CYBERNETICS AND PHYSICS, VOL. 2, NO. 3, 2013 , 133–142

THE MOTION PLANNING PROBLEM: DIFFERENTIAL FLATNESS AND NILPOTENT APPROXIMATION

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Abstract
This paper addresses the motion planning problem under two different view points: the flatness procedure and the nilpotent approximation along a reference non-admissible curve. The application of these techniques is carried out for two cases study, namely, an under actuated vibratory mechanical system and the nonholonomic car-like robot with a trailer. A general theoretical framework is presented for each technique and detailed calculations are exhibited for the two cases under consideration along with complementary numerical simulations.

Key words
Motion planning, differential flatness based control, nilpotent approximation.

1 Introduction

In real world situations when safety, obstacle avoidance and collision-free navigation are required, at the same time that kinematics restrictions are complied, it is necessary to develop techniques for finding admissible paths that can be automatically implemented within a reasonable man-machine interaction platform. A common wisdom suggest a general strategy that starts by proposing a reference trajectory that complies with the geographic-topological constraints of the workplace, and then use it as Ariadne thread for an approximating procedure by means of paths that satisfy admissibility with respect to a kinematic model of the system, see for instance [Li and Canny, 1993]

The motion planning problem is inspired in these sort of practical motivations, the problem is very rich in both theoretical and practical aspects: on one hand it has been the source of diverse geometric algorithms some of them real-time implemented, see for instance [LaValle, 2006], on the other hand it has given rise to very deep theoretical issues in geometric control theory [Fliess, Lévine, Martin, Martin and Rouchon, 1992] , differential algebra [Sira-Ramírez and Agrawal, 2004] and sub-Riemannian geometry [Boizot and Gauthier, 2013].

The motion planning problem consists, roughly speaking, in finding a collision-free admissible path for a non-linear control system, that steers the system from an initial position and velocity to a goal position and velocity; in some cases it might be requested that the trajectory minimizes certain cost functional such as time, length, fuel consumption etc. The motion planning problem is usually formulated through a fixed controllable control system, together with an arbitrary non-admissible but feasible (collision-free) trajectory, determined, by computational geometric methods such as Vöronoi diagrams or piano movers like strategies, see for instance [Berg, Kreveld, Overmars and Schwarzkopf, 2000]. The motion planning reduces then to the design of control strategies approximating the reference curve by means of admissible curves within appropriate tubular neighborhoods.

This paper addresses two different techniques for the motion planning problem, namely, differential flatness and nilpotent approximation, for each of these techniques we present a case study. As far as we know, there is no general theory for determining which is the best technique to be chosen in each particular case. It seems that there are systems that are naturally flat whereas some others are better suited for the implementation of nilpotent approximations.

For the differential flatness approach we consider a vibratory system inspired in a robotic mechanism, called the Elasto-Robot, consisting of a prismatic pair coupled with a revolute and containing an oscillating end-effector (Figure 1). We follow the approach of non-linear control of closed loop systems, and more specifically we describe it as a control systems whose trajectories can be parameterized by a finite number of functions and their time-derivatives.

For the case of nilpotent approximation we consider the model of a car with a trailer, which consists of a car-like robot towing a trailer attached to its rear at a distance that allows the free movement of the two parts (Figure 2). Nilpotent approximation is a geometric technique for nonlinear control systems affine in the control parameters and defined by means of distributions of a smooth vector fields on the state manifold. It consists, roughly speaking, in finding a new distribution that models the same kinematics but at the same time spans a nilpotent Lie algebra and in certain sense approximates the original distribution.

The paper is organized as follows in section 2 we present the framework of differential flatness along as the application of the technique to the motion planning problem. In section 3 we present the case study of a robot with an oscillatory end-effector. Following the same argumentation line of these two sections, sections 4 and 5 present the framework of nilpotent approximations and its application to the motion planning of a car with a trailer respectively. In section 6 we derive some conclusions and perspectives of future work, and at the end, for the sake of completeness, we present an appendix with the basic results and definitions of both differential flatness and nilpotent approximations.

2 Differentially Flat Systems

The concept of flat differential systems finds its mathematical foundations in D. Hilbert's 22th problem about the uniformization of analytic relations by means of meromorphic functions [Hilbert, 1902] and the equivalence method for differential systems of E. Cartan [Cartan, 1914]. That is a technique in differential geometry for determining whether two geometrical structures are the same up to a diffeomorphism.

The equivalence method is an essentially algorithmic procedure that has been successfully applied in differential geometry and control theory. More recently flat differential systems have been extensively studied within the non-linear control literature, see for instance M. Fliess et al. [Fliess, Lévine, Martin, Martin and Rouchon, 1992] and P. Rouchon treatment of control of oscillators [Rouchon, 2005].

In this section we present the main definitions concerning flatness, we restrict ourselves to the basic statements leaving aside formal demonstrations, we refer the reader to the book [Sira-Ramírez and Agrawal, 2004].

