix}$ one can perform a substitution $z = Xw$, where X is an invertible 2×2 block matrix, and we get a new control system $\dot{w} = \widetilde{\mathfrak{A}}w + \widetilde{\mathfrak{B}}u$. Here,

$$\widetilde{\mathfrak{A}} = X^{-1}\mathfrak{A}X - X^{-1}\dot{X}, \text{ and } \widetilde{\mathfrak{B}} = X^{-1}\mathfrak{B}.$$

If X is Lipschitz continuous with respect time t, then such a transformation does not have an essential in¤uence on the behavior of the reachable sets, but allows us to simplify the system matrix so that the Lipschitz continuity and stability of the corresponding system coef£cients are preserved.

We aim at reducing the system matrix to a block-diagonal form $\widetilde{\mathfrak{A}} = \begin{pmatrix} \widetilde{A} & 0 \\ 0 & \widetilde{D}/\varepsilon \end{pmatrix}$, to separate slow and fast variables. Suf£ce it to do an approximate reduction, so that the blocks $\widetilde{B}$ and $\widetilde{C}$ are $O(\varepsilon)$, because this gives us approximate reachable sets $\mathcal{D}_\varepsilon(T)$ with the same order $O(\varepsilon)$ of precision.

By using the lower-triangular transformation $X = \begin{pmatrix} 1 & 0 \\ -D^{-1}C & 1 \end{pmatrix}$, we obtain a new block $\widetilde{C}/\varepsilon$, where

$$\widetilde{C} = \varepsilon\frac{d}{dt}(D^{-1}C) + \varepsilon D^{-1}C(A - BD^{-1}C) = O(\varepsilon).$$

Note that the Lipschitz continuity implies that the derivative $\frac{d}{dt}(D^{-1}C) = O(1)$ is bounded. Similarly, by applying the upper-triangular transformation $X = \begin{pmatrix} 1 & \varepsilon BD^{-1} \\ 0 & 1 \end{pmatrix}$, one can ensure that $\widetilde{B} = O(\varepsilon)$. Then, we arrive at the split case:

$$\begin{aligned} \dot{x} &= \widetilde{A}x + \widetilde{F}u, \\ \varepsilon\dot{y} &= \widetilde{D}y + \widetilde{G}u, \ u \in U, \end{aligned} \tag{2}$$

where all matrices and the convex compact U are Lipschitz continuous with respect to time, and $\widetilde{D}$ is a stable matrix at each time instant.

The study of the asymptotic behavior of the reachable set $\widetilde{\mathcal{D}}_\varepsilon(T)$ for the split system (2) is to a large extent equivalent to the original problem related to system (1). More precisely, the reachable sets to systems (1) and (2) coincide up to an error of order $O(\varepsilon)$. This follows immediately from the stability of $\widetilde{D}$ and the classical Tikhonov–Levinson theorem (see, e.g., [Kokotovich, 1984; Flatto and Levinson, 1955; Vasilieva and Butuzov, 1973]). The asymptotics we are looking for is, in fact, rather crude. Its remainders are of order $o(\varepsilon \log 1/\varepsilon)$ so that an error of order $O(\varepsilon)$ is negligible.

The direct computations show that the matrix coef£cients of systems (1), and (2) are related by

$$\begin{aligned} \widetilde{A} &= A - BD^{-1}C, \ \widetilde{D} = D, \\ \widetilde{F} &= F - BD^{-1}G, \ \widetilde{G} = G. \end{aligned}$$

In the next section, we state our main results in terms of the original system (1), while, in the proofs, the system splitting is heavily used.

3 Asymptotics of Support Functions to Reachable Sets

Denote by $H_\varepsilon(\xi, \eta)$ the support function of the reachable set $\mathcal{D}_\varepsilon(T)$ to system (1), where $\varepsilon > 0$, and ξ, η are dual to the phase variables x, y. De£ne the function

$$\begin{aligned} H_0(\xi, \eta) = &\int_0^T h_t(\widetilde{F}(t)^*\Phi(T,t)^*\widetilde{\xi})\,dt + \\ &+ \int_0^\infty h_T(G(T)^* e^{D(T)^* t}\eta)\,dt, \end{aligned} \tag{3}$$

where the function Φ is the fundamental matrix for the linear system

$$\dot{x} = (A - BD^{-1}C)x, \tag{4}$$

$\widetilde{\xi} = \xi - C^* D^{*-1}\eta$, and $h = h_t$ is the support function of the set $U = U_t$ of controls. The function H_0 is, in fact, the support function for the limit reachable set

$$\mathcal{D}_0(T) = \lim_{\varepsilon \to 0} \mathcal{D}_\varepsilon(T).$$

Theorem 1. *Let $H_\varepsilon(\xi, \eta)$ be the support functions of the reachable sets $\mathcal{D}_\varepsilon(T)$ to system (1), then $H_\varepsilon(\xi, \eta)$ converge $H_0(\xi, \eta)$ uniformly on compacts as $\varepsilon \to 0$. Moreover, we have the asymptotic equivalence:*

$$H_\varepsilon(\xi, \eta) = H_0(\xi, \eta) + O(\varepsilon \log 1/\varepsilon(|\xi| + |\eta|)) \text{ as } \varepsilon \to 0.$$

