Impulsive Time-Optimal Control of Structure-Variant Rigidbody Mechanical Systems

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1. INTRODUCTION

The optimal control of rigid-body mechanical systems with discontinuous states is a non-investigated area. An impact in mechanics is defined as a discontinuity in the generalized velocities of a mechanical system which is induced by impulsive forces, therefore optimal control of impulsive systems inevitably encompasses optimal control with discontinuous states. In this work, necessary conditions for the time-optimal control of such rigid-body systems will be stated. The underlying Lagrangian structure, however, enables the consideration of broader classes of physical systems, as well. The relation between time-optimal control and impulsive control is emphasized, because in the philosophy of time-optimal control, it is taken advantage of any excessive control action in order to attain the goal and impulsive control action is the utmost excessive control action that can be applied to a dynamical system, since impulsive control forces can grow to infinite on a single time instant. The sources of impulsive forces are manifold. An introduction to impacts is provided in [4] and a literature survey on impactive systems is provided in [5]. The dynamics and modeling of structure-variant rigidbody mechanical systems is extensively treated in [1], [2] and [3]. The control of structure-variant mechanical systems is an active research area and some references include [14, 16, 17, 18, 19, 20, 21]. In the derivation of the necessary conditions for a strong local minimizer in the impulsive time-optimal control problem of rigidbody mechanical systems, it is assumed that the instant of discontinuity is reduced to an instant with Lebesgue measure zero, instead of taking an interval opening approach (see for an example [22]) contrary to the approach taken in literature so far. The approach requires different system modes and their order to be specified in advance. The proposed necessary conditions provide criteria for the optimal transition time and location. The smooth dynamics of the rigidbody mechanical system is characterized in every interval of motion (t_i^+, t_{i+1}^-) by a different differential equation system depending on the closed-directions of motion by the structure-varying controller, in general. Another issue therefore is the representation of the Lagrangian dynamics in different modes. A method that relies on projecting the mechanical dynamics into subspaces without additionally introducing algebraical constraints and generalized coordinates is discussed in section 3. In the modeling framework considered in this work, impulsive forces can arise autonomously, due to contact interactions such as collisions or controlled/nonautonomously, due to actions as blocking of DOF of manipulators impactively. The introduced framework will have the ability to model and control of hybrid mechanical systems with discontinuous state transitions among different system modes.

2. Preliminaries

By the application of subdifferential calculus techniques to extended-valued lower semi-continuous functionals, Pontryagin's Maximum Principle (PMP) like conditions are obtained. The considered functional is a generalized Bolza functional that is evaluated on multiple intervals. The well-known PMP entails the necessary conditions for optimal control problems with differential constraints and end-point constraints with sufficient regularity properties in the space of absolutely continuous arcs ($\mathscr{AC}$). However, impulsive optimal control requires to search extremizing arcs in the space of bounded variation arcs ($\mathscr{BV}$). So the obtained necessary conditions will encompass PMP conditions under mild hypotheses since the class of $\mathscr{BV}$ arcs totally encompass the class of $\mathscr{AC}$ arcs. Let us consider a problem in Bolza form (GPB), in which the objective is to choose an absolutely continuous arc $\mathbf{x} \in \mathscr{AC}$ in order to minimize

$$J(x) = l(\mathbf{x}(a), \mathbf{x}(b)) + \int_a^b L(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))\, dt \qquad (1)$$

where the function $L : [a,b] \times \mathscr{R}^n \times \mathscr{R}^n \to \mathscr{R} \cup \{+\infty\}$ is $\mathscr{L} \times \mathscr{B}$ measurable. Here $\mathscr{L} \times \mathscr{B}$ denotes the σ-algebra of subsets of $[a,b] \times \mathscr{R}^n$ generated by product sets $\mathscr{M} \times \mathscr{N}$, where $\mathscr{M}$ is a Lebesgue measurable subset of $[a,b]$ and $\mathscr{N}$ is a Borel subset of $\mathscr{R}^{2n}$. For each $t \in [a,b]$, the function l and L are lower semi-continuous on $\mathscr{R}^n \times \mathscr{R}^n$, with values in $\mathscr{R} \cup \{+\infty\}$. For each $(t, \mathbf{x})$ in $[a,b] \times \mathscr{R}^n$, the function $L(t, \mathbf{x}, \cdot)$ is convex and l represents the endpoint cost. GPB concerns the minimization of a functional whose form is identical to that in the classical calculus of variations. The

GPB is distinguished from its classical version, by the extremely mild hypotheses imposed on the endpoint cost l and the integrand L. Both are allowed to take the value $+\infty$. An important class of optimal control problems constrain the derivative of an admissible arc and they can be stated as the following Mayer problem (M):

$$\min\{l(\mathbf{x}(a),\mathbf{x}(b)) : \dot{\mathbf{x}}(t) \in \mathscr{F}(t,\mathbf{x}(t)), \quad a.e. \quad t \in [a,b]\}. \tag{2}$$

The problem (M) can be seen as minimizing the Bolza functional J over all arc $\mathbf{x}$. To cover the Mayer problem, it suffices to choose:

$$L(t,\mathbf{x},\mathbf{v}) = \Psi_{\mathscr{F}(t,\mathbf{x})}(\mathbf{v}) = \begin{cases} 0, \mathbf{v} \in \mathscr{F}(t,\mathbf{x}) \\ +\infty, \text{otherwise.} \end{cases} \tag{3}$$

The function $\Psi_{\mathscr{C}}$ is called the indicator function of the set $\mathscr{C}$. It is evident that for any arc $\mathbf{x}$, one has

$$\int_a^b L(t,\mathbf{x},\dot{\mathbf{x}})\,dt = \begin{cases} 0, \dot{\mathbf{x}}(t) \in \mathscr{F}(t,\mathbf{x}) \quad a.e. \quad t \in [a,b] \\ +\infty, \text{otherwise.} \end{cases} \tag{4}$$

The Mayer type variational problem can arise from a typical dynamic constraint in controls such as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t,\mathbf{x}(t),\tau(t)),\, \tau(t) \in \mathscr{C}_\tau, a.e. \quad t \in [a,b]. \tag{5}$$

If a control state-pair $(\tau,\mathbf{x})$ satisfies equation (5), then

$$\begin{aligned} \dot{\mathbf{x}}(t) &\in \mathscr{F}(t,\mathbf{x}(t)) \qquad (6)\\ &:= \{\mathbf{f}(t,\mathbf{x}(t),\tau(t)) : \tau(t) \in \mathscr{C}_\tau \quad a.e. \quad t \in [a,b]\} \end{aligned}$$

certainly does. The well-known Fillipov's theorem is the statement that the reversal of the above statement is true as well.

