OPTIMAL CONTROL OF STATE-DEPENDENT IMPULSE SYSTEMS

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Abstract
We study an optimal control problem for a measure-driven dynamic system, where jumps of a state trajectory may occur only at the moments of hitting a given set. The model can be described by using the complementarity formalism. The system is not assumed to satisfy correctness conditions. A time reparameterization technique is developed to reduce the optimization problem to the one with bounded controls. Necessary conditions for optimality are obtained by interpreting the Maximum Principle in the reduced problem.

Key words
Optimal control, impulsive hybrid systems, complementary systems, necessary conditions of optimality.

1 Introduction

A number of hybrid models can be naturally formalized by means of systems with impulsive impacts [Branicky et al., 1998; Matveev and Savkin, 2000; Kurzhanski and Tochilin, 2009; Miller and Rubinovich, 2003]. In the literature (see, e.g., [Branicky et al., 1998; Haddad et al., 2002; Sanfelice et al., 2006]), one can meet hybrid systems of the following type:

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{1}$$

$$[x(t)] = \Psi(t, x(t-), \nu(t)), \text{ if } x(t-) \in \mathcal{Z}. \tag{2}$$

Here $[x(t)] = x(t) - x(t-)$ stands for a jump of a function x at the point t, $\mathcal{Z}$ is a *resetting set*, and u and ν are controls. As opposed to systems with purely exogenous impulsive controls, in model (1), (2), impulses can occur only at the moments, when a state trajectory hits the resetting set $\mathcal{Z}$. In other words, impulses in this system are state-dependent. This fact reveals a distinctive hybrid feature of the model. For instance, an important subclass of (1), (2) is constituted by switched systems with controlled switchings, or logic-dynamic control systems ([Branicky et al., 1998; Boccadoro et al., 2005])

$$\dot{x} = f_q(t, x) := f(t, x(t), q(t)),$$
$$\dot{q} = 0, \quad q(\tau) = \Psi(\tau, x(\tau-), q(\tau-), \nu_\tau).$$

Here $q(t) \in \mathcal{Q}$ is a “logic” variable distinguishing the mode, in which the state $x(t)$ of the system evolves, $\mathcal{Q}$ is a £nite subset of $\mathbb{N}$, the function Ψ represents a £nite automaton as follows: $\Psi(\tau, x(\tau-), i, \nu_\tau) = j(\tau, x(\tau-), \nu_\tau) \in \mathcal{Q}$, where $j(\tau, x(\tau-), \nu_\tau) = j_k(\nu_\tau)$ if $(\tau, x(\tau-)) \in \mathcal{Z}_k$.

Model (1), (2) can be treated as the discrete-continuous system with intermediate state constraints

$$\dot{x} = f(t, x, u) + \sum_{\tau_i \le t} \Psi(\tau_i, x(\tau_i-), \nu(\tau_i))\delta(t-\tau_i), \tag{3}$$

$$x(\tau_i) \in \mathcal{Z} \ \ \forall\, i. \tag{4}$$

An optimal control problem for systems of this kind was investigated in [Dykhta and Samsonyuk, 2000], when the points τ_i in (4) and the instants of impulses in (3) are independent of each other. We address the case, when intermediate constraints (4) are to be satis£ed precisely at the instants of impulses. Therefore, an additional mixed constraint appears.

On the other hand, one can see a hidden complementary nature of system (1), (2). Indeed, put $\nu_i = \nu(\tau_i)$ and de£ne the discrete vector measure

$$d\mu(t) = \sum_{\tau_i \le t} \nu_i \delta(t-\tau_i) dt.$$

We can treat (4) as a constraint of the form

$$\tau_i = \tau_i(x) \in \{t : x(t-) \in \mathcal{Z}\},$$

which is supposed to hold for all τ_i in the summation above. Assume that $\Psi(t, x, 0) = 0$ for all t and

x, that is no switching may happen without applying a nonzero control ν. Such an assumption conforms to the practical sense of many hybrid models. Then, the de£ned discrete measure $d\mu$ is localized on the set $A = \{t : x(t-) \in \mathcal{Z}\}$. In other words, $d\mu(E) = 0$ for any Borel subset E of the complement of A.