A differential system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \quad m \leq n$$

is said to be differentially flat if there is a vector $y \in \mathbb{R}^m$ such that

1. $y, \dot{y}, \ddot{y}, \dots$ are linearly independent: they are not related by any differential equation.
2. y is a function of x and a finite number of derivatives of u
3. There are two smooth maps Θ and Ψ such that

$$x = \Theta(y, \dot{y}, \dots, y^{(\alpha)}), \quad u = \Psi(y, \dot{y}, \dots, y^{(\alpha+1)}),$$

for certain multi-index $\alpha = (\alpha_1, \dots, \alpha_m)$ and

$$y^{(\alpha)} = \left(\frac{d^{\alpha_1} y_1}{dt^{\alpha_1}}, \dots, \frac{d^{\alpha_m} y_m}{dt^{\alpha_m}}\right)$$

Roughly speaking, a control system is flat if we can find functions (flat outputs) of the state and control variables and their time-derivatives, so that the state and the control can be expressed in terms of that flat outputs and their derivatives. By consequence, the trajectories for y can be chosen freely.

2.1 Flatness and Motion Planning

Given the system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \quad m \leq n \tag{1}$$

and two configurations x_I, x_F in the space $\mathbb{R}^n$, the motion planning problem consists in finding an admissible trajectory $t \mapsto (x(t), u(t))$, $t \in [t_I, t_F]$ for the system (1), connecting those configurations, avoiding obstacles and with low cost.

For differentially flat systems it is possible to generate admissible paths joining two given states since there is a smooth 1-1 correspondence between solutions $x(t)$ of the system and the functions $y(t)$. Also, it suffices to control the flat outputs to control the whole system.

Once the terminal conditions over $x(t)$ and $u(t)$ are given, through the surjectivity of the mappings Θ and Ψ between sufficiently smooth trajectories of the output and feasible trajectories of the system, we can find a trajectory $t \mapsto y(t)$, sufficiently differentiable that satisfies the corresponding conditions for the flat output. To find a trajectory of the flat output satisfying the conditions

$$\left.\begin{array}{l} y(t_I) = y_I, \;\; \dot{y}(t_I) = 0, \;\; \cdots y^{(r+1)}(t_I) = 0 \\ y(t_F) = y_F, \; \dot{y}(t_F) = 0, \;\; \cdots y^{(r+1)}(t_F) = 0 \end{array}\right\} \tag{2}$$

we construct $(2r+3)$ *th* degree interpolation polynomials for the reference trajectories y_i of each variable of the flat output y:

$$\eta(t) = \eta_I + (\eta_I - \eta_F)\Big(\frac{t - t_I}{t_F - t_I}\Big)^{r+2} \sum_{j=0}^{r+1} a_j \Big(\frac{t - t_I}{t_F - t_I}\Big)^j \tag{3}$$

where $\eta_I = \eta(t_I)$, $\eta_F = \eta(t_F)$ and the coefficients a_j are independent of t_I, t_F, $\eta(t_I)$, $\eta(t_F)$ and [Levine,

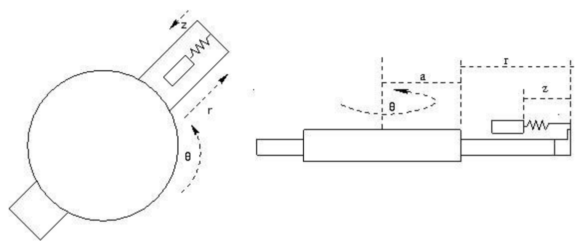

Figure 1. Robot with vibratory end-effector

2009] satisfy $r+2$ linear equations in $r+2$ unknown coefficients

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ r+2 & r+3 & & 2r+3 \\ \vdots & \vdots & & \vdots \\ (r+2)! & \frac{(r+3)!}{2} & \cdots & \frac{(2r+3)!}{(r+2)!} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{r+1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4}$$

We note that the above linear system always has a unique solution because the matrix have all its columns independent.

3 The Case Study of a Robot with Oscillatory End-Effector

The Elasto-Robot is a mechanism consisting of a circular base body, which can perform freely two movements: rotation and translation, and a prismatic pair coupled with a revolute and containing an vibratory element in the end-effector, moving on a horizontal plane. The motion planning problem for this systems is to move the robot between any given initial and final configurations such that the vibrating can be controlled, see Figure 1.

The parameters involved in this model are the following: a is the disk radius of the circular base body; θ is the angular displacement of the circular base body; r is the parallel displacement of the end-effector arm; m_2 is the prismatic-pair mass; m_3 is the terminal-effector mass and z is the coordinate associated to the vibration. the base body have mass negligible, κ denotes the spring constant associated to the vibration and the rotational inertia I. The torque forces $(u,v)=(\tau_1,\tau_2)$ are control parameters. We consider the kinetic and potential energies for the revolute, prismatic pair and the terminal-effector, so the Lagrangian $\mathcal{L}=\mathcal{L}(\theta,r,z,\dot\theta,\dot r,\dot z)$ of the system is

$$\left.\begin{array}{c} \mathcal{L} = I\dot\theta^2 + (m_2+m_3)\dot r^2 + (m_2+m_3)r^2\dot\theta^2 \\ +m_3\dot z^2 - r^2\kappa - z^2\kappa + 2rz\kappa \end{array}\right\} \tag{5}$$