In other words, $d_H(\mathcal{D}_\varepsilon, \mathcal{D}_0) = O(\varepsilon \log 1/\varepsilon)$, where d_H is the Hausdorff metric.

The idea underlying the proof goes back at least to [Dontchev and Veliov, 1983], and, basically, says that the reachable set to a linear control system can be decomposed in accordance with the decomposition of the spectrum of the system matrix into stable, unstable, and neutral components. Each composing part is formed by using controls supported on non-overlapping time intervals. In our case, there are just two spectral components: the neutral one corresponding to "slow" variables, and the stable one corresponding to "fast" variables. We divide the time interval $[0, T]$ into the two subintervals $[0, T-\delta]$ and $[T-\delta, T]$, where δ is a small positive parameter. The controls supported on the "long" interval are responsible for the "slow" part of the reachable set, while the controls supported on the "short" interval form the "fast" part of it. The proper choice of δ is crucial for the accuracy of approximation. By choosing a small $\delta > 0$ such that $\delta \to 0$ and $\delta/\varepsilon \to \infty$ as $\varepsilon \to 0$, we get an approximation to the set $\mathcal{D}_0(T)$. The main difference of the present paper with [Dontchev and Slavov, 1988] stems from the £nal choice $\delta \sim \varepsilon \log 1/\varepsilon$ instead of $\delta \sim \varepsilon^{\alpha}$.

Our main asymptotic result consists in £nding the remainder in the previous theorem in a more precise form $\mathbf{c}(\xi,\eta)\varepsilon\log 1/\varepsilon + o(\varepsilon \log 1/\varepsilon)$. We can do this under an extra assumption that the support function h_T of the set U_T of controls is C^1-smooth outside the origin. Denote the argument of the function $\bar{h} = h_T$ by ζ, and consider the average

$$\mathrm{Av}_{\tau}(\frac{\partial \bar{h}}{\partial \upsilon})(\eta) = \frac{1}{\tau}\int_0^{\tau} \frac{\partial \bar{h}}{\partial \upsilon}(G(T)^* e^{D(T)^* t}\eta)\, dt.$$

The limit $\mathrm{Av}(f) = \lim_{\tau\to\infty} \mathrm{Av}_{\tau}(f)$ does exist for any homogeneous of zero degree function f which is continuous for any $\upsilon \neq 0$. Indeed, the function $\phi(t) = G(T)^* e^{D(T)^* t}\eta$ has the form of a vector-valued quasipolynomial, and, therefore,

$$\phi(t) = e^{(\mathrm{Re}\,\lambda) t} t^N \sum e^{i\omega t} a_{\omega} + o(e^{(\mathrm{Re}\,\lambda)t} t^N),$$

where λ is an eigenvalue of the matrix $D(T)$, $N+1$ is the maximal size of the corresponding Jordan block, sum is taken over all real ω such that $\mathrm{Re}\,\lambda + i\omega$ is an eigenvalue of $D(T)$, and a_{ω} is a time-independent vector. Since the function f is homogeneous of zero degree, we have

$$\begin{aligned}\mathrm{Av}_{\tau}(f) &= \frac{1}{\tau}\int_0^{\tau} f(\phi(t))\, dt = \\ &= \frac{1}{\tau}\int_0^{\tau} f\left(\phi_0(t)\right)\, dt + o(1),\end{aligned}$$

where $\phi_0(t) = \sum e^{i\omega t} a_{\omega}$ is a trigonometric polynomial. "Generically", the trigonometric polynomial $\phi_0(t)$ is comprised either of one or two harmonics. The limit

$$\mathrm{Av}\,(f) = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} f\left(\phi_0(t)\right)\, dt,$$

does exist, since the integrand in (3) is an almost periodic function which can be averaged.

