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NUMERICAL RESEARCH OF THE CONCRETE DYNAMIC SYSTEMS BY METHODS OF PATTERN RECOGNITION AND STATISTICAL MODELLING

Yu.I. Neimark, I.S. Gel'fer, I.V. Kotel'nikov, L.G. Teklina
The Research Institute for Applied Mathematics and Cybernetics
Nizhni Novgorod State University
Russia
neymark@pmk.unn.ru

Abstract

In this work the new approach to numerical research of the concrete multidimensional dynamic systems on the basis of use of the pattern recognition methods and the statistical modeling is submitted. This approach allows to overcome the difficulties connected with the high dimensions of phase space and space of parameters of researched system. The offered approach is based on consideration not ideal but the real and computable mathematical model having the physical and technical importance. The concept of a coarsened computer phase portrait for dynamic system is entered, and the methods of the coarsened phase portrait construction are offered. The potentialities of the new approach are demonstrated on the example of mathematical model for the immune response of a organism to an infection invasion.

Key words

Dynamic system, phase portrait, pattern recognition.

1 Introduction

The theory of dynamic systems and the theory of the differential equations adjoining to it is one of the most beautiful, extensive and rich by appendices mathematical theories. Research of the dynamic systems is one of the ways for knowledge of the world surrounding us and for perfection of the modern technics. The theory of the dynamic systems which have arisen in works of great scientists A.Poinkare and D.Birkgof has received powerful development however its successes in research of the concrete dynamic systems are modest enough. In the thirtieth years of the last century academician A.Andronov has paid attention to insufficiency of the classical theory of dynamic systems for research of real systems of dimension $n > 2$. Certainly, from that time the mathematical theory of dynamic systems has essentially extended, but its opportunities lag still behind modern requirements, so that today there are actual A.Andronov's words: "It is necessary to find the apparatus which would adequately display processes and besides which would be effective enough, i.e. would allow answers to the questions which are put forward by physics and technics".

In the basic doubtless successes in research of the dynamic systems are connected with use of the high-speed computers. But can a computer all?

2 The concept of coarsened numerical research for the concrete dynamic systems

The theory of the dynamic systems studies the ideal mathematical model, finding out what are its possible phase portraits, movements and bifurcations. However, requirements of physics and technics concern directly not to ideal mathematical model of the dynamic system but to real and computable. But even concrete phase trajectories can be calculated not always. At physical realization of the dynamic system the unreiteration answers to this fact, and only it is possible to try to find the statistical description. Furthermore, some features of a phase portrait for the ideal dynamic system can turn out practically not computable. But from our point of view, what is not computable cannot be physically or technically significant. Moreover, it is necessary to distinguish based on principle and real potentialities of calculations. So, the exponentially unstable trajectory is computable in principle for any finite time interval but it is really computable only for time that is less than some T_0. The possibility of numerical research for the concrete dynamic systems is limited by real potentialities of calculations. Exact research of the ideal dynamic system by numerical methods is impossible and a certain roughness in the description of the phase space structure for these system is inevitable.

The realized calculations, the necessary restrictions and the initial data are put in a basis of the offered concept of computer research for the concrete dynamic systems. First of all clearly that numerical research is realizable only in the finite region of the phase space in spite of a theoretical opportunity of its infinite size. This finite region G should be specified proceeding from a real problem. Further in it the part D which phase trajectories do not leave can be allocated. The further research is carried out only in this finite subregion D of region G. Thus, the phase space is supposed by finite, the operator of the dynamic system is by computable, the notions of stability and instability are made more exact, the ideas of small and large quantities for space and time are specified, the conceptions of attractors and their attraction regions are defined, the final purpose of calculations is formulated. Moreover it is necessary to distinguish when attractor is a stable equilibrium or periodic movement and when it is chaotic attractor.