In order to guarantee the well-behaving of $\mathscr{F}$ and l let following hypotheses hold:

Hypotheses 1. An arc $\bar{\mathbf{x}} : [a,b] \to \mathscr{R}^n$ is given. On some relatively open subset $\Omega \subseteq [a,b] \times \mathscr{R}^n$ containing the graph of $\bar{\mathbf{x}}$, the following statements hold:

- The multifunction $\mathscr{F}$ is $\mathscr{L} \times \mathscr{B}$ measurable on Ω. For each $(t,\mathbf{x})$ in Ω, the set $\mathscr{F}(t,\mathbf{x})$ is nonempty, compact and convex.
- There are nonnegative integrable functions k and Φ on $[a,b]$ such that
  1. $\mathscr{F}(t,\mathbf{x}) \subseteq \Phi(t)B$ for all $\mathbf{x}$ in Ω_t, almost everywhere, and
  2. $\mathscr{F}(t,\mathbf{x}) \subseteq \mathscr{F}(t,\mathbf{x}) + k(t)|\mathbf{y}-\mathbf{x}|\mathrm{cl}B$ for all $\mathbf{x}$, $\mathbf{y} \in \Omega_t$, almost everywhere.
- The endpoint cost function l is Lipschitz on $\Omega_a \times \Omega_b$, with Lipschitz constant K_l.

where $\Omega_t = \{\mathbf{x} \in \mathscr{R}^n : (t,\mathbf{x}) \in \Omega\}$ for each t in $[a,b]$ and B is the unit sphere.

The generalized problem of many practical problems place constraints not only on the derivative of the state trajectory, but also on its endpoints. The differential inclusion problem (M) will be augmented with the additional constraint $(\mathbf{x}(a),\mathbf{x}(b)) \in \mathscr{S}$, where $\mathscr{S}$ is a given target set in $\mathscr{R}^n \times \mathscr{R}^n$ and is assumed to be closed. The new problem will be called $M_{\mathscr{S}}$. Suppose that there is a function $\varphi(t,\mathbf{x})$ with the following properties:

1. $\varphi(t,\mathbf{x}) \in \mathscr{F}(t,\mathbf{x})$ for all $\mathbf{x} \in \Omega_t$, almost everywhere;
2. $\varphi(t,\mathbf{x})$ is a Carathéodory function, i.e., φ is $\mathscr{L} \times \mathscr{B}$ measurable on Ω, and for almost every t the function $x \mapsto \varphi(t,\mathbf{x})$ is Lipschitz on Ω_t with Lipschitz rank $k(t)$;
3. $\dot{\bar{\mathbf{x}}}(t) = \varphi(t,\bar{\mathbf{x}}(t))$ almost everywhere on $[a,b]$.

Theorem -Pontryagin's Maximum Principle Consider the optimal control problem of minimizing the endpoint function

$$l(\mathbf{x}(a),\mathbf{x}(b)) + \Psi_{\mathscr{S}}(\mathbf{x}(a),\mathbf{x}(b)) \tag{7}$$

over all arcs $\mathbf{x}$ satisfying the differential constraint

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t,\mathbf{x}(t),\tau(t)),\, \tau(t) \in \mathscr{C}_\tau, \quad a.e. \quad t \in [a,b]. \tag{8}$$

In addition, suppose that f is a Carathéodory function for which the velocity sets $\mathscr{F}(t,\mathbf{x}(t)) = \{\mathbf{v}|\mathbf{v} = \mathbf{f}(t,\mathbf{x}(t),\tau(t)),\, \tau(t) \in \mathscr{C}_\tau\}$ satisfy Hypotheses 1. If an arc $\bar{\mathbf{x}}$, together with a corresponding control function $\bar{\tau}$, solves this problem, then there exist an arc $\mathbf{p} \in \mathscr{AC}$ on $[a,b]$ and a scalar λ equal to either 0 or 1 for which one has, for almost every $t \in [a,b]$,

- the adjoint equation,

$$-\dot{\mathbf{p}}(t) \in \bar{\partial}_x \langle \mathbf{p}(t), \mathbf{f}(t,\bar{\mathbf{x}}(t),\tau(t)) \tag{9}$$

- the maximum condition

$$\langle \mathbf{p}(t), \mathbf{f}(t,\bar{\mathbf{x}},\bar{\tau}) \rangle = \sup\{\langle \mathbf{p}, \mathbf{f}(t,\bar{\mathbf{x}},\tau) \rangle : \tau \in \mathscr{C}_\tau\} \tag{10}$$

- the transversality conditon

$$(\mathbf{p}(a),-\mathbf{p}(b)) \in \lambda \partial l(\bar{\mathbf{x}}(a),\bar{\mathbf{x}}(b)) + \mathscr{N}_{\mathscr{S}}(\bar{\mathbf{x}}(a),\bar{\mathbf{x}}(b)). \tag{11}$$

Here $\mathscr{N}_{\mathscr{S}}(\bar{\mathbf{x}}(a),\bar{\mathbf{x}}(b))$ denotes the limiting normal cone to the set $\mathscr{S}$ at $(\bar{\mathbf{x}}(a),\bar{\mathbf{x}}(b))$. The operator ∂ denotes generalized subdifferential in the sense of Clarke [9]. The above stated form of the Pontryagin's maximum principle (PMP) defines the necessary conditions for an arc