Theorem 2. *Assume that the support function $\bar{h} = h_T$ of the control set U_T is C^1-smooth outside the origin. Then, the support function $H_{\varepsilon}(\xi,\eta)$ of the reachable set $\mathcal{D}_{\varepsilon}(T)$ to system (1) has the following asymptotic representation:*

$$\begin{aligned}H_{\varepsilon}(\xi,\eta) &= H_0(\xi,\eta) + \mathbf{c}(\xi,\eta)\,\varepsilon\log 1/\varepsilon + \\ &+ o(\varepsilon\log 1/\varepsilon(|\xi| + |\eta|)) \quad \textit{as } \varepsilon \to 0, \qquad (5)\end{aligned}$$

where

$$\mathbf{c}(\xi,\eta) = \frac{1}{\Lambda}\left[\mathrm{Av}\left\langle \widetilde{F}(T)^*\widetilde{\xi}, \frac{\partial \bar{h}}{\partial \upsilon}\right\rangle(\eta) - \bar{h}(\widetilde{F}(T)^*\widetilde{\xi})\right],$$

$\Lambda = \Lambda(\eta)$ is the absolute value of the £rst Lyapunov exponent of the function $t \mapsto e^{D(T)^* t}\eta$ (this is the modulus of the real part of an eigenvalue of $D(T)$), $\widetilde{\xi} = \xi - C(T)^* D(T)^{*-1}\eta$, and $\widetilde{F}(T) = F(T) - B(T)D(T)^{-1}G(T)$.

Note that the coef£cient $\mathbf{c}(\xi,\eta)$ of $\varepsilon\log\frac{1}{\varepsilon}$ in the asymptotic expansion (5) is nonpositive. Indeed, $\langle\zeta, \frac{\partial \bar{h}}{\partial \upsilon}(\upsilon)\rangle \leq \bar{h}(\zeta)$ for any ζ and $\upsilon \neq 0$, and the averaging operation preserves the inequality.

Proof. Consider system (2) and divide the proof into the several steps.

Decomposition of the time interval. First, we de£ne an optimal value of $\delta = \delta(\eta,\varepsilon)$ by the condition $|e^{D(T)^*\frac{\delta}{\varepsilon}}\eta| = \varepsilon|\eta|$. Here, $|\cdot|$ stands for an arbitrary norm. Of course, δ depends on the choice of the norm, but not essentially: for any norm

$$\delta \sim \frac{1}{\Lambda}\varepsilon\log\frac{1}{\varepsilon} \quad \text{as } \varepsilon \to 0.$$

Integration over the "long" interval. In what follows, we rely on the Lipschitz inequality

$$|h_t(\zeta+\theta) - h_t(\zeta)| \leq C|\theta|$$

for the support function h of the set U. The inequality is immediately implied by the uniform bound $|x| \leq C$ for any $x \in U$. In particular,

$$\begin{aligned}\left|\int h_t(\zeta(t)+\theta(t))\, dt - \int h_t(\zeta(t))\, dt\right| &\leq \\ \leq C\int |\theta(t)|\, dt, \qquad (6)\end{aligned}$$

where ζ, θ are arbitrary integrable vector functions. Consider the support function H_{ε} to the reachable set $\mathcal{D}_{\varepsilon}(T)$ to the split system (2) de£ned by

$$H_{\varepsilon}(\xi,\eta) = \int_0^T h_t(F(t)^*\Phi(T,t)^*\xi + \frac{1}{\varepsilon}G(t)^*\Psi_{\varepsilon}(T,t)^*\eta)dt.$$

Due to the Lipschitz inequality (6), we have

$$\begin{aligned}I_0^{T-\delta} &= \int_0^{T-\delta} h_t(F^*(t)\Phi^*(T,t)\xi + \frac{1}{\varepsilon}G^*(t)\Psi_{\varepsilon}^*(T,t)\eta)\, dt \\ &= \int_0^{T-\delta} h_t(F^*(t)\Phi^*(T,t)\xi)\, dt + \\ &+ O(\int_0^{T-\delta} \frac{1}{\varepsilon}|\Psi_{\varepsilon}^*(T,t)\eta|\, dt).\end{aligned}$$

The remainder

$$\begin{aligned}R &= \int_0^{T-\delta} \frac{1}{\varepsilon}|\Psi_{\varepsilon}^*(T,t)\eta|\, dt = \\ &= O(\frac{1}{\varepsilon}\int_0^{T-\delta} |\Psi_{\varepsilon}^*(T,t)||\eta|\, dt)\end{aligned}$$

can be estimated as follows. We have

$$\Psi^*_\varepsilon(T,t) = \Psi^*_\varepsilon(T-\delta,t)\Psi^*_\varepsilon(T,T-\delta), \text{ and}$$
$$\Psi^*_\varepsilon(T,T-\delta)\eta = O(\varepsilon)|\eta|.$$