The really computable features of behavior for phase trajectories of researched system (the presence of stable equilibriums, periodic movements and also chaotic or stochastic attractors) are put in the base of the coarsened description. For the limited region of the phase space, which the phase trajectories do not leave, the concept of a coarsened computer phase portrait of the dynamic system is entered. To construct a coarsened phase portrait at preset values of parameters means to define and describe the stable movements (attractors) and their regions of attraction, more precisely, their some parts adjoining to corresponding attractors. And the purpose of calculations is to obtain not reliable results, but statistically reliable with probability $p < 1$ and p is sufficiently close to 1.

Thus the result of the coarsened numerical research of the dynamic system is the determination of the phase space structure as some set of attractors J_s and their attraction regions $O(J_s)$. All results are within the limits of determined precisions and time, which can be made more exact and extended if it is necessary. Concerning the attraction regions the statistical reliability of the received results can be estimated. With increase in calculating time the reliability of result comes nearer to unit.

Similarly to a coarsened phase portrait for the limited region in the space of parameters the concept of a coarsened computer parametrical portrait of the dynamic system is entered. To construct a coarsened parametrical portrait means to determine and describe sets of the parameters values, adequate to various kinds of the stable movements (attractors). The coarsened parametrical portrait describes the change of a coarsened phase portrait depending on values of parameters.

But even such coarsened research for the large dimensions of the phase and parametrical spaces demands the qualified, long and laborious work. Sometimes difficulties are so great that research is practically unrealizable. In fact in case of two-dimensional system the researcher recognizes the images of equilibriums, periodic movements, separatrixes, attraction regions, bifurcations on the computer screen representing the pieces of the phase trajectories for the researched system, and thus he studies the phase portrait of system, i.e. carries out its research. It is inconvenient for three-dimensional systems, let alone dynamic systems of the greater dimension. But the research of n-dimensional systems ($n \geq 3$) can be algorithmically formalized by use of ideas and methods of recognition.

The recognition methods allow to overcome the difficulties connected with dimension of space at the analysis of the stable movements, at the description of attractors, at construction of the attraction regions, at the allocation and the description of sets of the parameters values corresponding to various kinds of the stable movements. All recognition problems are solved on the basis of the training data set. The methods of statistical modeling are used for a formation of the representative sample and for an estimation of results of the decision-making.

Methods of the pattern recognition in a combination with statistical modeling enable to carry out not simply the research but to automate a significant part of these researches and to construct quickly enough the coarsened phase and parametric portraits. To the present time the methods of the coarsened phase portrait construction for the concrete dynamic systems at preset values of parameters are algorithmically realized and tested, and the methods of the coarsened parametric portrait construction are in a stage of development.

3 Statement of the research problem of phase portrait structure for the dynamic system as the pattern recognition problem with active experiment

Numerical research of the concrete dynamic system at preset values of parameters consists in construction of a computer phase portrait and is reduced to solving of three basic problems:

I) the determination of the kind and number of the stable limit subsets in the phase space (attractors) for the researched dynamic system;

II) the description and discrimination of the attractors in the phase space;

III) the allocation of the attraction regions for each of attractors.

All process of the computer phase portrait construction by methods of pattern recognition can be presented as the consecutive solving of the following problems of the data analysis and recognition, in which the elements of unknown coarsened computer phase portrait (phase trajectories, attractors, their attraction regions) are the patterns to be recognized:

1) the analysis of the separate phase trajectories, set by coordinates values of a trajectory in phase space for consecutive time moments equidistant from each other, with the purpose of feature extraction, that are informative for solving recognition problems;

2) the recognition of a phase trajectory type on the basis of features formed at solving of problem (1);

3) the determination of a kind and number of attrac-

tors (the quantity of recognized patterns) as a solution of classification problem (the problem of recognition without the teacher) on the basis of the data received at solving of problems (1) and (2);

4) the description and separation of the attractors in the phase space by solving of the discrimination problem for pattern constructed in a problem (3) (a classical problem of recognition with a teacher);

5) the construction of a rule for decision-making about belonging any trajectory to certain attractor;

6) the allocation of the attraction regions for each of attractors as a solution of the problem of recognition with the teacher on the basis of a sample formed according to results received at solving of problems (3), (4) and (5).