Now we consider a more general model

$$dx = f(t,x,u)dt + G(t,x)d\mu, \qquad (5)$$

$$d\mu \text{ is localized on } \{t : x(t-) \in \mathcal{Z}\}, \qquad (6)$$

by admitting that a control measure possesses also a continuous component in its Lebesgue decomposition. In its turn, system (5), (6) can be classi£ed as a complementary one [Van der Schaft and Schumacher, 2000], where constraint (6) is interpreted as the *complementary slackness condition*. Indeed, it is well known that given a closed set $\mathcal{Z}$ in a £nite-dimensional space, one can constructively determine a continuous (and, even, in£nitely smooth) function Q, characterizing $\mathcal{Z}$ in the following sense: $Q(x) \geq 0$ for all x, and $Q(x) = 0$ iff $x \in \mathcal{Z}$. Introduce the new variable

$$\widetilde{x}(t) = Q(x(t-)) \geq 0.$$

By using the complementary formalism (see, e.g., [Van der Schaft and Schumacher, 2000]), system (5), (6) can be rewritten as follows

$$dx = f(t,x,u)dt + G(t,x)d\mu,\ dV = |d\mu|, \qquad (7)$$

$$\widetilde{x}(t) = Q(x(t-)), \qquad (8)$$

$$0 \leq \widetilde{x} \perp dV \geq 0. \qquad (9)$$

Here $|d\mu|$ stands for the measure induced by the total variation of the function μ. Inequalities in (9) hold trivially, $dV \geq 0$ means that the measure is nonnegative for any Borel subset of the time interval. The notation $\widetilde{x} \perp dV$, which is typical in the complementary formalism, means here that $\widetilde{x}$ vanishes almost everywhere with respect to the measure dV.

There is a subtle but essential difference between model (7)–(9) and mechanical systems with unilateral constraints [Brogliato, 2000; Miller and Bentsman, 2006]. The latter systems also form a certain class of complementary hybrid systems and can be described by measure differential equations. In such models, the complementary forces (e.g., elastic or friction ones) act, when a body comes into contact with an "obstacle", and are aimed at preventing the violation of the unilateral constraints. Measures describing actions of the complementary forces are not regarded as actual controls, they are rather required for physically adequate de£ning the system dynamics. The space-time reparameterization proposed in [Miller and Rubinovich, 2003] takes into account that "fast motions" occur only in a forbidden domain. However, such technique is not applicable in many switched systems, say, in population models with impulsive effects. In our model, measures play the role of control inputs, and the complementarity reveals itself in the form of a speci£c constraint on the control measure.

2 Problem Formulation

Consider a problem (P) of minimization of a functional $I = F(x(T), V(T))$ under the following constraints:

$$dx = f(t,x,V,u)\,dt + G(t,x,V)\,d\mu, \qquad (10)$$

$$dV = |d\mu|, \qquad (11)$$

$$x(0-) = x^0,\ V(0-) = 0, \qquad (12)$$

$$V(T) \leq M, \qquad (13)$$

$$u(t) \in U, t \in [0,T],\ d\mu(E) \in W\ \forall E \in \mathcal{B}_{[0,T]}, \qquad (14)$$

$$(t, x(t-), V(t-)) \in \mathcal{Z}_- \ |d\mu|\text{-a.e. on } [0,T]. \qquad (15)$$

Here, $x(\cdot) \in BV([0,T], \mathbb{R}^n)$, $V(\cdot) \in BV([0,T], \mathbb{R})$ are right continuous functions of bounded variation, controls $u(\cdot)$ are Borel measurable bounded functions, the set $U \subset \mathbb{R}^k$ is compact, the inclusion $u(t) \in U$ holds $\mathcal{L}$-a.e. (almost everywhere with respect to the Lebesgue measure). Controls $d\mu$ are regular measures induced by functions $\mu(\cdot) \in BV([0,T], \mathbb{R}^m)$, $\mu(0-) = 0$. The set W is a closed convex cone in the nonnegative orthant $\mathbb{R}^m_+$, $\mathcal{B}_{[0,T]}$ stands for the σ-algebra of Borel subsets of the interval $[0,T]$, $M > 0$. The set $\mathcal{Z}_- \subset [0,T] \times \mathbb{R}^n \times \mathbb{R}$ is supposed to be closed, and called the *resetting set*. The functions f and G are Lipschitz continuous in all variables and satisfy the linear growth conditions with respect to x and V, F is continuous.