Indeed, $\Psi^*_\varepsilon(T,T\text{–}\delta)\eta = e^{D(T)^*\delta/\varepsilon}\eta + O(\frac{\delta^2}{\varepsilon}e^{-\alpha\delta/\varepsilon}|\eta|)$, and $\frac{\delta^2}{\varepsilon}e^{-\alpha\delta/\varepsilon} = C\varepsilon\left(\log^2\frac{1}{\varepsilon}\right)\varepsilon^\beta = o(\varepsilon)$, while $e^{D(T)^*\delta/\varepsilon}\eta = O(\varepsilon)|\eta|$, where C and β are positive constants. Therefore,

$$R = O(\varepsilon)O(\frac{1}{\varepsilon}\int_0^{T-\delta}|\Psi^*_\varepsilon(T-\delta,t)||\eta|\,dt).$$

The integral $\frac{1}{\varepsilon}\int_0^{T-\delta}|\Psi^*_\varepsilon(T-\delta,t)|\,dt$ is bounded, because it can be estimated via an integral of the form

$$\frac{1}{\varepsilon}\int_0^{T-\delta}e^{-\alpha\frac{\tau}{\varepsilon}}\,d\tau \le \frac{1}{\varepsilon}\int_0^{\infty}e^{-\alpha\frac{\tau}{\varepsilon}}\,d\tau = \frac{1}{\alpha}.$$

Thus, we conclude that $I = O(\varepsilon|\eta|)$, and

$$I_0^{T-\delta} = \int_0^{T-\delta}h_t(F^*(t)\Phi^*(T,t)\xi)\,dt + O(\varepsilon|\eta|). \quad (7)$$

Since we do not need a greater precision than $o(\varepsilon\log\frac{1}{\varepsilon})$, this estimate is wholly satisfactory.

Integration over the "short" interval. The integral over the complementary interval

$$I_{T-\delta}^T = \int_{T-\delta}^T h_t(F^*(t)\Phi^*(T,t)\xi + \frac{1}{\varepsilon}G^*(t)\Psi^*_\varepsilon(T,t)\eta)\,dt$$

is equal to

$$\int_{T-\delta}^T h_t(F^*(T)\xi + \frac{1}{\varepsilon}G^*(T)e^{D(T)^*\frac{T-t}{\varepsilon}}\eta)\,dt + $$
$$+ O(\delta^2|\xi|) + O(\varepsilon|\eta|). \quad (8)$$

Indeed, first we have

$$\frac{1}{\varepsilon}\int_{T-\delta}^T|\Psi^*_\varepsilon(T,t) - e^{D(T)^*\frac{T-t}{\varepsilon}}|dt = O(\varepsilon),$$

and this allows us to substitute $e^{D(T)^*\frac{T-t}{\varepsilon}}$ for $\Psi^*_\varepsilon(T,t)$. Second, we have to take into account that

$$\left|\int_{T-\delta}^T f(t)\,dt\right| = O(\delta^2)$$

if f is a Lipschitz function such that $f(T) = 0$. This allows us to substitute $F^*(T)$ for $F^*(t)\Phi^*(T,t)$, and we obtain that

$$I_{T-\delta}^T = \int_{T-\delta}^T h_t(F^*(T)\xi + \frac{1}{\varepsilon}G^*(t)e^{D(T)^*\frac{T-t}{\varepsilon}}\eta)\,dt + $$
$$+ O(\delta^2|\xi|) + O(\varepsilon|\eta|).$$

Third, if we substitute $G^*(T)$ for $G^*(t)$ in the last integral, there arises an error of order

$$\int_{T-\delta}^T\frac{1}{\varepsilon}(T-t)\left|e^{D(T)^*\frac{T-t}{\varepsilon}}\right|\,dt =$$
$$= \varepsilon\int_0^{\delta/\varepsilon}\tau|e^{D(T)^*\tau}|\,d\tau = O(\varepsilon),$$

and we arrive at (8).

Estimation formula for the support function. Due to the estimates (7), (8), we can summarize that

$$H_\varepsilon(\xi,\eta) = I_0^{T-\delta} + I_{T-\delta}^T = \int_0^{T\text{–}\delta}h_t(F^*(t)\Phi^*(T,t)\xi)\,dt$$
$$+ \int_{T\text{–}\delta}^T h_t(F^*(T)\xi + \frac{1}{\varepsilon}G^*(T)e^{D(T)^*\frac{T-t}{\varepsilon}}\eta)\,dt$$
$$+ O(\delta^2|\xi|) + O(\varepsilon|\eta|). \quad (9)$$