By writing the Euler-Lagrange equations

$$\left.\begin{array}{l} \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\theta}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = \tau_1 \\ \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot r}\right) - \frac{\partial\mathcal{L}}{\partial r} = \tau_2 \\ \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot z}\right) - \frac{\partial\mathcal{L}}{\partial z} = 0, \end{array}\right\} \tag{6}$$

defining the state variables

$$\begin{array}{l} x_1=\theta,\ x_4=\dot x_1, \\ x_2=r,\ x_5=\dot x_2, \\ x_3=z,\ x_6=\dot x_3, \end{array}$$

for the coordinates $x=(x_1,x_2,x_3,x_4,x_5,x_6)$ in the manifold

$$\mathcal{M}=(0,2\pi)\times(0,R)\times(0,Q)\times(0,2\pi)\times(0,R)\times(0,Q)$$

for certain fixed values for R and Q, and by setting

$$I+m_2x_2^2+m_3(x_2-x_3)^2=J$$

with $J>I$, assuming $J=1$, $m_2=1$, $a^2-x^2>0$, with $a=\sqrt{J-I}$, we obtain ([Monroy, Romero and Vázquez-González, 2011]) the following non-linear control system

$$\dot x = X_0(x)+X_1(x)u+X_2(x)v, \tag{7}$$

for the drift vector field

$$X_0=\begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ -2x_2x_4x_5-2\sqrt{m_3}x_4(x_5-x_6)\sqrt{a^2-x_2^2} \\ x_2x_4^2 \\ \frac{\kappa}{\sqrt{m_3^3}}\sqrt{a^2-x_2^2}+x_3x_4^2 \end{pmatrix}$$

and $X_1=\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}$, $X_2=\begin{pmatrix}0\\0\\0\\0\\1\\1\end{pmatrix}$ the control vector fields.

3.1 Flatness of the Model

We know [Monroy and Romero, 2012] that the Elasto-robot is flat and the position $(\theta,r)=(x_1,x_2)$ of the base body and the end-effector arm is a flat output:

$$y=(y_1,y_2)=(x_1,x_2). \tag{8}$$

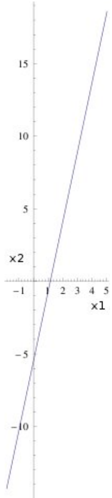

Figure 2. Motion of the point (x_1, x_2) for the Elasto-robot.

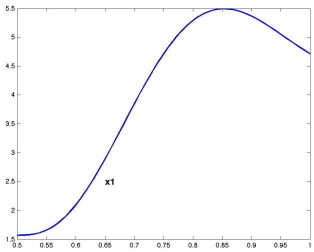

Figure 3. Reference trajectory for angle θ for the Elasto-robot.

We get

$$\begin{aligned} x_1 &= y_1 \\ x_2 &= y_2 \\ x_3 &= y_2 - \frac{1}{m_3}\sqrt{a^2 - y_2^2} \\ x_4 &= \dot{y}_1 \\ x_5 &= \dot{y}_2 \\ x_6 &= \frac{\ddot{y}_1 - \ddot{y}_2 + 2y_2\dot{y}_1\dot{y}_2 + y_2 y_1^2}{2\sqrt{m_3}\dot{y}_1\sqrt{a^2 - y_2^2}} + \dot{y}_2 \\ u &= \ddot{y}_1 - 4y_2\dot{y}_1\dot{y}_2 \\ v &= \ddot{y}_2 - y_2\dot{y}_1^2 \end{aligned} \quad (9)$$

The applied transformation

$$\begin{aligned} x &= \Theta(y, \dot{y}, \ddot{y}) \\ u &= \phi(y, \dot{y}, \ddot{y}) \\ v &= \psi(y, \dot{y}, \ddot{y}) \end{aligned} \quad (10)$$

is invertible, so y is a flat output of the Elasto-robot.

Now, we illustrate the solution of the the motion planning problem for the Elasto-Robot, in order to prevent

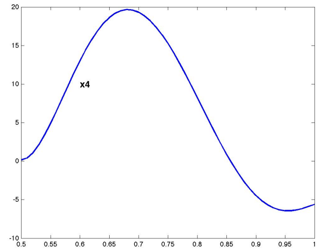

Figure 4. Coordinate x_4, angular velocity for the Elasto-robot.

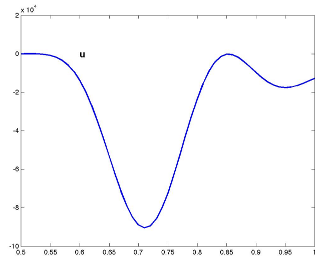

Figure 5. Control input u for the Elasto-robot.