We do not impose Frobenius type correctness conditions. As is well known, this implies that the reaction of the system to impulses is not unique and strictly depends on a particular kind of approximation of a generalized impulsive control by conventional ones. This results in arising an integral funnel of the measure-driven system under a given control.

By (P_Q) we denote problem (P), where (15) is replaced with the following equivalent mixed constraint:

$$Q(\tau, x(\tau-), V(\tau-)) = 0\ |d\mu|\text{-a.e. on } [0,T]. \qquad (16)$$

Here, a scalar nonnegative continuous function Q is assumed to vanish only on the resetting set $\mathcal{Z}_-$.

Given controls u, and $d\mu$, we de£ne a *trajectory* of system (10)–(12) as an individual curve of the integral funnel of (10)–(12). In other words, by a trajectory of (10)–(12) we mean a couple of functions (x, V), satisfying everywhere on $[0,T]$ the conditions

$$x(t) = x^0 + \int_0^t f(\theta,x,V,u)d\theta + \\ + \int_0^t G(\theta,x,V)d\mu_c(\theta) + \sum[x(\tau)], \quad (17)$$

$$V(t) = \int_0^t |d\mu_c(\theta)| + \sum[V(\tau)]. \quad (18)$$

The sums in (17), (18) are taken over all $\tau \in D_\mu$ such that $\tau \le t$. By D_μ we denote the set $\{\tau \in [0,T] : [\mu(\tau)] \ne 0\}$ of all points of jumps of the function μ. The notation $d\mu_c$ stands for the continuous component of the measure $d\mu$, meaning the sum of its absolutely continuous and singular continuous parts. Jumps of the functions x and V at the points τ are de£ned by $[x(\tau)] = \varkappa_\tau(\nu_\tau) - x(\tau-)$ and $[V(\tau)] = r_\tau(\nu_\tau) - V(\tau-)$. Here $\nu_\tau = |[\mu(\tau)]|$, while $\varkappa_\tau$ and r_τ satisfy the system

$$\dot\varkappa(\theta) = G(\tau,\varkappa(\theta),r(\theta))e_\tau(\theta),\ \dot r(\theta) = 1, \\ \varkappa(0) = x(\tau-),\ r(0) = V(\tau-), \quad (19)$$

under a control $e_\tau(\cdot)$, which meets the constraints

$$e_\tau(\theta) \in W \cap B,\ \theta \in [0,\nu_\tau], \\ \int_0^{\nu_\tau} e_\tau(\theta)d\theta = [\mu(\tau)]. \quad (20)$$

We assume that functions e_τ are Borel measurable on $[0,\nu_\tau]$, B is the m-dimensional unit ball (centered at zero) with respect to the norm $|\cdot|$, $|e| = \sum_{i=1}^m |e_i|$.

We call system (19), (20) the *limit* one. Denote $\kappa_\tau = (\varkappa_\tau, r_\tau)$. A tuple

$$\sigma = \left(x(\cdot), V(\cdot), u(\cdot), d\mu, \{\kappa_\tau(\cdot), e_\tau(\cdot)\}_{\tau\in D_\mu}\right),$$

satisfying the conditions (13), (14), and (16)–(20), is called an *admissible control process* of system (10)–(15). The set $\Sigma(P_Q)$ of all admissible processes of problem (P_Q) is nonempty, at least processes corresponding to the null measure $d\mu$ are admissible.

3 Problem Transformation

The time reparameterization method is a well-known and powerful tool in impulsive systems theory. In optimal impulsive control such a technique is effectively applied to derive necessary conditions of optimality (see, e.g., [Vinter and Pereira, 1988; Bressan and Rampazzo, 1994; Silva and Vinter, 1997; Zavalischin and Sesekin, 1997; Pereira and Silva, 2000; Miller and Rubinovich, 2003]). In optimization of discrete–continuous systems and measure driven equations it is used to design computational algorithms for optimal control [Goncharova and Staritsyn, 2010]. For optimal control problems of hybrid systems with unilateral constraints the technique was suggested as a development of the penalization method.