The right-hand side of (9) can be rewritten as

$$\int_0^T h_t(F^*(t)\Phi^*(T,t)\xi)\,dt + \int_0^{\delta/\varepsilon}\bar h(G^*(T)e^{D(T)^*\tau}\eta)\,d\tau$$
$$+ \int_0^{\delta/\varepsilon}\Big(\bar h(\varepsilon F^*(T)\xi + G^*(T)e^{D(T)^*\tau}\eta) -$$
$$-\bar h(G^*(T)e^{D(T)^*\tau}\eta)\Big)\,d\tau - \delta\bar h(F^*(T)\xi) +$$
$$+O(\delta^2|\xi|) + O(\varepsilon|\eta|) = K + L + M - N + J, (10)$$

where $\bar h = h_T$, and $J = O(\delta^2|\xi|) + O(\varepsilon|\eta|)$ is the remainder. The integral $K = \int_0^T h_t(F^*(t)\Phi^*(T,t)\xi)\,dt$ does not need any further transformation. The integral $L = \int_0^{\delta/\varepsilon}\bar h(G^*(T)e^{D(T)^*\tau}\eta)\,d\tau$ has the limit $\int_0^\infty \bar h(G^*(T)e^{D^*(T)\sigma}\eta)\,d\sigma$ as $\varepsilon\to 0$. More precisely,

$$\int_0^{\delta/\varepsilon}\bar h(G^*(T)e^{D^*(T)\sigma}\eta)\,d\sigma =$$
$$= \int_0^\infty \bar h(G^*(T)e^{D^*(T)\sigma}\eta)\,d\sigma + O(\varepsilon),$$

because of the defining equality $|e^{\widetilde D^*\frac{\delta}{\varepsilon}}\eta| = \varepsilon|\eta|$ for δ, and the estimate

$$\int_T^\infty|f(t)|dt = O(f(T)) \text{ as } T\to+\infty,$$

for $f(t) = e^{Dt}\eta$, where D is a stable matrix.

Linearization. The asymptotic computation of the integral

$$M = \int_0^{\delta/\varepsilon}\Big(\bar h(\varepsilon F^*\xi + G^*e^{D^*\tau}\eta) - \bar h(G^*e^{D^*\tau}\eta)\Big)\,d\tau$$

is the heart of the proof. The idea is to treat the term $\varepsilon F^*\xi$ in the argument of the integrand of (3) as a small perturbation of $G^*e^{D^*\tau}\eta$, and linearize the difference:

$$\bar h(\varepsilon F^*\xi + G^*e^{D^*\tau}\eta) - \bar h(G^*e^{D^*\tau}\eta) =$$
$$= \varepsilon\left\langle F^*\xi, \frac{\partial\bar h}{\partial v}(G^*e^{D^*\tau}\eta)\right\rangle + \text{remainder}.$$

Here, the notations F, G, D stand for $F(T), G(T), D(T)$, respectively. Unfortunately, $\dfrac{\partial \bar{h}}{\partial \upsilon}$ is discontinuous and even unde£ned at the origin. Therefore, in order to use the linearization, we have to make sure that $G^* e^{D^*\tau}\eta$ is much greater than $\varepsilon F^*\xi$. Certainly, $\varepsilon F^*\xi = O(\varepsilon)$ is small. Still the term $G^* e^{D^*\tau}\eta$ can be very small at some points τ of the interval of integration $[0, \delta/\varepsilon]$. We show that the interval $[0, \delta/\varepsilon]$ can be divided into two subsets such that on the one subset the term $G^* e^{D^*\tau}\eta$ is relatively large, while the other part has a relatively small Lebesgue measure. Before we proceed let us consider a trivial case. If the function $\tau \mapsto G^*(T)e^{D(T)^*\tau}\eta$ is identically zero in $[0, \delta/\varepsilon]$, then, as we can see from (10), the integral M is simply equal to N:

$$M = \int_0^{\delta/\varepsilon} \Big(\bar{h}(\varepsilon F^*\xi + G^* e^{D^*\tau}\eta) - \bar{h}(G^* e^{D^*\tau}\eta) \Big)\, d\tau \\ = \delta \bar{h}(F^*\xi).$$

In what follows we assume that the function $G^*(T)e^{D(T)^*\tau}\eta$ is not identically zero.

"Bad" and "good" time instants. Denote $A_\varepsilon = A(\eta, \varepsilon) := [0, \delta/\varepsilon]$. This is an interval of length

$$l_\varepsilon = \mathrm{length}(A_\varepsilon) = \delta/\varepsilon \sim \frac{1}{\Lambda} \log 1/\varepsilon.$$

Let us divide the interval of integration A_ε into the two subsets $S_\varepsilon = S(\eta, \varepsilon) = \{\tau \in A_\varepsilon : |G^* e^{D^*\tau}\eta| \geq \varepsilon \log 1/\varepsilon\}$, and $\bar{S}_\varepsilon = \bar{S}(\eta, \varepsilon)$ de£ned by the opposite inequality.