$\bar{\mathbf{x}} \in \mathscr{AC}$ to extremize the problem $M_{\mathscr{S}}$. However, impulsive optimal control requires to seek extremizing arcs in the space of bounded variation functions $\mathscr{BV}$. Every function $\mathbf{x} : [t_0, t_1] \to \mathscr{R}^n$ of bounded variation is associated with an $\mathscr{R}^n$-valued regular Borel measure $d\mathbf{x}$ on $[t_0, t_1]$. The atoms for $d\mathbf{x}$ occur only at discontinuities of $\mathbf{x}$, of which there are at most countably many. Trajectories of bounded variation in $\mathscr{R}^n$ are defined to be an equivalence class, and the space of all arcs is denoted by $\mathscr{BV}$. The space of absolutely continuous arcs $\mathscr{AC}$ is a subspace of $\mathscr{BV}$. There are uniquely determined functions $\mathbf{x}^+(t)$ and $\mathbf{x}^-(t)$ in $[t_0, t_1] \to \mathscr{R}^n$, right and left continuous respectively, such that $\mathbf{x}^+(t) = \mathbf{x}^-(t) = \mathbf{x}(t)$ at all the non-atomic points, and at the end points $\mathbf{x}^-(t_0) = \mathbf{x}(t_0)$ and $\mathbf{x}^+(t_f) = \mathbf{x}(t_f)$ are valid. Therefore, a further classification of $\mathbf{x} \in \mathscr{BV}$ is to subdivide these functions into left-continuous bounded variation ($\mathscr{LCBV}$) and right-continuous bounded variation ($\mathscr{RCBV}$) functions. The quantity $\Delta\mathbf{x}(t) = \mathbf{x}^+(t) - \mathbf{x}^-(t)$ is called the jump of the arc $\mathbf{x}$ at t, and if it is nonzero there is an atom of $d\mathbf{x}$ at t with this value. The absolutely continuous part of the measure $d\mathbf{x}$ is denoted by $\dot{\mathbf{x}}\,dt$. The singular part of $d\mathbf{x}$, can be represented as $(\frac{d\mathbf{x}}{d\sigma})\,d\sigma$, where $d\sigma$ is some nonnegative singular measure (a regular Borel measure), and $\frac{d\mathbf{x}}{d\sigma}$ is the Radon-Nikodym derivative of $d\mathbf{x}$ with respect to $d\sigma$, which is also denoted as $\mathbf{x}'$. Having set the stage, the necessary conditions for the impulsive optimal control of structure-variant rigid-body mechanical systems is formally derived by considering a problem in Bolza form (GPB), in which the objective is to choose an arc $\mathbf{x} \in \mathscr{BV}$ in order to minimize

$$J(x) = \Sigma_{i=1}^{N} l_i(\mathbf{x}(t_{i+1}^-), \mathbf{x}(t_i^+)) + \int_{t_i^+}^{t_{i+1}^-} L_i(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))\,dt. \tag{12}$$

Here it is assumed that the control horizon is composed of N different phases, which are separated from each other by $N-1$ possibly discontinuous transitions. The set of transition times is defined as $\mathscr{I}_T$. Any transition time t_i is Lebesgue-negligible.

3. Projected Newton-Euler Equations

The interaction of the mechanical system with the surroundings as well as the control actions imposed on the system necessitates to allow discontinuity events in the velocities and accelerations of the system. The projected Newton-Euler equations have to be supplemented with some force laws that relate the external forces $\mathbf{f}$ and controls τ with the system's state $(\mathbf{q}, \mathbf{u})$. The existence of the accelerations $\dot{\mathbf{u}}$ on an time interval $\left(t_i^+, t_{i+1}^-\right)$ are limited to the instants where $\mathbf{u}$ is continuous. Because of the set of discontinuity points $\{t_i\}$ of $\mathbf{u}$ where $\dot{\mathbf{u}}$ does not exist, the projected Newton-Euler equations should be stated in the following form:

$$\mathbf{M}(\mathbf{q})\dot{\mathbf{u}} - \mathbf{h}(\mathbf{q}, \mathbf{u}) = \mathbf{f} + \mathbf{B}(\mathbf{q})\,\tau, \quad [dt] - \text{a.e.} \tag{13}$$

Here $\mathbf{M}$ is the symmetric and positive definite generalized mass matrix depending smoothly on $\mathbf{q}$, and $\mathbf{h}$ is a smooth function of $\mathbf{q}$ and $\mathbf{u}$ containing the gyroscopical accelerations of the multibody system. The linear operator $\mathbf{B}(\mathbf{q})$ includes the generalized directions of control force action on the system. In order to investigate the discontinuity points of the velocities $\mathbf{u}$ and accelerations $\dot{\mathbf{u}}$ properly, equation (13) is replaced by the corresponding equality of measures as in [7]:

$$\mathbf{M}(\mathbf{q})\,d\mathbf{u} - \mathbf{h}(\mathbf{q}, \mathbf{u})\,dt = d\mathbf{R} + \mathbf{B}(\mathbf{q})\,d\Gamma, \tag{14}$$

This form of representation of the projected Newton-Euler equations has wider range of validity such that it is valid "everywhere" instead of " almost everywhere". For the force measure $d\mathbf{R}$ following decomposition is valid:

$$d\mathbf{R} = \mathbf{f}\,dt + \mathbf{F}'\,d\sigma + d\mathbf{R}_C, \tag{15}$$

such that $\mathbf{f}$ and $\mathbf{F}'$ represent Lebesgue-measurable forces and Borel-measurable forces, respectively. The singular force measure $d\mathbf{R}_C$ is assumed to vanish. Similarly the differential measure of controls is decomposed as:

$$d\Gamma = \tau\,dt + \zeta'\,d\sigma, \tag{16}$$

Here τ and ζ' represent the Lebesgue-measurable controls and the Borel-measurable controls, respectively. The substitution of (15) into (14) along with $d\mathbf{u} = \dot{\mathbf{u}}\,dt + (\mathbf{u}^+ - \mathbf{u}^-)\,d\sigma$ reveals:

$$\begin{aligned}&\mathbf{M}(\mathbf{q})\,\dot{\mathbf{u}}\,dt + \mathbf{M}(\mathbf{q})\,(\mathbf{u}^+ - \mathbf{u}^-)\,d\sigma - \mathbf{h}(\mathbf{q}, \mathbf{u})\,dt = \\ &(\mathbf{f} + \mathbf{B}(\mathbf{q})\,\tau)\,dt + \left(\mathbf{F}' + \mathbf{B}(\mathbf{q})\,\zeta'\right)\,d\sigma.\end{aligned} \tag{17}$$