A classical approach to treat state discontinuities consists in regarding them as results of motions in a "fast time scale". The main idea of the time change technique is to make such fast motions comparable in duration with motions in the natural time scale by means of an appropriate extension of the instants of impulses into intervals. In the present section we use this idea to reduce problem (P_Q) to the one with conventional bounded controls. For problem (P_Q), the reduction [Miller and Rubinovich, 2003] is inadequate, since we are to meet constraint (16). To keep the information on hitting the resetting set over the intervals of fast motions, we extend the state space.

Assume that the functions f and G satisfy the Lipschitz and linear growth conditions also in t. On an un£xed time interval $[0,S]$, $T \le S \le T+2M$, we consider the following optimal control problem (RP): We are to minimize the functional $J = F(y_1(S), \eta_1(S))$ under the constraints

$$\dot\xi = \alpha,\ \ \dot\eta_i = (1-\alpha)\beta_i|e|,\ \ \dot\zeta_i = (1-\alpha)\beta_i\, e, \quad (21)$$

$$\dot y_i = \alpha f(\xi,y_1,\eta_1,v) + (1-\alpha)\beta_i\, G(\xi,y_i,\eta_i)\, e, \quad (22)$$

$$\xi(0) = \eta_i(0) = 0,\ \ \zeta_i(0) = 0,\ \ y_i(0) = x^0, \quad (23)$$

$$\xi(S) = T, \quad \eta_2 - \eta_1 \le 0, \quad (24)$$

$$v \in U,\ e \in W\cap B,\ \alpha,\beta_i \in [0,1],\ \beta_1+\beta_2 = 1, \quad (25)$$

$$J_1 = \int_0^S \Phi(\xi,y,\eta,\zeta,\alpha,\beta,e)\, ds = 0. \quad (26)$$

Here, trajectories ξ, y, η, and ζ are absolutely continuous, controls v, α, β, and e are Borel measurable bounded functions, $y = (y_1, y_2)$ and similar notations are used for η, ζ and β. The sets U and W are the same as in (14). The function Φ has the form

$$\Phi = \alpha\left(\rho(\Delta y) + \varphi(\Delta\zeta)\right) + (1-\alpha)\beta_1|e|Q(\xi,y_2,\eta_2),$$

where $\rho(x)$ and $\varphi(\mu)$ are scalar nonnegative continuous functions vanishing only at zero, $\Delta y = y_1 - y_2$ and $\Delta\zeta$ is de£ned similarly.

By $\Sigma(RP)$ we denote the set of all control processes $\bar\sigma = (\gamma,\omega;S)$, which are admissible in problem (RP), i.e., satisfy constraints (21)–(26). Here $\gamma = (\xi,y,\eta,\zeta)$ and $\omega = (v,\alpha,\beta,e)$.

Theorem 1. *Given $\sigma \in \Sigma(P_Q)$, there exists a process $\bar\sigma \in \Sigma(RP)$ such that, for all $t \in [0,T]$, the equalities*

$$x(t) = y_i(\Gamma(t)),\ V(t) = \eta_i(\Gamma(t)),\ \ i = 1,2, \quad (27)$$

hold, where $\Gamma(t) = t + 2V(t)$, and $S = \Gamma(T)$.

The proof is based on a change of variable under the sign of Lebesgue–Stieltjes integral and invokes the properties of the time reparameterization ([Miller and Rubinovich, 2003]). The function Γ plays the role of the "inverse time transformation". It is right continuous and strictly monotone increasing on $[0,T]$. De£ne a function ξ by the relation

$$\xi(s) = \inf\{t : \Gamma(t) > s\},\ s \in [0,S].$$

Then, $\xi(\Gamma(t)) = t$ for all $t \in [0,T]$, and $\Gamma(\xi(s)) = s$, if $t = \xi(s)$ is the point of continuity of Γ.