We will show that the reduced Lebesgue measure $\bar{\lambda}(\bar{S}_\varepsilon) = \lambda(\bar{S}_\varepsilon)/\lambda(A_\varepsilon)$ of the set $\bar{S}_\varepsilon$ is $o(1)$. For this purpose, we reduce the problem to another one pertaining to quasiperiodic functions.

Clearly, the condition $|G^* e^{D^*\tau}\eta| \leq \varepsilon \log 1/\varepsilon$ can be rewritten in notations (3) as

$$|\phi_0(\tau) + \phi_1(\tau)| \leq e^{\Lambda\tau} \frac{1}{\tau^N} \varepsilon \log 1/\varepsilon, \qquad (11)$$

where $\phi_0(\tau) = \sum e^{i\omega\tau} a_\omega$ is a trigonometric polynomial, while $\phi_1(\tau) = O(1/\tau)$ as $\tau \to \infty$. One can get rid of the term ϕ_1 as follows. If one removes a relatively short interval (the reduced measure of which is $o(1)$), e.g., $[0, l_\varepsilon^{1/2}]$, from A_ε, then condition (11) reduces to

$$|\phi_0(\tau)| \leq e^{\Lambda\tau} \frac{1}{\tau^N} \varepsilon \log 1/\varepsilon + o(1),$$

where $o(1) = O(1/\log^{1/2}(1/\varepsilon))$. It is enough to show that the reduced Lebesgue measure $\bar{\lambda}(\sigma_\varepsilon)$ of the set

$$\sigma_\varepsilon = \{\tau \in A_\varepsilon : |\phi_0(\tau)| \leq e^{\Lambda\tau} \frac{1}{\tau^N} \varepsilon \log 1/\varepsilon + o(1)\},$$

which is asymptotically equivalent to $\bar{\lambda}(\bar{S}_\varepsilon)$, is $o(1)$ as $\varepsilon \to 0$. Now, we £x an arbitrary small $\kappa > 0$ and split the set $\sigma_\varepsilon = \sigma'_\varepsilon \cup \sigma''_\varepsilon$ into two subsets, where $\sigma'_\varepsilon = \{\tau \in \sigma_\varepsilon : \tau \in [l_\varepsilon^{1/2}, (1-\kappa) l_\varepsilon]\}$ is the intersection of σ_ε with the "long" interval $[l_\varepsilon^{1/2}, (1-\kappa) l_\varepsilon]$, while $\sigma''_\varepsilon = \{\tau \in \sigma_\varepsilon : \tau \in [(1-\kappa) l_\varepsilon, l_\varepsilon]\}$ is contained in the "short" interval $[(1-\kappa) l_\varepsilon, l_\varepsilon]$. The maximum of $e^{\Lambda\tau}\tau^{-N}$ in the "long" interval $[l_\varepsilon^{1/2}, (1-\kappa) l_\varepsilon]$ is less than $\varepsilon^{\kappa - 1}$. Therefore, for $\tau \in \sigma'_\varepsilon$,

$$e^{\Lambda\tau} \frac{1}{\tau^N} \varepsilon \log 1/\varepsilon + o(1) \leq \varepsilon^\kappa \log 1/\varepsilon + o(1) = o(1),$$

and the set σ'_ε is contained in the set

$$\sigma'''_\varepsilon = \{\tau \in A_\varepsilon : |\phi_0(\tau)| \leq o(1)\},$$

which is de£ned via the quasiperiodic function ϕ_0. We know that $\bar{\lambda}(\sigma_\varepsilon) \leq \bar{\lambda}(\sigma''_\varepsilon) + \bar{\lambda}(\sigma'''_\varepsilon)$, where $\bar{\lambda}(\sigma''_\varepsilon) \leq \kappa$. The reduced Lebesgue measure $\bar{\lambda}(\sigma'''_\varepsilon)$ can be estimated by using averaging:

Averaging quasiperiodic sets. The function ϕ_0 is de£ned as follows. We have an analytic function Ψ and a dense straight winding on a torus $\mathcal{T}$. The function ϕ_0 is the restriction of Ψ on the winding. If we £x any number $\alpha > 0$, then the reduced measure of the set $\{\tau \in A_\varepsilon : |\phi_0(t)| \leq \alpha\}$ tends to the canonical Haar measure of the set $\{\mathbf{t} \in \mathcal{T} : |\Psi(t)| \leq \alpha\} \subset \mathcal{T}$ as $\varepsilon \to 0$. Since the set $\{\mathbf{t} \in \mathcal{T} : \Psi(\mathbf{t}) = 0\}$ is an analytic hypersurface in $\mathcal{T}$, its Haar measure is zero. This implies that $\bar{\lambda}(\sigma'''_\varepsilon) = o(1)$ as $\varepsilon \to 0$, and the same is true for $\bar{\lambda}(\sigma_\varepsilon)$.