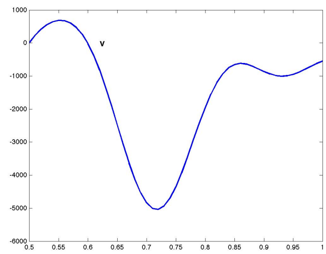

Figure 6. Control input v for the Elasto-robot

vibrations of the small mass. In this case we have the constraints

$$\left.\begin{array}{l} y(t_I) = y_I, \quad \dot{y}(t_I) = 0, \quad \ddot{y}(t_I) = 0 \\ y(t_F) = y_F, \; \dot{y}(t_F) = 0, \quad \ddot{y}(t_F) = 0 \end{array}\right\} \quad (11)$$

and the reference trajectory is

$$x_i(t) = x_i^I - (x_i^I - x_i^F)(\frac{t - t_I}{t_F - t_I})^4 \sum_{j=0}^{3} a_j (\frac{t - t_I}{t_F - t_I})^j \quad (12)$$

for the variables x_i, $i = 1, 2$ of the flat output $y = (x_1, x_2)$, where $x_i^I = x_i(t_I)$, $x_i^F = x_i(t_F)$. The values of the coefficients a_j are $a_0 = 35$, $a_1 = -84$, $a_2 = 70$ and $a_3 = -20$. Then, by using the interpolation polynomials (12) for each variable x_i, $i = 1, 2$ of the flat output $y = (x_1, x_2)$, we obtain, as solution for the motion planning problem for the Elasto-Robot, connecting the two rest-to-rest configurations $x_1^I = 0.5$,

$x_1^F = \pi/2$, $x_2^I = 0.5$ and $x_2^F = 2$, a straight line trajectory for the point (x_1, x_2) (Figure 2). Figure 3 illustrates our reference trajectory solution $x_1(t)$. Figure 4 shows the angular velocity of the Elasto-robot. Figure 5 and Figure 6 show the control outputs.

4 Nilpotent Approximations and the Motion Planning Problem

Nilpotent approximation is a technique that is very well suited for non-linear control systems that are affine in the control parameters, and that are defined by means of a finite family of smooth vector field on the state manifold.

Let $\mathcal{M}$ be a n-dimensional smooth manifold and let $\Delta \subset T\mathcal{M}$ be a co-rank $k = n - p$ distribution of smooth vector fields, we assume that Δ is generated by an orthonormal frame of vector fields, say $\mathcal{F} = \{X_1, \ldots, X_p\}$, we assume also that the control system $\dot{m} = \sum u_i X_i(m)$ is controllable, which is tantamount of saying that $\Delta_m = T_m\mathcal{M}$ for all $m \in \mathcal{M}$. The motion planning problem addresses the issue of approximating uniformly non-admissible paths by admissible ones. Observe that the orthonormality of the frame $\mathcal{F}$ defines a smooth varying inner product on the vector spaces Δ_m, in such a way that the energy of admissible curves defines the cost functional $\int \sum u^i$, and poses a natural optimal control problem on the state manifold $\mathcal{M}$. The controllability of the system along with Filippov's theorem, see [Vinter R. 2000], guarantee the existence of optimal solutions, therefore the manifold $\mathcal{M}$ is endowed with the structure of metric space, (the so- called Carnot-Carathéodory metric).

In this setting, a motion planning problem on $\mathcal{M}$ is formally defined as a pair $P = (\Delta, \Gamma)$, where $\Gamma : [0, T] \to \mathcal{M}$ is a fixed non-admissible curve.

For a small $\varepsilon > 0$ let T_ε and C_ε denote respectively a tubular neighborhood of Γ and its corresponding cylinder. The metric complexity $M(\varepsilon)$ of P is defined as $1/\varepsilon$ times the minimum length of an admissible curve connecting the points $\Gamma(0)$ and $\Gamma(T)$ meanwhile remaining within T_ε. The asymptotic optimal synthesis of P is a 1-parameter family of admissible curves that realizes $M(\varepsilon)$, and consequently solves the motion planning problem, for details see [Boizot and Gauthier, 2013].

The first step for the nilpotent approximation is the existence of normal coordinates for a $k-$ dimensional smooth surface S transversal to Δ and defined in a neighborhood of Γ. Such local coordinates are written as $(x, y, z) \in \mathbb{R}^p \times \mathbb{R}^{k-1} \times \mathbb{R}$ and satisfy $S(y, z) = (0, y, z)$, $\Gamma(z) = (0, 0, z)$, $\Delta|_S = \ker dz \cap \ker dy_1 \cap \cdots \cap \ker dy_{k-1}$, and furthermore the metric on S is Euclidean and the geodesics of the Pontryagin maximum principle are orthogonal lines to S, see for instance [Romero, Monroy and Gauthier, 2004]. By using these normal coordinates it can be shown in the elements of $\mathcal{F}$ can be written as follows

$$X_j = \sum_{i=1}^{p} Q_{ij} \frac{\partial}{\partial x_i} + \sum_{i=1}^{k-1} L_{ij} \frac{\partial}{\partial y_i} + M_j \frac{\partial}{\partial z}$$

with, Q symmetric, $Q(x, y, z)x = x, Q(0, y, z) = I$ and $L(x, y, z)x = M(x, y, z)x = 0$, and furthermore inside T_ε it holds that $\|x\|_2 \leq \varepsilon$ and $\|y\|_2 \leq \alpha\varepsilon^2$ for certain constant $\alpha > 0$. The nilpotent approximation of P along Γ is obtained by considering inside T_ε terms of order -1 only.