Equation (17) can be split into a Lebesgue and Borel part as given below:

$$\begin{aligned}\mathbf{M}(\mathbf{q})\,(\mathbf{u}^+ - \mathbf{u}^-)\,d\sigma &= \left(\mathbf{F}' + \mathbf{B}(\mathbf{q})\,\zeta'\right)d\sigma, \\ \mathbf{M}(\mathbf{q})\,\dot{\mathbf{u}}\,dt - \mathbf{h}(\mathbf{q}, \mathbf{u})\,dt &= (\mathbf{f} + \mathbf{B}(\mathbf{q})\,\tau)\,dt.\end{aligned}$$

An impact in mechanics is defined as a discontinuity in the generalized velocities of a mechanical system which is induced by impulsive forces. From the atomic part one obtains after evaluation of the Lebesgue-Stieltjes integral over atom of impact time following impact equation:

$$\mathbf{M}(\mathbf{q}(t_i))\,(\mathbf{u}_i^+ - \mathbf{u}_i^-) = \mathbf{F}_i^+ - \mathbf{F}_i^- + \mathbf{B}(\mathbf{q}(t_i))\left(\zeta_i^+ - \zeta_i^-\right), \tag{18}$$

where t_i is an element of discontinuity points of the velocity $\mathbf{u}$. The Lebesgue part which remains unaffected by the points of discontinuity can be expressed in two equivalent forms as below:

$$\begin{aligned}\mathbf{M}(\mathbf{q})\,\dot{\mathbf{u}}^{+}\,dt-\mathbf{h}(\mathbf{q},\mathbf{u}^{+})\,dt &= \left(\mathbf{f}^{+}+\mathbf{B}(\mathbf{q})\,\tau^{+}\right)\,dt,\\ \mathbf{M}(\mathbf{q})\,\dot{\mathbf{u}}^{-}\,dt-\mathbf{h}(\mathbf{q},\mathbf{u}^{-})\,dt &= \left(\mathbf{f}^{-}+\mathbf{B}(\mathbf{q})\,\tau^{-}\right)\,dt.\end{aligned}$$

The points of discontinuity are Lebesgue negligible. Based on this evidence, for $\mathbf{f}$ and τ it is assumed that $\mathbf{f}=\mathbf{f}^{+}=\mathbf{f}^{-}$ and $\tau=\tau^{+}=\tau^{-}$ for $[dt]-a.e.\quad t$, where $\mathbf{f}^{+}$ and $\mathbf{f}^{-}$ are meant to be the right and left limits of $\mathbf{f}$ with respect to time, respectively. As a corollary, the directional Newton-Euler equations can be stated as follows:

$$\mathbf{M}(\mathbf{q})\,\dot{\mathbf{u}}^{+}-\mathbf{h}(\mathbf{q},\mathbf{u}^{+}) = \mathbf{f}^{+}+\mathbf{B}(\mathbf{q})\,\tau^{+}, \tag{19}$$
$$\mathbf{M}(\mathbf{q})\,\dot{\mathbf{u}}^{-}-\mathbf{h}(\mathbf{q},\mathbf{u}^{-}) = \mathbf{f}^{-}+\mathbf{B}(\mathbf{q})\,\tau^{-}. \tag{20}$$

3.1. Equations of Motion in Different Phases of Motion

After a possibly impactive transition the equations of motion on acceleration level may differ from the pre-transition equations of motion based on the closed directions of motion. The equations of motion can be projected to a subspace such that the dynamics do not evolve in the restrained directions of motion, without changing the number of generalized degrees of freedom (DOF) or generating additional algebraical constraints. In the sequel, representation after transition will be derived, the advantage of which being that a new set of minimal DOF is not required. Further, it will be assumed that the interaction of the mechanical system with the surroundings (unilateral contacts, etc.) do not interfere during the course of control action ($d\mathbf{R}=\mathbf{0}$). A direction of interest for the controller strategy γ_i, which for example can be the relative velocity at a blockable joint, can be stated as a linear combination of generalized velocities as in equation (21):

$$\gamma_i = w_i^T(\mathbf{q})\,\mathbf{u}. \tag{21}$$

The directions of interest can be expressed vectorially as:

$$\gamma = W^T(\mathbf{q})\,\mathbf{u}. \tag{22}$$

where γ is such that $w_i(\mathbf{q})\in \text{col}\{W\}$. Here $\text{col}\{\cdot\}$ denotes the set of column vectors of the relevant matrix. The generalized acceleration of the system in a blocked phase, is given by (23):

$$\dot{\mathbf{u}} = M^{-1}(\mathbf{q})\left(\mathbf{h}(\mathbf{q},\mathbf{u})+W_b(\mathbf{q})\,\tau_b+B(\mathbf{q})\,\tau\right). \tag{23}$$

The controls τ_b represent the forces which are required to keep the blocked directions closed. The linear operator W_b denotes the generalized force direction of the blocking forces, such that $\text{col}\{W_b\}\subset\text{col}\{B\}$ and $\gamma_b = W_b^T(\mathbf{q}(t))\,\mathbf{u}$. They can be seen as bilateral constraint forces, which are imposed on the system as a control action. After the transition, the relative acceleration in the blocked directions must be zero:

$$\dot{\gamma}_b = W_b^T\,\dot{\mathbf{u}}+\dot{W}_b^T\,\mathbf{u}=\mathbf{0}. \tag{24}$$

where the notation $\dot{a}$ denotes, the total time derivative of a. The substitution of (23) in equation (24) reveals:

$$\begin{aligned}&W_b^T\,\dot{\mathbf{u}}+\dot{W}_b^T\,\mathbf{u}=\\ &W_b^T\,M^{-1}\left(\mathbf{h}+W_b\,\tau_b+B\,\tau\right)+\dot{W}_b^T\,\mathbf{u}=\mathbf{0}.\end{aligned} \tag{25}$$