Asymptotic form of integral M. Now we come back to the integral M from (10):

$$M = \int\limits_{S(\eta,\varepsilon)} \Big(\bar{h}(\varepsilon F^*\xi + G^* e^{D^*\tau}\eta) - \bar{h}(G^* e^{D^*\tau}\eta) \Big)\, d\tau \\ + \int\limits_{\bar{S}(\eta,\varepsilon)} \Big(\bar{h}(\varepsilon F^*\xi + G^* e^{D^*\tau}\eta) - \bar{h}(G^* e^{D^*\tau}\eta) \Big)\, d\tau \\ = I_1(\varepsilon) + I_2(\varepsilon).$$

We know now that the vectors $\varepsilon F^*\xi$ and $G^* e^{D^*\tau}\eta$ on $S(\eta, \varepsilon)$ have different orders of magnitude: $\varepsilon F^*\xi = O(\varepsilon)$, while $|G^* e^{D^*\tau}\eta| \geq \varepsilon \log 1/\varepsilon$. Therefore, $\upsilon(\lambda) = \lambda \varepsilon F^*\xi + G^* e^{D^*\tau}\eta \neq 0$ for all $\lambda \in [0, 1]$. Then, in view of the absolute continuity of $h(\upsilon(\lambda))$, we have

$$\bar{h}(\varepsilon F^*\xi + G^* e^{D^*\tau}\eta) - \bar{h}(G^* e^{D^*\tau}\eta) = \\ = \int_0^1 \left\langle \varepsilon F^*\xi, \frac{\partial \bar{h}}{\partial \upsilon}(\upsilon(\lambda)) \right\rangle d\lambda \\ = \varepsilon \left\langle F^*\xi, \frac{\partial \bar{h}}{\partial \upsilon}(G^* e^{D^*\tau}\eta) \right\rangle + o(\varepsilon|\xi|), \qquad (12)$$

where $o(\varepsilon)$ is uniform with respect to $\tau \in S(\eta, \varepsilon)$. Indeed, the natural domain of the function $f(\upsilon) = \dfrac{\partial \bar{h}}{\partial \upsilon}$ is the sphere $\sigma = \{|\upsilon| = 1\}$, because f is homogeneous of degree 0. Since $\bar{h}$ is assumed to be C^1 on the sphere, the function f has a modulus of continuity ω, so that $|f(x) - f(y)| \leq \omega(|x - y|)$ if $x, y \in \sigma$. Now, consider vectors $x = x_\lambda = \upsilon(\lambda)/|\upsilon(\lambda)|$ in σ. By de£nition of the set $S(\eta,\varepsilon)$ we have that

$$\frac{|\upsilon(\lambda)-\upsilon(0)|}{|\upsilon(0)|} \le \frac{C|\xi|}{\log 1/\varepsilon} = o(1)$$

for any $\lambda \in [0,1]$. This implies that $|x_\lambda - x_0| \sim (\log 1/\varepsilon)^{-1} = o(1)$, and, therefore,

$$|f(x_\lambda) - f(x_0)| \le \omega((\log 1/\varepsilon)^{-1}) = o(1)$$

uniformly over $\tau \in S(\eta,\varepsilon)$, $\lambda \in [0,1]$. The integral from (12) is equal to

$$\left\langle \varepsilon F^*\xi, \int_0^1 f(x_\lambda)d\lambda \right\rangle = \langle \varepsilon F^*\xi, f(x_0)\rangle + \\ + \varepsilon O(\omega((\log 1/\varepsilon)^{-1})|\xi|),$$

what makes transparent the uniformity of $o(\varepsilon)$ in (12).

Now, we turn to integral $I_1(\varepsilon)$. We know that the Lebesgue measure $\lambda(S(\eta,\varepsilon)) \sim \delta/\varepsilon$, while the measure $\lambda(\overline{S}(\eta,\varepsilon))$ is $o(\delta/\varepsilon)$. By virtue of (12) we have

$$I_1(\varepsilon) = \varepsilon \int\limits_{S(\eta,\varepsilon)} \left\langle F^*\xi, \frac{\partial \bar h}{\partial \upsilon}(G^* e^{D^*\tau}\eta) \right\rangle d\tau + o(\delta|\xi|)$$
$$= \delta \text{Av} \left\langle F^*\xi, \frac{\partial \bar h}{\partial \upsilon} \right\rangle (\eta) + o(\delta|\xi|).$$

Note that the remainder $o(\delta) = \delta o(1)$, where $o(1)$ may decrease rather slowly. The rate depends on the Diophantine properties of eigenvalues of matrix D.