The desired admissible control ω in the reduced problem can be constructively de£ned by the following direct space-time transformation. Consider the union $\Omega = \bigcup_{\tau\in D_\Gamma} \Omega_\tau$ of the intervals $\Omega_\tau = [\Gamma(\tau-), \Gamma(\tau)]$, where $D_\Gamma = \{\tau \in [0,T] : [\Gamma(\tau)] > 0\}$, and introduce the notations

$$\Omega^0_\tau = \Gamma(\tau-) + [0, [V(\tau)]\,),$$
$$\Omega^1_\tau = \Gamma(\tau-) + [\,[V(\tau)], [\Gamma(\tau)]\,],$$
$$\Omega^j = \textstyle\bigcup_{\tau\in D_\Gamma} \Omega^j_\tau,\ j = 0,1.$$

Controls v, β and e can be determined as follows:

$$v(s)\begin{cases} \in U, & s \in \Omega,\\ = u(\xi(s)), & s \in [0,S]\setminus\Omega,\end{cases}$$

$$\beta_1(s) = \begin{cases} 1, & s \in \Omega^0,\\ 0, & s \in \Omega^1,\\ 1/2, & s \in [0,S]\setminus\Omega,\end{cases} \qquad \beta_2 = 1 - \beta_1,$$

$$e(s) = \begin{cases} e_\tau(\theta^0_\tau(s)), & s \in \Omega^0_\tau,\\ e_\tau(\theta^1_\tau(s)), & s \in \Omega^1_\tau,\ \tau \in D_\Gamma,\\ l(\xi(s)), & s \in [0,S]\setminus\Omega,\end{cases}$$

where the functions $\theta^0_\tau(s) = s - \Gamma(\tau-)$ and $\theta^1_\tau(s) = \theta^0_\tau(s) - [V(\tau)]$ are de£ned on the corresponding sets, and $l(t) = \dfrac{d\mu(t)}{dV(t)}$, $t \in [0,T]$, is the Radon–Nikodym derivative of the measure $d\mu$ with respect to the measure dV. For the sought control α one can take the Borel measurable function $\dot{\xi}$.

The following inverse statement is also valid:

Theorem 2. *Given $\bar\sigma \in \Sigma(RP)$, there exists a process $\sigma \in \Sigma(P_Q)$, such that (27) hold, where Γ is de£ned by the relations $\Gamma(t) = \inf\{s \in [0,S] : \xi(s) > t\}$, $t \in [0,T)$, and $\Gamma(T) = S$.*

The desired controls in problem (P_Q) can be de£ned as follows

$$u(t) = v(\Gamma(t)),\ \mu(t) = \zeta_1(\Gamma(t)),\ t \in [0,T],$$
$$e_\tau(\theta) = e(s_\tau(\theta))|e(s_\tau(\theta))|^{\oplus},\ \text{where}$$
$$s_\tau(\theta) = \theta^{-1}_\tau(s) := \inf\{s \in \Omega_\tau : \theta_\tau(s) > \theta\},$$
$$\theta_\tau(s) = \int_{\Gamma(\tau-)}^{s} \beta_1(\vartheta)|e(\vartheta)|d\vartheta.$$

Here we use the pseudoinversion symbol $\oplus$: $a^{\oplus} = a^{-1}$, if $a \neq 0$, and $a^{\oplus} = 0$, if $a = 0$.

We also can state the following result.

Theorem 3. *In problem (P_Q) there exists an optimal process σ^*, iff there is an optimal one $\bar\sigma^*$ in problem (RP), moreover,*

$$I(\sigma^*) = \min_{\Sigma(P_Q)} I = \min_{\Sigma(RP)} J = J(\bar\sigma^*).$$

If σ is optimal in (P_Q), then the process $\bar\sigma$ obtained by the direct space-time transformation is optimal in reduced problem (RP). If $\bar\sigma$ is optimal in (RP), then the process σ obtained by the inverse transformation is optimal in problem (P_Q).

Now, armed with the results of Theorems 1 and 3, we can formulate necessary conditions of optimality in problem (P_Q). The conditions are obtained by interpreting the Maximum Principle [Ioffe and Tikhomirov, 1979] in problem (RP).