5 The Case Study of a Car with a Trailer

In this case the state space is the four dimensional manifold $\mathcal{M} = \mathbb{R}^2 \times S^1 \times S^1$, with local coordinates (x, y, θ, φ), where $(x, y) \in \mathbb{R}^2$ gives the position of the mid-point of the rear wheels, θ is the angle between the main direction of the car and the x-axis, and φ is the angle between the front wheels and the x-axis. The control parameters u_1 and u_2 allow displacements forward-backward and turning respectively, following [Berret *et al.*, 2006] one has that the kinematic equations can be written as follows:

$$\begin{aligned} \dot{x} &= u_1 \cos\theta \\ \dot{y} &= u_1 \sin\theta \\ \dot{\theta} &= -u_2 \sin\varphi \\ \dot{\varphi} &= u_2 \end{aligned}$$

This system can be written as a control-affine system on $\mathcal{M}$ given by two vector fields

$$\begin{aligned} \dot{m} &= u_1 X_1(m) + u_2 X_2(m) \\ X_1 &= \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} - \sin\varphi \frac{\partial}{\partial \varphi} \\ X_2 &= \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \varphi} \end{aligned}$$

It follows directly that at each $m \in \mathcal{M}$ the $\{X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]]\}$ is a basis of the tangent space $T_m\mathcal{M}$.

We compute the nilpotent approximation for the system by taking the reference parametrized trajectory $\Gamma = (0, 0, 0, t)$, normal coordinates and the terms of homogeneous degree -1 in the Taylor expansions of X_1 and X_2 to obtain:

$$\dot{\widetilde{m}} = u_1 \widetilde{X}_1(\widetilde{m}) + u_2 \widetilde{X}_2(\widetilde{m})$$

where $\widetilde{m} = (x, y, z, w)$ and $\widetilde{X}_1$, $\widetilde{X}_2$ are given by

$$\begin{aligned} \widetilde{X}_1 &= \frac{\partial}{\partial x} + \frac{y}{2} \frac{\partial}{\partial z} + \frac{y^2}{2} \frac{\partial}{\partial w} \\ \widetilde{X}_2 &= \frac{\partial}{\partial y} - \frac{x}{2} \frac{\partial}{\partial z} + \frac{xy}{2} \frac{\partial}{\partial w} \end{aligned}$$

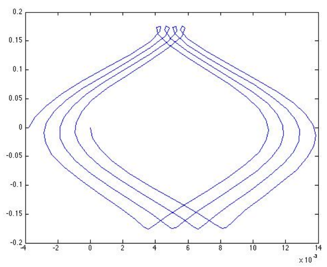

Figure 7. Motion of the point (x, y) of the car.

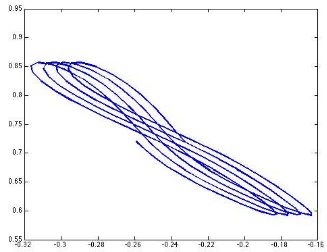

Figure 8. Motion from a different starting point.

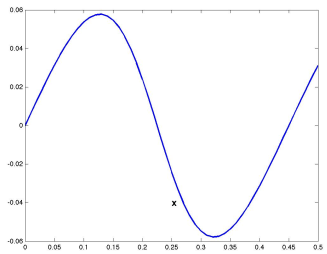

Figure 9. Coordinate x for the Car-like robot.

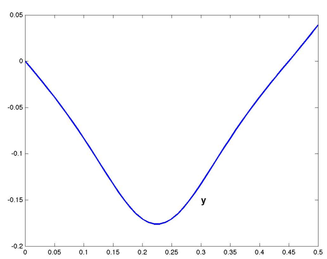

Figure 10. Coordinate y for the Car-like robot.

That system is nilpotent of order 3.

We address now the motion planning, to go from the configuration $(-0.4, 0.9, 0, 0)$ to the configuration $(-0.37, 0.9, 0, 0)$ in a minimal time, we use the Pontryagin Maximum Principle for a fixed time.

Taking the adjoint variable as $\psi = (p, q, r, p_0)$ we

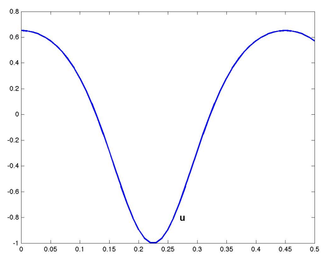

Figure 11. Control u_1 for the Car-like robot.