Since $\bar h(\upsilon)$ satis£es the Lipschitz inequality, the integral $I_2(\varepsilon)$ can be estimated as follows:

$$I_2(\varepsilon) = O(\varepsilon|\xi|)\,\lambda(\overline{S}(\eta,\varepsilon)) = o(\delta|\xi|).$$

Asymptotically best estimate of the support function. Summing up we obtain that

$$H_\varepsilon(\xi,\eta) = H_0(\xi,\eta) + \left(\text{Av}\left\langle F^*\xi, \frac{\partial \bar h}{\partial \upsilon}\right\rangle(\eta) - \bar h(F^*\xi)\right)\delta \\ + o(\delta|\xi|) + O(\varepsilon|\eta|),$$

where

$$H_0(\xi,\eta) = \int_0^T h_t(F^*(t)\Phi^*(T,t)\xi)dt + \int_0^\infty \bar h(G^* e^{D^*t}\eta)dt.$$

By passing to the original coordinates, we get the similar estimate for the support function of the reachable set to the original system in accordance with the theorem statement.

4 Conclusion

In the generic case, when the set U_T of admissible controls at the time instant T is not a singleton, the coef£cient $\mathbf{c}(\xi,\eta)$ is not identically (for all ξ,η) equal to zero. Otherwise, this would imply the equality $\bar h(\zeta) = \langle \zeta, \varphi\rangle$, $\varphi = \text{Av}\,(\frac{\partial \bar h}{\partial \zeta}(G^* e^{D^*t}\eta))$, for all ζ and η. The latter necessarily means that $\bar h(\zeta)$ is the support function for the singleton $U_T = \{\varphi\}$. The fact that $c(\xi,\eta) \not\equiv 0$ means that the estimate given by Theorem 1 is sharp, i.e., for some ξ, η, we have

$$|H_\varepsilon(\xi,\eta) - H_0(\xi,\eta)| \ge C\varepsilon \log 1/\varepsilon,$$

where $C > 0$ does not depend on ε.

The above results can be illustrated by the following simple example of a singularly perturbed linear system:

$$\dot x = u \\ \varepsilon \dot y = -y + u,$$

where x, y, u are scalars, and $|u| \le 1$. This example is also presented in [Dontchev and Slavov, 1988]. An easy calculation reveals that in this case the difference of the support functions of the prelimit and limit reachable sets equals

$$\Delta H = H_\varepsilon(\xi,\eta) - H_0(\xi,\eta) = \\ = -2t_\varepsilon|\xi| - |\eta|(2e^{-t_\varepsilon/\varepsilon} - e^{-T/\varepsilon})$$

provided that $\xi\eta < 0$. Here, $t_\varepsilon = \varepsilon \log \frac{1}{\varepsilon}\frac{|\eta|}{|\xi|}$. Thus, for £xed ξ,η in this range, the difference ΔH has the form $-2|\xi|\varepsilon \log \frac{1}{\varepsilon} + C\varepsilon + r$, where C is a constant, and the remainder r is exponentially small as $\varepsilon \to 0$. This proves again that the estimate in Theorem 1 is sharp.

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References

Dontchev, A. L. and Slavov, J. I. (1988). Lipschitz properties of the attainable set of singularly perturbed linear systems. *Systems & Control Letters*, **11(5)**, pp. 385–391.

Dontchev, A. L. and Veliov, V. M. (1983). Singular perturbations in Mayer's problem for linear systems. *SIAM J. Control and Optim.*, **21**, pp. 566–581.

Kokotovich, P. V. (1984). Singular Perturbation Techniques In Control Theory. *SIAM Review*, **6(4)**, pp. 501–550.

Flatto, L. and Levinson, N. (1955). Periodic solutions of singularly perturbed systems. *J. Rat. Mech. Anal.*, **4**, pp. 943–950.

Vasilieva, A. B. and Butuzov, V. F. (1973). *Asymptotic Expansions of Solutions of Singularly Perturbed Differential Equations*. Nauka. Moscow.

Dontchev, A. L. and Veliov, V. M. (1983). On the behaviour of solutions of linear autonomous differential inclusions at in£nity. *C. R. Acad. Bulg. Sci.* **36**, pp. 1021–1024.