4 Necessary Conditions of Optimality

Note that in reduced problem (RP) there are terminal, phase and functional constraints (24), and (26). Moreover, the problem is considered on an un£xed time interval $[0,S]$, $T \leq S \leq T + 2M$. Denote by $(\widetilde{RP})$ an optimal control problem which is obtained from (RP) by replacing the constraint $S \leq T + 2M$ with $\eta_2(S) \leq M$. Notice that the conditions $\xi(S) = T$ and $\eta_2(S) \leq M$ imply $S \leq T + 2M$. Given an optimal control process $\sigma^* \in \Sigma(P)$, the corresponding (determined by the direct space-time transformation) tuple $\bar\sigma^* \in \Sigma(RP)$ will be optimal in $(\widetilde{RP})$.

In what follows we assume that the functions f, G, Q, and F are continuously differentiable in all variables. For problem (P_Q) introduce the Pontryagin functions

$$H^1(t, X, \psi^t, \psi^x, u) = \langle \psi^x, f(t, X, u)\rangle + \psi^t,$$
$$H^0(t, X, \psi^X, l) = \langle \psi^x, G(t, X)l\rangle + \psi^V,$$

and the Hamiltonians

$$\mathcal{H}^1 = \max_{u\in U} H^1,\ \mathcal{H}^0 = \max_{l\in W\cap\partial B} H^0.$$

Here $\psi = (\psi^t, \psi^x, \psi^V)$ is the dual vector corresponding to (t, x, V), $X = (x, V)$, and $\psi^X = (\psi^x, \psi^V)$.

For simplicity we can also assume that function Q is such that $\nabla Q = 0$ on $\mathcal{Z}_-$, and take $\rho(x) = \|x\|^2$ and $\varphi(\mu) = \|\mu\|^2$, where $\|\cdot\|$ stands for the Euclidean norm. Necessary conditions of optimality in problem (P_Q) take the following form:

Theorem 4. *Given an optimal control process $\sigma^* \in \Sigma(P_Q)$, there exists a tuple $(\Lambda, \{c_\tau\}_{\tau\in D^*_V}, dw, \phi)$ of Lagrange multipliers, with $\Lambda = (\Lambda_0, \ldots, \Lambda_3)$,*

$\Lambda_j, c_\tau \in \mathbb{R}$, $\sum_{\tau\in D_V^*} c_\tau < \infty$, *$dw$ is a regular scalar measure induced by a non-decreasing function $w(\cdot)$, $w(0-) = 0$, and $\phi(\cdot) \in BV([0,T], \mathbb{R}^{2n+3})$, such that the following set of conditions holds:*

$(\mathbf{C}_1)$ *Nonnegativity and nontriviality:*

$$\Lambda_0, \Lambda_2 \geq 0,\ c_\tau \geq 0\ \forall \tau \in D_V^*,$$
$$|\Lambda| + w(T) + \sum_{\tau\in D_V^*} c_\tau > 0.$$

$(\mathbf{C}_2)$ *Complementary slackness condition associated with the constraint on the total impulse of control:*

$$\Lambda_2(V^*(T) - M) = 0.$$

$(\mathbf{C}_3)$ *The adjoint system and transversality conditions: The vector function $\phi = (\phi^t, \phi_1^X, \phi_2^X)$, with $\phi_i^X = (\phi_i^x, \phi_i^V)$, $i = 1,2$, satis£es the following system of measure differential equations:*

$$\begin{aligned} d\phi^t &= -H_t^1 dt - \bar{H}_t^0 dV^*, \\ d\phi_1^X &= -H_X^1 dt - H_X^0 dV^* - dw^X, \\ d\phi_2^X &= -\widetilde{H}_X^0 dV^* + dw^X, \end{aligned} \qquad (28)$$

with the initial conditions

$$\phi(T) = -(\Lambda_1, \Lambda_0 F_x, \Lambda_0 F_V, 0, \Lambda_2). \qquad (29)$$