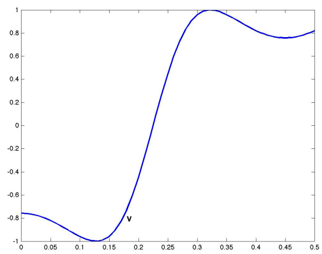

Figure 12. Control u_2 for the Car-like robot.

have that Hamiltonian of the system $H = \langle \psi, \widetilde{X}_1 \rangle u_1 + \langle \psi, \widetilde{X}_2 \rangle u_2$ which can be written in coordinates as follows

$$H = \frac{p_0}{2}(y^2 u_1 - xyu_2) + pu_1 + qu_2 + \frac{r}{2}(yu_1 - xu_2),$$

where $p_0 < 0$, because we are interested in the normal extremals and the abnormal extremals, $p_0 = 0$, are straight lines in the plans $w = w_0$ to which our reference trajectory is transversal. The necessary condition for the optimality of H, implies that the projection to the first two coordinates satisfies

$$\begin{aligned} \dot{x} &= u_1(t) = -\cos\varphi \\ \dot{y} &= u_2(t) = -\sin\varphi \end{aligned}$$

and for the other coordinates

$$\begin{aligned} \dot{z} &= \tfrac{1}{2} y(t) u_1(t) - \tfrac{1}{2} x(t) u_2(t) \\ \dot{w} &= \tfrac{1}{2} y^2(t) u_1(t) - \tfrac{1}{2} x(t) y(t) u_2(t) \end{aligned}$$

So, we obtain, according to ([Love, 1927]) the solutions for $x(t)$, $y(t)$ and therefore for controls $u_1(t)$, $u_2(t)$, in terms of Jacobi elliptic function, therefore the geodesic curves are the famous Euler elastic curves. In Figure 7 we show the path followed by (x, y). Figure 9 and Figure 10 illustrate our reference trajectory solution for $x(t)$ and $y(t)$. Figure 11 and Figure 12 show the control outputs. We have obtained the solution for the motion planning problem for the Car-like robot towing a trailer, connecting the configurations $(0, 0, \frac{\pi}{2}, 0)$ to $(-0.004, 0, \frac{\pi}{2}, 0)$. To ensure that the initial configuration is within the neighborhood of the reference trajectory Γ, we define the linear transformation:

$$(x, y, \theta, \varphi) \mapsto (y, \theta - \frac{\pi}{2}, \varphi - \theta + \frac{\pi}{2}, x - \varphi + \theta - \frac{\pi}{2})$$

This transformation straightens the curve $\hat{\Gamma} = (t, 0, \frac{\pi}{2}, 0)^t$ into the curve $\Gamma = (0, 0, 0, t)^t$. In Figure 8 we show the trajectory followed by (x, y) from a different starting point.

6 Conclusions and Perspectives

In this paper we present the motion planning problem under two different approaches: differential flatness and nilpotent approximations. We illustrate these methods by means of two specific examples: a arm-robot with an oscillatory end-effector and a car with a trailer, in the first case a flat control is explicitly exhibited whereas in the second optimal admissible trajectories appearing in the guise of elasticæ are presented as the admissible curves for approximating the reference curve within a tubular neighborhood. Numerical simulations are carried out in both cases.

The results in the paper are in some sense preliminary, and have to be completed in a further comparative study between flatness and nilpotent approximation.

7 Appendix

In this section we give some definitions and results related to the different approaches we have presented in this paper.

7.1 Differential Fields

It is a commutative ring $\mathcal{R}$ with a derivation $\frac{d}{dt} : \mathcal{R} \to \mathcal{R}, \quad a \mapsto \frac{d}{dt}(a) =: \dot{a}$

$$\left.\begin{array}{l} \frac{d}{dt}(a+b) = \dot{a} + \dot{b} \\ \frac{d}{dt}(ab) = \dot{a}\, b + a\, \dot{b} \end{array}\right\} \tag{13}$$

An element $c \in \mathcal{R}$ is a constant if $\dot{c} = 0$.

$L/K$for two given fields $K \subset L$, in such a way that the derivation of L in K coincides with the derivation of K.

An element $\xi \in L$ is differentially K-algebraic, if there exists a $p \in K[x_1, \ldots, x_n]$ such that

$$p(\xi, \dot{\xi}, \ldots, \xi^{(n)}) = 0 \tag{14}$$

The extension L/K is said to be algebraic if all the elements in L are K-algebraic.

$\xi \in L$ is K-transcendent if and only if is not K-algebraic. The extension L/K is said to be transcendent if there exist at least an element L that is transcendent.

A set $\{\xi_i\}_{i\in I}$ is differentially K- algebraic independent if $\{\xi_i^{(\nu)} \mid \nu \in \mathbb{N}\}_{i\in I}$ is K-algebraic independent.

Maximal independent sets with respect to the inclusion. The cardinality of a basis is the transcendence differential degree of the extension. Let K be a differential field then

$$K\left[\frac{d}{ds}\right] = \left\{ \sum_{finita} a_\nu \frac{d^\nu}{ds^\nu} \right\} \tag{15}$$

is a principal ideals ring. It is commutative if and only if K is a field of constants.

7.2 Field of Differential Operators

Let $\mathcal{C} = \{f : [0, +\infty) \longrightarrow \mathbb{C}\}$ be a ring of functions with respect to sum and convolution

$$(f \star g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau \tag{16}$$

$\mathcal{C}$ has no zero divisors (). The field of differential operators is the quotient field of $\mathcal{C}$.