The part of these problems (3, 4, 6) is directed on the immediate decision of the problems connected with research of the phase space structure for the dynamic system, and others (1, 2, 5) have applied meaning, they are necessary for formation of training sample and for automation of the research process.

All problems are solved on the basis of the data set consisting of the phase trajectories segments. Primary sample is created by a casual uniform choice of initial points in the given limited region of phase space. During the research, that is carried out in adaptive mode, training sample is extended and corrected depending on solved current problems, i.e. all recognition problems are the problems of recognition with active experiment. Planning of experiment means not only a choice of an initial point for construction of a phase trajectory but also the choice of its duration, of a step of digitization and of an account precision. All control samples are formed by a casual choice.

The pattern recognition problems are solved by methods based on using of two supplementing each other approaches to the data analysis, namely: the logic approach taking into account the concrete features of analyzed sets and the statistical approach basing on average estimation of the training set characteristics. The logic approach is realized with the help of the optimal irreducible fuzzy tests [Kotel'nikov, 2001], and the statistical analysis is carried out with use of the recurrent least squares method [Neimark and Teklina, 2003]. These methods possess wide adaptive potentialities and are convenient for solving of problems with active experiment when the data sample changes and can achieve very large sizes.

Let's tell in a few words about peculiarity of some from the enumerated recognition problems. The features describing behavior of trajectories at their approach to attractor and the attributes characterizing stability or instability of the phase trajectories are chosen as features informative for solving recognition problem on determination of the phase trajectory type. The kind of the stable movement is defined by type of trajectories approaching to it, and the number of the same kind attractors is determined by their description in the phase space of a dynamic system. The algorithms based on a covering of points from training sample, belonging to the trajectories approaching to corresponding attractor, by either parallelepipeds (the syndrom decision rules [Kotel'nikov, 2001]) or spheres or ellipsoids (the orthogonal components method on the base of the universal recurrent form of the least squares method [Neimark and Teklina, 2003]) are used for search of the attraction regions. The syndrom decision rules, in which the attraction region is described by inequalities set of a kind $a_i \leq x_i \leq b_i$ for all variables of the researched system, are the most convenient and easy in interpretation for the researcher.

The possibilities of the new approach to research of the dynamic systems we shall illustrate on the example of mathematical model for the immune response of an organism to a infection invasion. [Neimark, 2003]

4 Presentation of the mathematical model

Immune systems of a living organisms are very complex, but despite all complexity of the immune response of an organism to an infectious attack there arise the following three fundamental factors: an infection; its reproduction and its infecting of the organism; the counteraction of an organism and its potential counteracting capabilities dependent upon its state. Quantitatively these factors may be represented by three magnitudes: a quantity of the infection x, an extent of the organism counteraction y and a potential of the organism z. Besides its quantity x, any infection is characterized by the rate of its reproduction within the organism environment and by its suppressive action upon the organism. As for the counteraction it is characterized by its extent y and its counteracting efficiency, fastness and the replenishing rate w, which depend upon the organism and its potential z. The dynamics of the immune response (the variation of the magnitudes x, y, z and w) is described by the following four differential equations:

$$\dot{x} = \lambda x - axy/(1+\alpha x) - \epsilon x^2$$

$$\dot{y} = \begin{cases} -bxy/(1+\alpha x) + w = K & if \quad y > 0 \\ & or\ y = 0 \& K \geq 0 \\ 0 & if\ y = 0 \& K < 0 \end{cases}$$

$$\dot{z} = \begin{cases} c(z_0 - z)/(1+\gamma x) - & \\ -dy - e = F & if \quad z \geq 0 \\ & or\ y = 0 \& F \geq 0 \\ 0 & if\ z = 0 \& F < 0 \end{cases}$$

$$\tau \dot{w} + w = \begin{cases} 0 & if\ x \leq x_0 \\ Bz(2z_0 - z)(x + \beta x^2) & if\ x > x_0 \end{cases}$$

Parameters of system are the effectiveness factors of the described processes and the thresholds used in them. In these equations x_0 is the threshold of the organism sensitivity against the infection, τ is the time delay of the supplementary immune response, z_0 is the limit value of the organism potential, $z_0 \geq z \geq 0$. Converting z to zero is understood as the complete organism exhaustion bringing about its destruction.