Here we use the notations $\widetilde{H}^0 = H^0 - \Lambda_3 Q$, and

$$\bar{H}^0(t,X,\phi,l) = H^0(t,X,\phi_2^X,l) + \widetilde{H}^0(t,X,\phi_1^X,l).$$

In (28) the partial derivatives of H^1, H^0, and $\widetilde{H}^0$ are computed at the points $(t, X^*, \phi^t, \phi_1^x+\phi_2^x, u^*)$, (t, X^*, ϕ_1^X, l^*), and (t, X^*, ϕ_2^X, l^*), respectively, the derivative $\bar{H}_t^0$ is taken along (X^*, l^*), with $l^* = d\mu^*/dV^*$, and $dw^X = (dw^x, dw^V)$ denotes a vector measure, with the null measure dw^x and $dw^V = dw$. In (29) the derivatives of F are computed at $(x^*(T), V^*(T))$.
Jumps $[\phi(\tau)]$ of ϕ at $\tau \in D_V^*$ are de£ned as follows

$$[\phi^t(\tau)] = \phi^t(\tau) - p_\tau^t(0),$$
$$[\phi_i^x(\tau)] = \phi_i^x(\tau) - p_{i\tau}^x(0),$$
$$[\phi_i^V(\tau)] = (-1)^{i+1}c_\tau + \phi_i^V(\tau) - p_{i\tau}^V(0).$$

The functions $p_{i\tau}^t$, and $p_{i\tau}^X$, with $p_{i\tau}^X = (p_{i\tau}^x, p_{i\tau}^V)$, satisfy on $[0,\nu_\tau^*]$, $\nu_\tau^* = [V^*(\tau)]$, the following adjoint limit system:

$$\dot{p}_i^t = -H_t^0(\tau, \kappa_\tau^*, p_i^X, e_\tau^*),$$
$$\dot{p}_i^X = -H_X^0(\tau, \kappa_\tau^*, p_i^X, e_\tau^*),$$

and the conditions $p_1^t(\nu_\tau^*) = p_2^t(0)$, $p_2^t(\nu_\tau^*) = \phi^t(\tau)$, and $p_i^X(\nu_\tau^*) = \phi_i^X(\tau)$.

$(\mathbf{C}_4)$ *Maximum conditions:*

$$\begin{aligned} &H^1(t, X^*, \phi^t, \phi_1^x + \phi_2^x, u^*) = \\ &\quad = \mathcal{H}^1(t, X^*, \phi^t, \phi_1^x + \phi_2^x), \text{ and} \\ &\max\Big\{H^0(t, X^*, \phi_2^X, l^*), \widetilde{H}^0(t, X^*, \phi_1^X, l^*)\Big\} \\ &\quad = \max\Big\{\mathcal{H}^0(t, X^*, \phi_2^X), \widetilde{\mathcal{H}}^0(t, X^*, \phi_1^X)\Big\}, \end{aligned}$$

hold $\mathcal{L}$-a.e. and dV_c^*-a.e. on $[0,T]$. Here $\widetilde{\mathcal{H}}^0 = \max_{l\in\partial B\cap W} \widetilde{H}^0$.

$(\mathbf{C}_5)$ *Conditions of optimality with respect to the support of the continuous part of the control measure:*

$$\begin{aligned} &\mathcal{H}^1 \geq \max\Big\{\mathcal{H}^0, \widetilde{\mathcal{H}}^0\Big\}, \\ &\qquad\qquad \mathcal{L}\text{-a.e. on } [0,T] \setminus \operatorname{supp}\{dV_c^*\}, \\ &\mathcal{H}^1 \leq \max\Big\{\mathcal{H}^0, \widetilde{\mathcal{H}}^0\Big\}, \\ &\qquad\qquad dV_c^*\text{-a.e. on } \operatorname{supp}\{dV_c^*\}. \end{aligned}$$

Here functions $\mathcal{H}^1$, $\mathcal{H}^0$, and $\widetilde{\mathcal{H}}^0$ are computed at the points $(t, X^*, \phi^t, \phi_1^x + \phi_2^x)$, (t, X^*, ϕ_2^X), and (t, X^*, ϕ_1^X), respectively.