1. Identity element: Dirac in $t = 0$
2. The inverse of the Heaviside function: is the derivation operator

$$\mathbf{1}(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \tag{17}$$

7.3 Equivalence

Let M be a differential manifold and let $F \in C^\infty(TM, \mathbb{R}^{n-m})$, an implicit system is written as follows

$$F(x, \dot{x}) = 0, \quad \text{rank}\left(\frac{\partial F}{\partial \dot{x}}\right) = n - m \tag{18}$$

Any system $\dot{x} = f(x, u)$ can be taken into this form: rank $\left(\frac{\partial f}{\partial u}\right) = m$ implies $u = \mu(x, \dot{x}_{n-m+1}, \ldots, x_n)$, for then

$$F_i(x, \dot{x}) = \dot{x}_i - f_i(x, \mu(x, \dot{x}_{n-m+1}, \ldots, x_n)) \tag{19}$$

Two systems $(M, F), (N, G)$ with rank $\left(\frac{\partial F}{\partial \dot{x}}\right) = n - m$ and rank $\left(\frac{\partial G}{\partial \dot{y}}\right) = p - q$ are equivalent in $x_0 \in M$ and $y_0 \in N$ if:

1. There is $\Phi = (\varphi_1, \varphi_2, \ldots) \in C^\infty(N, M)$ such that

$$\Phi(y_0) = x_0, \quad \frac{d\varphi_i}{dt} = \varphi_{i+1} \tag{20}$$

and any solution $t \mapsto y(t)$ of $G(y, \dot{y}) = 0$ satisfies $F(\varphi_1(y(t)), \varphi_2(y(t))) = 0$.
2. There is $\Psi = (\psi_1, \psi_2, \ldots) \in C^\infty(M, N)$ such that

$$\Psi(x_0) = y_0, \quad \frac{d\psi_i}{dt} = \psi_{i+1} \tag{21}$$

and any solution $t \mapsto y(t)$ of $F(x, \dot{x}) = 0$ satisfies

$$G(\psi_1(x(t)), \psi_2(x(t))) = 0. \tag{22}$$

If two systems are equivalent then they have the same co-ranks $m = q$.

Given a trajectory $t \mapsto x(t)$ of system $F(x, \dot{x}) = 0, x \in M$ and $\xi \in TM$, the implicit system

$$\left(\frac{\partial F}{\partial x}(x, \dot{x})\right) \xi(t) + \left(\frac{\partial F}{\partial \dot{x}}(x, \dot{x})\right) \dot{\xi}(t) = 0 \tag{23}$$

is called *the linear approximation* around x.

Proposition 7.1. *If two systems are equivalent then the corresponding linear approximations are also equivalent.*

Definition 7.1. *(M, F) is flat in x_0 if it is equivalent to $(\mathbb{R}^m, 0)$, that is, if trajectories $t \mapsto x(t)$ are the image of a trivialization Φ, such that, $\Phi(y_0) = x_0$. Equivalently, for each curve $t \mapsto y(t)$*

$$x(t) = (x, \dot{x}, \ldots) = \Phi(\varphi_1(y(t)), \varphi_2(y(t)), \ldots) \tag{24}$$

Proposition 7.2. *If a system is flat then it is equivalent to its linear approximation.*

Proposition 7.3. *If (M, F) is flat in x_0, then*

1. *Its linear approximation is controllable.*
2. *If x_0 is an equilibrium point, the system is locally controllable around x_0.*

7.4 Nilpotent Approximation, Normal Coordinates and the Normal Form

In this section we introduce the nilpotent approximation, normal coordinates and the normal form. We consider a fixed system of vector fields $X_1, X_2, \ldots, X_m$ on a manifold $\mathcal{M}$.

Definition 7.2. *The order of an analytical function f at $p_o \in \mathcal{M}$ is $\geq s$ if for any $i_1, i_2, \ldots i_q, \quad q \leq s-1$ the nonholonomic partial derivatives of order q of f, $X_{i_1} \ldots, X_{i_q} f$, vanish at p_0 (with respect to the system $\{X_1, X_2, \ldots, X_m\}$).*

Definition 7.3. *A vector field X is said to be of order $\geq q$ at p_0 if for every s and every function f having order s at p_0, the function Xf has order $\geq q + s$ at p_0.*

So, the vector fields X_i have order ≥ -1, the brackets $[X_i, X_j]$ have order $\geq -2, \ldots$. Usually, $X_{i_1} \cdots X_{i_q}$ have order -1.

Let $L^s(p)$ the subspace of $T_p\mathcal{M}$ spanned by values at p of the brackets of length $\geq s$ of vector fields $X_1, X_2, \ldots, X_m$. By the Chow-Rashevsky theorem, by each $p \in \mathcal{M}$ there is a smallest integer $r = r(p)$ such that $L^{r(p)}(p) = T_p\mathcal{M}$. This integer is called the degree of nonholonomy at p.

Definition 7.4. *We call the weight of the coordinate y_i the number w_j defined as follow. Let $y_1, y_2, \ldots, y_n$ be a system of linearly adapted coordinates at p of $L^1(p) \subset L[2(p) \subset \cdots \subset L^r(p) = T_p\mathcal{M}$, then $w_i = s$, if $y_i \in L^s(p)$ and $y_i \notin L^{s-1}(p)$.*

Notice we always have w_i=1 and $w_n = r$, the degree of nonholonomy at p.