Equation (25) can be solved for the blocking forces/moments as below:

$$\begin{aligned}\tau_b &= -G_{bb}^{-1}\left(W_b^T M^{-1}\mathbf{h}+W_b^T M^{-1} B\,\tau+\dot{W}_b^T\mathbf{u}\right)\\ &= L_{bb}\left(\mathbf{h}+B\,\tau\right)-G_{bb}^{-1}\dot{W}_b^T\mathbf{u},\end{aligned} \tag{26}$$

where L_{bb} and G_{bb} are defined as given below:

$$G_{bb}=W_b^T M^{-1} W_b,\quad L_{bb}=-G_{bb}^{-1}W_b^T M^{-1}.$$

Substitution of equation (26) into (23) for τ_b reveals:

$$M\,\dot{\mathbf{u}}-\mathbf{h}-W_b L_{bb}\left(\mathbf{h}+B\,\tau\right)+W_b G_{bb}^{-1}\dot{W}_b^T\mathbf{u}-B\,\tau=\mathbf{0}. \tag{27}$$

The equations of motion after the directions W_b are blocked can be rearranged as below:

$$M\,\dot{\mathbf{u}}-(I+W_b L_{bb})\mathbf{h}-(I+W_b L_{bb})B\,\tau+W_b G_{bb}^{-1}\dot{W}_b^T\,\mathbf{u}=\mathbf{0}. \tag{28}$$

The new $\mathbf{h}$ vector as well as the matrix of generalized control directions $\mathbf{B}$ can be redefined as:

$$\mathbf{h}_b = (I+W_b L_{bb})\mathbf{h}-W_b G_{bb}^{-1}\dot{W}_b^T\mathbf{u}, \tag{29}$$
$$B_b = (I+W_b L_{bb})B, \tag{30}$$

to yield

$$M(\mathbf{q})\,\dot{\mathbf{u}}-\mathbf{h}_b(\mathbf{q},\mathbf{u})-B_b(\mathbf{q})\,\tau=\mathbf{0}. \tag{31}$$

The equations of motion for different modes can be derived by making use of the above procedure.

4. Statement of the Optimal Control Problem

The time-optimal control problem with free end-time t_f and free transition times t_i and location $\mathbf{q}(t_i)$, $\mathbf{u}(t_i^-)$, $\mathbf{u}(t_i^+)$ is considered. The assumptions during a possibly impactive transition are given as follows:

Hypotheses 2

- The transitions may be impactively.
- The generalized position remain unchanged during transition.

- The impulsive control action acts on the system at a time instant t_i which is Lebesgue-negligible.
- At a possibly impactive transition, the pre-transition controller configuration is assumed to be effective.
- There are no transitions at t_0 and t_f.

The goal function is given by:

$$\min \quad \int_{t_0}^{t_f} dt, \tag{32}$$

subject to the mechanical system dynamics stated in the first-order measure-differential equation form:

$$\begin{aligned} d\mathbf{q} &= \mathbf{y}\,dt, & (33)\\ d\mathbf{y} &= (\mathbf{f}_i(\mathbf{q}(t),\mathbf{y}(t)) + G_i(\mathbf{q}(t))\tau(t))\,dt & (34)\\ &\quad +V_i(\mathbf{q}(t))\zeta'\,d\sigma. \end{aligned}$$

The smooth dynamics of the rigidbody mechanical system is characterized in every interval of motion (t_i^+, t_{i+1}^-) by a triplet $(\mathbf{f}_i(\mathbf{q}(t),\mathbf{y}(t)), G_i(\mathbf{q}(t)), V_i(\mathbf{q}(t)))$. The vector controls τ is assumed to be constrained in a polytopic convex set denoted by $\mathscr{C}_\tau$. Here the sets are defined as below:

$$\begin{aligned} \mathscr{C}_f &= \left\{\left(\mathbf{q}(t_f),\mathbf{u}(t_f)\right) | \,\mathbf{q}(t_f) = \mathbf{q}_f, \quad \mathbf{u}(t_f) = \mathbf{u}_f\right\}\\ \mathscr{C}_\tau &= \{\tau\,|\,\tau \in \mathscr{K}, convex, polytopic\}. \end{aligned}$$

The set of transition conditions at each transition instant t_i are denoted by $\mathscr{T}_i$ are stated in terms of generalized positions $\mathbf{q}(t_i)$, and generalized post-, and pre-transition velocities $\mathbf{u}(t_i^+)$, $\mathbf{u}(t_i^-)$. The overall value function is given by:

$$\begin{aligned} J &= \Sigma_i \Psi_{\mathscr{T}_i} + \Psi_{\mathscr{C}_f} & (35)\\ &+ \int_{(t_0,t_f)} \Psi_{\mathscr{C}_\tau}\,dt + dH - \eta_1\,d\mathbf{q} - \eta_2\,d\mathbf{y} \end{aligned}$$

which is equivalent to (12) under Hypotheses (2).

Further, the differential measure of the Hamiltonian is defined as:

$$\begin{aligned} dH &= dt + \eta_1(t)\,\mathbf{y}(t)\,dt + \eta_2(t)\left(V_i(\mathbf{q}(t))\zeta'\,d\sigma\right)\\ &+ \eta_2(t)\,((\mathbf{f}_i(\mathbf{q}(t),\mathbf{u}(t)) + G_i(\mathbf{q}(t))\tau(t))\,dt) & (36)\\ &= H_t\,dt + H_\sigma\,d\sigma & (37) \end{aligned}$$

where $\eta_1(t)$ and $\eta_2(t)$ are the dual states.

Following structure for various differential measures is noted:

$$\begin{aligned} d\mathbf{q} &= \dot{\mathbf{q}}\,dt + \rho'\,d\sigma, \quad d\mathbf{y} = \dot{\mathbf{u}}\,dt + \pi'\,d\sigma,\\ d\eta_1 &= \dot{\eta}_1\,dt + \xi_1'\,d\sigma, \quad d\eta_2 = \dot{\eta}_2\,dt + \xi_2'\,d\sigma. \end{aligned}$$

The necessary conditions are derived by making use of following hypotheses:

Hypotheses 3

- the conditions of Hypotheses 1 are valid, for the continuous-part of the dynamics given in equations (33-34).
- It is assumed that $\tau(t)$ is confined to a convex set $\mathscr{C}_\tau$.
- The transition conditions are tangentially regular.
- The interior of the intersection of all sets involved in the problem considered is nonempty such that the problem is feasible in the sense of [6].
- the dual states η_1 and η_2 will be assumed left-continuous bounded variation functions ($\mathscr{LCBV}$), and the generalized velocities $\mathbf{y}$ of the mechanical system will be assumed right-continuous bounded variation functions ($\mathscr{RCBV}$), whereas the generalized positions are in class $\mathscr{AC}$.