The model contains 4 variables and 14 parameters and is rather complicated for studying.

5 The brief statement of the basic results of model studying

Research of the dynamic system containing parameters consists both in the phase portraits construction, answering to concrete values of parameters included in system, and in the bifurcations portraits describing the variations of the phase space structure depending on parameters. Last problem is rather difficult for a model with 14 parameters and for the present it is not completely solved by us. Research of system was based on an opportunity of the prompt construction of a phase portrait with use of the new approach and has allowed to receive not full but capacious enough results.

For the various values sets of parameters the coarsened phase portraits have been received. These portraits include the stable equilibriums, the variety of stable equilibriums, the periodic movements and the specific attractor adequate to a lethal outcome and also their attraction regions. These data allow to reveal the basic types of the diseases outcome, the variants of their course and also the influence of various factors and parameters. The possible outcomes of disease (the recovery, the bacilli-carrying, the relapsing disease, the chronic disease, the lethal outcome) are determined on the attractor type. The disease course (the recovery by way of gradual destruction of an infection, the recovery through an exacerbation, the change to bacilli-carrying, the death from an emaciation or from an intoxication, etc.) is defined by behavior of the phase trajectories at their approach to an attractor and by location of the attractors in the phase space. The dependence of the disease outcome on number of an infection and an initial state of an organism is determined by the attraction regions of attractors. The modification of a phase portrait with a variation of parameters testifies to dependence of the course and outcome of disease on character of an infection and on a common state of an organism, its possibilities of counteraction to an infection.

More than 10 phase portraits answering to various forms of the disease course have been stood out during research. For an example we shall consider four of them containing the basic types of attractors to have been found.

A) There is revealed the set of parameters for which the corresponding coarsened phase portrait has in the phase space only one variety of the equilibriums of a kind $(0, y^*, z^*, 0)$. To such attractor the full recovery corresponds at any initial conditions. And the recovery is realized either by way of gradual destruction of an infection or through an exacerbation of disease.

B) There are the states of system at which for various initial conditions two positive outcomes of disease are possible, namely: the full recovery with destruction of an infection and the recovery with preservation of an infection quantity at a level of the sensitivity threshold of an organism to an infection x_0 (the bacilli-carrying). Threat of a lethtal outcome arises only at very large value $x >> x_0$ and the smallest initial values of y, z and w. In the region of the phase space limited on x there are two stable attractors: the variety of stable equilibriums of type $(0, y^*, z^*, 0)$, adequate to recovery with destruction of an infection, and stable knot in a point $(x_c = x_0, y_c > 0, z_c > 0, w_c > 0)$, corresponding to the bacilli-carrying. The variety of stable equilibriums is represented by a set of points on plane yz to be approximated by a straight line and $y^* \geq y_0^* > 0$ at all initial conditions. The process of full recovery for a researched condition of system is carried out always by gradual destruction of an infection, but the transition to the bacilli-carrying occurs both through the gradual reduction of an infection and through an exacerbation of disease when at some stage the increase in number of x is observed.

C) At some sets of parameters four outcomes of disease are possible, namely:

- recovery without the complications, represented by a variety of the stable equilibriums on the plane yz;

- relapsing disease with the corresponding stable limit cycle in the fase portrait, in which x varies in some neighborhood of a threshold value x_0 (from $x_1 < x_0$ to $x_2 > x_0$);

- lethal outcome as results of an intoxication of an organism (a stable knot in a point $(x^* >> x_0, 0, 0, 0)$);

- death from an emaciation of an organism (a variety of the stable equilibriums on axis y).