Definition 7.5. *A system of privileged coordinates is a system of local coordinates $z_1, \ldots, z_n$, defined along a curve Γ, such that*

1. *$z_1, \ldots, z_n$ are linearly adapted at p.*
2. *The weight of z_j at p is w_j.*

In a system of local coordinates $z_1, \ldots, z_n$ we can express a vector field X like

$$X(z) = \Sigma_{j=1}^n X^j(z) \partial z_j \sim \Sigma_{j=1}^n \Sigma_\alpha (a_{\alpha,j} z^\alpha) \partial z_j,$$

where $\Sigma_\alpha a_{\alpha,j} z^\alpha$ is the Taylor development of the vector field X along Γ. We associate the weight $-w_j$ to the terms ∂z_j.

Definition 7.6. *The nilpotent approximation of a system $X_1, X_2, \ldots, X_m$ along Γ is the set of vector fields $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_m$, such that*

$$X_i = X_i^{-1} + X_i^0 + X_i^1 + X_i^2 + \cdots,$$

where X_i^s is homogeneous of degree s and $\hat{X}_i = X_i^{-1}$.

7.5 Pontryagin Maximum Principle for Fixed Time

In this section we present the Pontryagin maximum principle in the case of a sub-Riemannian metric for the optimization of the energy:

Theorem 7.1. *If a non-admissible trajectory $(x(t), u(t))$ for a control system*

$$\dot{x} = f(x, u), \quad x \in \mathcal{M}, \quad u \in \mathbb{R}^m \quad m \leq dim(\mathcal{M})$$

minimize the energy between two points $p, q \in \mathcal{M}$ in the fixed time problem, then there exists an extension (x, φ) of x in $T^*\mathcal{M}$, such that $\varphi \neq 0$, and there exists $\varphi_0 \leq 0$, such that

$$\dot{x}(t) = \frac{\partial H}{\partial \varphi}, \quad \dot{\varphi}(t) = -\frac{\partial H}{\partial x}$$

$$H(x(t), \varphi(t), u(t), \varphi_0) = max\{H(x(t), \varphi(t), v(t), \varphi_0)\},$$

with the maximum over $v \in \mathbb{R}^m$.

A trajectory $(x(t), u(t))$ that satisfies the above theorem is called extremal. If $\varphi_0 < 0$, the extremal trajectory is called normal and if $\varphi_0 = 0$, it's called abnormal.

Theorem 7.2. *(Normal coordinates) Let $P = (\Delta, \Gamma)$ be a motion planning problem and g a sub-Riemannian metric. There exists a coordinate system $(x, y, z, w) \in \mathbb{R}^4$, defined on a neighborhood of $\Gamma([0, 1])$, such that*

1. *$\Gamma(t) = (0, 0, 0, t)$, $\Delta(\Gamma(t)) = ker\, dw$ and $g|_\Gamma(t) = dx^2 + dy^2 + dz^2 + dw^2$*
2. *Geodesics satisfying the Pontryagin Maximum Principle'transversality conditions with respect to Γ are straight lines through Γ contained in the planes $S_{w_0} = \{(x, y, z, w)|w = w_0\}$*
3. *For ε small enough, the sub-Riemannian cylinder $C_\varepsilon = \{q \mid d(q, \Gamma) = \varepsilon\}$ is the Riemannian cylinder $\{||(x, y, z, w)||_2 = \varepsilon\}$*

Theorem 7.3. *(Normal form) Let $P = (\Delta, \Gamma)$ be a motion planning problem, g a sub-Riemannian metric and let $(x, y, z, w) \in \mathbb{R}^4$ be a fixed normal coordinate system. Then, there is a unique sub-Riemannian's orthonormal frame $\mathcal{F} = (Q, L)$, where*

$$Q = \begin{pmatrix} q_{11} & q_{21} \\ q_{21} & q_{22} \end{pmatrix} \; L = \begin{pmatrix} l_{11} & l_{21} \\ l_{21} & l_{22} \end{pmatrix} \; F = \begin{pmatrix} q_{11} \\ q_{21} \\ l_{11} \\ l_{21} \end{pmatrix}$$

$$G = \begin{pmatrix} q_{12} \\ q_{22} \\ l_{12} \\ l_{22} \end{pmatrix}$$ *and the following properties:*

1. *Q is symmetric*
2. *$Q(0, 0, z, w) = id$*
3. *$Q(x, y, z, w) \cdot (x, y) = (x, y)$*
4. *$L(x, y, z, w) \cdot (x, y) = (0, 0)$*
5. *$Q_1 = 0$*
6. *$L_0 = 0$*

hold.

Acknowledgements

This paper was prepared during the sabbatical leave of the second author at the Laboratoire des Sciences de l'information et des Systèmes (LSIS, UMR 7296) in the Université du Sud Toulon-Var, France. This author was financially supported by the CONACYT under the program of sabbatical leaves abroad for the reinforcement of Research Groups, project number 204051.

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