5. Necessary Conditons in First-order Form

The value function J has some pleasant regularity properties if hypotheses (3) holds. Making use of these regularity properties, it turns out that the optimal trajectories and controls $(\mathbf{q}^*(t), \mathbf{y}^*(t), \eta_1^*(t), \eta_2^*(t), \tau^*(t)),\, t \in (t_0, t_f)$, discontinuous singular values and multiplier $({}_i\xi_1^{+*}, {}_i\xi_1^{-*}, {}_i\xi_2^{+*}, {}_i\xi_2^{-*}, \zeta_i^{-*}, \zeta_i^{+*}, \lambda)$, $\forall t_i^*$ and optimal transition locations $\left(\mathbf{q}^*(t_i), \mathbf{y}^*(t_i^-), \mathbf{y}^*(t_i^+)\right)$ and times $t_i^* \in \mathscr{I}_T$ have to fulfill the directional measure-differential equation of motion of the system, as given in equations (38-39) in the smooth parts of motion:

$$d\mathbf{q}^* = \mathbf{y}^*(t^+)\,dt \tag{38}$$

$$d\mathbf{y}^* = \left(\mathbf{f}_i\left(\mathbf{q}^*(t^+), \mathbf{y}^*(t^+)\right) + G_i(\mathbf{q}^*(t^+))\tau^*(t^+)\right)dt. \tag{39}$$

Further, the candidates of minimizers must fulfill directional differential equations for the directional costate dynamics in the non-transitional phase of motion and are element of ($\mathscr{LCBV}$) as given in equations (40-41):

$$\begin{aligned} \dot{\eta}_1^*(t^-) &= -\eta_2^*(t^-)\,\nabla_{\mathbf{q}}\mathbf{f}_i(\mathbf{q}^*(t^+), \mathbf{y}^*(t^+)) & (40)\\ &- \nabla_{\mathbf{q}}\left(\eta_2^*(t^-)\,G_i(\mathbf{q}^*(t^+))\right)\tau^*(t^+), \quad a.e.\\ \dot{\eta}_2^*(t^-) &= -\eta_1^*(t^-) - \eta_2^*(t^-)\,\nabla_{\mathbf{y}}\mathbf{f}_i(\mathbf{q}^*(t^+), \mathbf{y}^*(t^+))\\ &\qquad a.e.. & (41) \end{aligned}$$

The time-optimal control law is expected to fulfill following normal cone inclusion condition in every phase of motion:

$$-\eta_2^*(t^-)\,G_i(\mathbf{q}^*(t^+)) \in \mathscr{N}_{\mathscr{C}_\tau}(\tau^*(t^+)), \quad a.e. \quad \forall t \in \left(t_i^+, t_{i+1}^-\right). \tag{42}$$

The discontinuities in the costates η_1^* and η_2^* are supposed to fulfill some jump conditions at every transition time t_i of

Lebesgue-measure zero. The transition conditions which are expressed in terms of the pre-, and post-transition generalized velocities and generalized position are related to the discontinuities of the costate dynamics and the differential measure of controls by the following variational inequalities:

$$\Psi^{\uparrow}_{\mathscr{T}_i}\left(\cdot,\varepsilon\,\hat{\mathbf{y}}_i^{+}\right)\geq-\left({}_i\xi_2^{+*}-{}_i\xi_2^{-*}\right)\varepsilon\,\hat{\mathbf{y}}_i^{+},\qquad(43)$$

$$\Psi^{\uparrow}_{\mathscr{T}_i}\left(\cdot,\varepsilon\,\hat{\mathbf{q}}_i^{+}\right)\geq-\varepsilon\left[\left({}_i\xi_1^{*+}-{}_i\xi_1^{-*}\right)+\right.\qquad(44)$$
$$\left.\lambda\nabla_{\mathbf{q}}\left(V_i\left(\mathbf{q}(t_i^{+})\right)\left({}_i\zeta^{+*}-{}_i\zeta^{-*}\right)\right)\right]\hat{\mathbf{q}}_i^{+},$$

$$\Psi^{\uparrow}_{\mathscr{T}_i}\left(\cdot,\varepsilon\,\hat{\mathbf{y}}_i^{-}\right)\geq\quad-\left({}_i\xi_2^{+*}-{}_i\xi_2^{-*}\right)\varepsilon\,\hat{\mathbf{y}}_i^{-}.\qquad(45)$$

The notation $f^{\uparrow}(\cdot,\hat{\mathbf{v}})$ denotes the upper directional derivative of an extended-valued function f in the direction $\hat{\mathbf{v}}$ as defined in [6]. In addition the multiplier λ must fulfill:

$$\lambda\,V_i\left(\mathbf{q}(t_i^{+})\right)=0.\qquad(46)$$

The expression $\left({}_i\zeta^{+*}-{}_i\zeta^{-*}\right)$ is equivalent to the singular part of the differential measure of controls. The discontinuity in the generalized velocity of the system is expected to fulfill:

$$\pi_i^{+*}-\pi_i^{-*}=V_i(\mathbf{q}^*(t_i^{-}))\left(\zeta_i^{+*}-\zeta_i^{-*}\right),\qquad(47)$$

which is the recovered impact equation stated in first-order form. It turns out that the nonsingular part of the differential of the Hamiltonian exhibits following jump:

$$\begin{aligned}&H_t^{+}-H_t^{-}=\qquad(48)\\ -\;&\lambda\nabla_{\mathbf{q}}\left(V_i\left(\mathbf{q}^*\right)\left({}_i\zeta^{+*}-{}_i\zeta^{-*}\right)\right)-\kappa_1\dot{y}^*(t_i^{+})-\kappa_3\dot{y}^*(t_i^{-})\\ +\;&\lambda\left(\dot{y}^*(t_i^{+})-\dot{y}^*(t_i^{-})\right)-\kappa_2\left(y^*(t_i^{+})+y^*(t_i^{-})\right).\end{aligned}$$

where

$$\Psi^{\uparrow}_{\mathscr{T}_i}\left(\cdot,\varepsilon\,\hat{\mathbf{y}}_i^{+}\right)\geq\varepsilon\kappa_1\,\hat{\mathbf{y}}_i^{+},$$
$$\Psi^{\uparrow}_{\mathscr{T}_i}\left(\cdot,\varepsilon\,\hat{\mathbf{q}}_i^{+}\right)\geq\varepsilon\kappa_2\hat{\mathbf{q}}_i^{+},$$
$$\Psi^{\uparrow}_{\mathscr{T}_i}\left(\cdot,\varepsilon\,\hat{\mathbf{y}}_i^{-}\right)\geq\varepsilon\kappa_3\hat{\mathbf{y}}_i^{-}.$$