D) Among the coarsened phase portraits that have been researched a portrait with three attractors exists. They are:

- the variety of stable equilibriums on the plane yz, adequate to recovery without the complications;

- the stable equilibrium in the point with coordinates $(x_c > 0, y_c > 0, z_c > 0, w_c > 0)$, corresponding to chronic disease;

- the stable equilibrium with very major value x and very small values y and z corresponds to a state of patient with major threat for life.

At full research of a part of phase portraits the attraction regions for all attractors have been found with reliability of result $p \geq 0.99$. For an illustration on fig. 1 the attraction regions for three attractors in a phase portrait (d), described for simplicity of the image only by first two syndromes, are submitted in projections on the coordinate planes.

The attraction region for the variety of the stable equilibriums (the recovery) is represented by inclined lines, the attraction regions for two stable knots answering to a chronic disease and to a state with threat of a lethal outcome are represented by vertical and horizontal lines correspondingly. Though the extracted regions of an attraction are not crossed in phase space the projections of these regions on coordinate planes

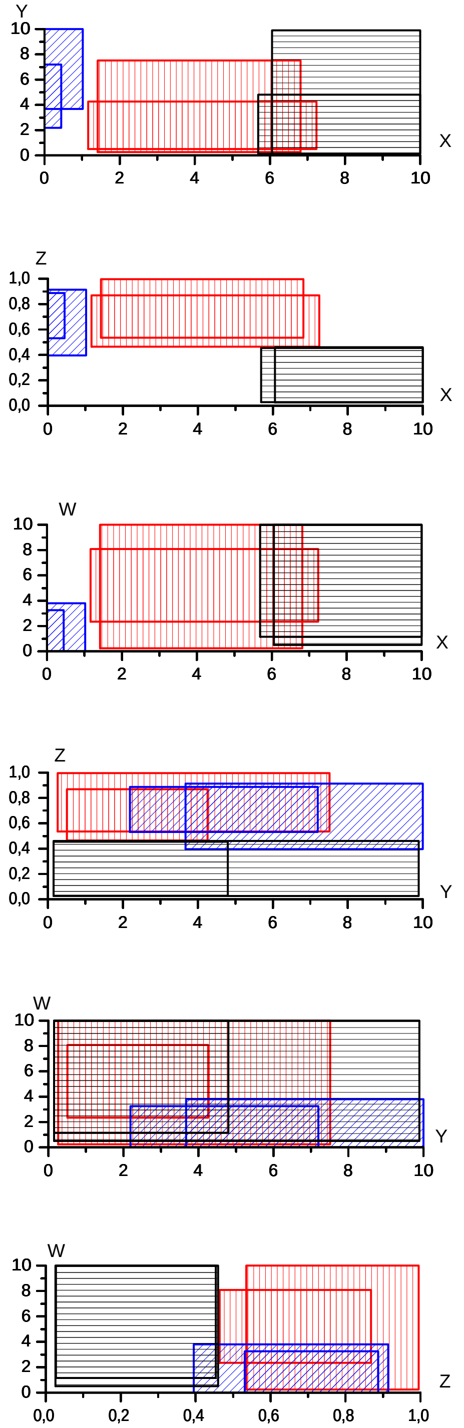

Figure 1. The attraction regions for three attractors in a phase portrait (d) in projections on the coordinate planes xy, xz, xw, yx, yw, zw.

are crossed (and very significantly on some variables), what complicates essentially the analysis of a phase portrait structure by means of habitual observation over behavior of phase trajectories on the computer screen.

6 Conclusion

The offered work has for an object to show that the problem of research of the concrete dynamic systems can be considered naturally and fruitfully as a problem of pattern recognition and that the essential advancement in its solution is possible on this way. First of all, it concerns overcoming the difficulties caused by the large dimension of the parameters space. Till now in fact this problem has no formalized methods of the decision. From our point of view, the successful decision of this problem is possible under two conditions:

- the automated analysis of a phase portrait structure for the concrete dynamic systems should become accessible;
- the further development of specific methods of recognition is necessary.

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