$(\mathbf{C}_6)$ *Optimality with respect to processes of the limit system:*
For each $\tau \in D_V^*$ the conditions

$$\begin{aligned} &H^0(\tau, \kappa_\tau^*, p_{i\tau}^X, e_\tau^*) = \mathcal{H}^0(\tau, \kappa_\tau^*, p_{i\tau}^X), \\ &\mathcal{H}^1(\tau, \kappa_\tau^*, p_{1\tau}^t, p_{1\tau}^x + p_{2\tau}^x(0)) \leq \\ &\quad H^0(\tau, \kappa_\tau^*, p_{1\tau}^X, e_\tau^*) + \\ &\quad \Lambda_3\{\rho(\varkappa_\tau^* - x^*(\tau-)) + \varphi(w_\tau^* - \mu^*(\tau-))\}, \\ &\mathcal{H}^1(\tau, X^*(\tau), p_{2\tau}^t, p_{1\tau}^x(\nu_\tau^*) + p_{2\tau}^x) \leq \\ &\quad H^0(\tau, \kappa_\tau^*, p_{2\tau}^X, e_\tau^*) + \\ &\quad \Lambda_3\{\rho(x^*(\tau) - \varkappa_\tau^*) + \varphi(\mu^*(\tau) - w_\tau^*)\} \end{aligned}$$

hold $\mathcal{L}$-a.e. on $[0,\nu_\tau^*]$. Here

$$w_\tau^*(\theta) = \mu^*(\tau-) + \int_0^\theta e_\tau^*(\vartheta)d\vartheta.$$

Theorem 4 is a result of a straightforward interpretation of the conditions of the Maximum Principle in the reduced problem. This implies the appearance of the coupled "adjoint" (as well as the "limit adjoint") trajectories due to the state space extension we applied to formulate (RP). In fact, in problem (P_Q) one can retrieve the proper adjoint trajectory (ψ^t, ψ^x, ψ^V) by means of the relations $\psi^t = \phi^t$, $\psi^x = \frac{1}{2}\{\phi_1^x + \phi_2^x\}$, and $\psi^V = \frac{1}{2}\{\phi_1^V + \phi_2^V\}$. Maximum conditions $(\mathbf{C}_4)$ are quite standard. Conditions $(\mathbf{C}_5)$ appear to set a right priority between the natural mode of the system behavior (no measure type control is applied), and the dynamics governed by the continuous part of a control measure. Conditions $(\mathbf{C}_6)$ are to validate optimality of instants of impulses and a certain kind of approximation of the impulsive control applied.

The conditions of optimality look complicated and hard to apply directly. In some particular cases (say, if G is a constant matrix), the result can be restated in a more familiar form, where dimensions of the phase and adjoint trajectories conform with each other.

5 Conclusion

The purpose of the work is to apply mathematical tools of impulsive control theory in optimization of hybrid models. In the paper we consider a particular class of impulsive hybrid systems described by a measure differential equation, namely, state-dependent impulse systems. To reduce the corresponding optimal control problem we proposed an appropriate space-time transformation, which is used to obtain necessary conditions for optimality. We expect that the approach can be useful for making a qualitative and numeric analysis of the considered problem.

With problem (P) we can associate the problem, where any effective impulsive control steers the system directly to a given closed set $\mathcal{Z}_+$. Then constraint (15) should be replaced with the following one:

$$(t, x(t), V(t)) \in \mathcal{Z}_+ \quad |d\mu|\text{-a.e. on } [0, T]. \qquad (30)$$

As distinct from (15), the condition is formulated in terms of the right limits of a trajectory at discontinuity points. In a certain sense, this problem is a counterpart of (P). In a similar way, we can reduce the problem to problem (RP), where phase constraint (24) is written in the form of an opposite inequality. Notice that Theorems 1–3 remain valid. Thus, the reduction proposed is also applicable to the counter problem. The respective Maximum Principle is easily obtained by employing the developed technique. Such a kind of problems is typical for optimization in mechanical systems with blockable degrees of freedom [Yunt and Glocker, 2006]. We have, in fact, more general results to be published elsewhere. The results are obtained for a problem under both constraints (15) and (30).

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