The following normal cone inclusion condition holds at the final state:

$$-\left(\eta_1^*(t_f^{-}),\eta_2^*(t_f^{-})\right)\in\mathscr{N}_{\mathscr{C}_f}(\mathbf{q}^*(t_f),\mathbf{y}^*(t_f)),\qquad(49)$$

along with the following necessary condition on the the nonsingular part of the differential of the Hamiltonian:

$$H_t\left(\mathbf{y}^*(t_f),\mathbf{q}^*(t_f),\eta_1^*(t_f),\eta_2^*(t_f),\tau^*(t_f)\right)=0.\qquad(50)$$

6. Case Study: Underactuated Robotic Manipulators with impulsively blockable DOF

In this section the transition conditions for a specific impulsive system is exemplified as a case study. In this type of systems, there are two type of transition actions, namely, blocking of some degrees of freedom and release of some blocked directions of motion. At an instant of transition both actions may concur. Let $p<r$ the number DOF which are being blocked impactively. The impact equation is given by the following expression:

$$M(\mathbf{q})\left(\mathbf{u}^{+}-\mathbf{u}^{-}\right)-W_b(\mathbf{q})\,\Gamma=\mathbf{0},\qquad(51)$$

where $\Gamma\in\mathscr{R}^p$ are blocking impulsive forces that can be generated at the joints, which participate in blocking and the matrix $W_b\in\mathscr{R}^{n\times p}$ denotes the generalized force direction of the blocking forces, such that $\mathrm{col}\{W_b\}\subset\mathrm{col}\{B\}$. Further, it is assumed that $\mathrm{col}\{B\}\subset\mathrm{col}\{W\}$ for convenience and without loss of generality. Here the notation $\mathrm{col}\{\cdot\}$ denotes the column set of the relevant linear operator. The difference between the pre-impact and post-impact relative joint contact velocities is related to the post-, and pre-impact generalized velocities of the mechanical system by expression (52):

$$\gamma^{+}-\gamma^{-}=W^T(\mathbf{q})\left(\mathbf{u}^{+}-\mathbf{u}^{-}\right).\qquad(52)$$

Let at a transition, which is accompanied by an impact, which is induced by the sudden blocking of directions of motion, p of the joints, characterized by their force directions, be active. Then, the vector γ can be decomposed in the following manner:

$$\begin{bmatrix}\gamma_b^{+}\\ \gamma_f^{+}\end{bmatrix}-\begin{bmatrix}\gamma_b^{-}\\ \gamma_f^{-}\end{bmatrix}=\begin{bmatrix}W_b\\ W_f\end{bmatrix}^T\left(\mathbf{u}^{+}-\mathbf{u}^{-}\right),\qquad(53)$$

where γ_b^{+} and γ_b^{-} denote the relative joint post-, and pre-transition velocities at the blocked/active joints, and γ_f^{+} and γ_f^{-} denote the relative joint post-, and pre-transition velocities at the free/passive joints. Here $W_b\in\mathscr{R}^{n\times p}$, $W_f\in\mathscr{R}^{n\times(n-p)}$ denote the matrices, consisting columnwise of blocked and unblocked generalized directions such that $\mathrm{col}\{W_f\}\bigcup\mathrm{col}\{W_b\}=\mathrm{col}\{W\}$ and $\mathrm{col}\{W_f\}\bigcap\mathrm{col}\{W_b\}=\emptyset$. The equation (51) can be solved for the jump in the generalized velocities of the system:

$$\mathbf{u}^{+}-\mathbf{u}^{-}=M^{-1}(\mathbf{q})\,W_b(\mathbf{q})\,\Gamma.\qquad(54)$$

Inserting this expression in (52) reveals the jump in the vector of relative joint velocity vector:

$$\gamma^{+}-\gamma^{-}=W^T\left(\mathbf{u}^{+}-\mathbf{u}^{-}\right)=W^T M^{-1}W\begin{bmatrix}\Gamma\\ \Delta\end{bmatrix}.\qquad(55)$$

The entity $\Delta\in\mathscr{R}^{n-p}$ denotes the impulsive forces at non-blocked joints. By making use of the decomposition of the relative joint velocities into blocked and free directions as introduced in equation (53), following is obtained:

$$\begin{bmatrix}\gamma_b^{+}-\gamma_b^{-}\\ \gamma_f^{+}-\gamma_f^{-}\end{bmatrix}=\begin{bmatrix}W_b^T M^{-1}W_b & W_b^T M^{-1}W_f\\ W_f^T M^{-1}W_b & W_f^T M^{-1}W_f\end{bmatrix}\begin{bmatrix}\Gamma\\ \Delta\end{bmatrix}.\qquad(56)$$

In order to simplify the notation matrices G_{bb}, G_{bf}, G_{fb} and G_{ff} are introduced as follows:

$$\begin{bmatrix} \gamma_b^+ - \gamma_b^- \\ \gamma_f^+ - \gamma_f^- \end{bmatrix} = \begin{bmatrix} G_{bb} & G_{bf} \\ G_{fb} & G_{ff} \end{bmatrix} \begin{bmatrix} \Gamma \\ \Delta \end{bmatrix}. \tag{57}$$

Noting that the impulsive control actions at joints which do not participate at the blocking and post-transition velocity at the blocked joints is zero:

$$\Delta = \mathbf{0}, \quad \gamma_b^+ = \mathbf{0}, \tag{58}$$

the impulsive control Γ can be eliminated after insertion of (58) into (57), and results in:

$$-\gamma_b^- = G_{bb}\Gamma, \gamma_f^+ - \gamma_f^- = G_{fb}\Gamma. \tag{59}$$

Eliminating the impulsive force vector, following relation between post-, and pre-impact relative joint velocities is established:

$$\gamma_f^+ = \gamma_f^- - G_{fb} G_{bb}^{-1} \gamma_b^-. \tag{60}$$

This equation can be rewritten in terms of the post-, and pre-impact generalized velocities by making use of equation (53) as given in (61):

$$W_f^T \mathbf{u}^+ = W_f^T \mathbf{u}^- - G_{fb} G_{bb}^{-1} W_b^T \mathbf{u}^-, \tag{61}$$

and by defining $K(\mathbf{q}) = \left(W_f^T - G_{fb} G_{bb}^{-1} W_b^T\right)$ reveals following expression:

$$W_f(\mathbf{q})\,\mathbf{u}^+ - K(\mathbf{q})\,\mathbf{u}^- = \mathbf{0}. \tag{62}$$

On the other hand by insertion of the impulsive force obtained in (59) into (54) reveals the relation between post- and pre-transition generalized velocities:

$$\mathbf{u}^+ - \mathbf{u}^- = -M^{-1} W_b G_{bb}^{-1} W_b^T \mathbf{u}^-. \tag{63}$$

This equation can be rewritten in the following form:

$$\mathbf{u}^+ = (I - M^{-1} W_b G_{bb}^{-1} W_b^T)\mathbf{u}^- = P(\mathbf{q})\mathbf{u}^-, \tag{64}$$

where I is the identity matrix of appropriate size. The value of impulsive force established through (59) represents the minimal value to induce full blocking at joint $i \in \mathscr{C}_B$, beyond which no difference in action will be observed, therefore this value is used in evaluating γ_f^+.

7. Necessary Transition Conditions for underactuated Manipulators with Blockable Degrees of Freedom

In this section the necessary conditions for the optimal control problem of a manipulators with blockable DOF will be presented. In the light of the analysis in section (6), the discontinuities in η_1 and η_2 shall fulfill: Further, at the transition, the discontinuities in η_1 and η_2 shall fulfill:

$$\begin{aligned} \eta_1(t_i^+) \quad - \quad & \eta_1(t_i^-) = \\ - \quad & \xi \begin{bmatrix} \nabla_{\mathbf{q}} \left(W_b^T \mathbf{y}(t_i^+)\right) \\ \nabla_{\mathbf{q}} \left(W_f^T \mathbf{y}(t_i^+)\right) - \nabla_{\mathbf{q}} \left(K \mathbf{y}(t_i^-)\right) \end{bmatrix}, \end{aligned} \tag{65}$$

and

$$\eta_2(t_i^+) - \eta_2(t_i^-) = -\xi \begin{bmatrix} W_b^T \\ G_{fb} G_{bb}^{-1} W_b^T \end{bmatrix}, \tag{66}$$

for some $\xi \in \mathscr{R}^n$. The jump in the generalized velocities shall fulfill:

$$\mathbf{y}(t_i^+) = P(\mathbf{q}(t_i))\,\mathbf{y}(t_i^-), \tag{67}$$

and

$$\mathscr{T}_i \cap \mathscr{C}_I = \left\{ \begin{pmatrix} \mathbf{q}(t_i) \\ \mathbf{y}(t_i^-) \\ \mathbf{y}(t_i^+) \end{pmatrix} \middle| \begin{bmatrix} W_b^T \mathbf{y}(t_i^+) \\ W_f^T \mathbf{y}(t_i^+) - K \mathbf{y}(t_i^-) \end{bmatrix} = \mathbf{0} \right\}.$$

At the instant of transition the jump in the value of the Hamiltonian is given by:

$$\begin{aligned} H_t^+ - H_t^- = -\xi & \left(W^T(\mathbf{q})\dot{\mathbf{y}}(t_i^+) + \begin{bmatrix} \mathbf{0} \\ -K(\mathbf{q}) \end{bmatrix} \dot{\mathbf{y}}(t_i^-) \right) \\ - \quad \xi\Omega & \left(\mathbf{y}(t_i^+) + \mathbf{y}(t_i^-)\right), \end{aligned} \tag{68}$$

where Ω is given by:

$$\Omega = \left(\nabla_{\mathbf{q}} \left(W^T(\mathbf{q})\,\mathbf{y}(t_i^+)\right) + \nabla_{\mathbf{q}} \left(\begin{bmatrix} \mathbf{0} \\ -K(\mathbf{q}) \end{bmatrix} \mathbf{y}(t_i^-) \right) \right). \tag{69}$$

8. Discussion and Conclusion

In this work, necessary conditions of strong local minimizers for the impulsive optimal control of rigidbody mechanical systems has been presented. The necessary conditions are derived by the application of subdifferential calculus to extended-valued lower-semicontinuous functionals. By making use of some regularity properties of the involved dynamics, sets and transition conditions; necessary conditions at a transition instance which can be impulsive are stated. The necessary conditions obtained enable the determination the optimal transition time and location. For the underlying non-convex problem the given conditions can only propose the candidates for minimizers, for the conditions of sufficiency further work has to be conducted. Indeed, there are two sets of necessary conditions that belong to the optimal control being considered. These two different sets of necessary conditions arise from the fact that the semigroup property is lost due to the nonreversible transitions, in general. The first set of necessary conditions can be obtained by taking the generalized positions and velocities being $\mathbf{q}(t), \mathbf{u}(t) \in \mathscr{RCBV}$, which necessitates the costates to be $\eta_1(t), \eta_2(t) \in \mathscr{LCBV}$. The second set of necessary conditions can be obtained by taking the generalized positions and velocities being $\mathbf{q}(t), \mathbf{u}(t) \in \mathscr{LCBV}$, which necessitates the costates to be $\eta_1(t), \eta_2(t) \in \mathscr{RCBV}$. These two sets of necessary conditions turn out to be identical if all possible transitions are reversible, which is naturally the case for dynamical systems with smooth vector fields.

The proposed necessary conditions are for strong local minimizers and are valid in singular intervals as well. The optimal control law as stated in equation (42) is valid in singular intervals, since the zero vector at the origin belongs to the normal cone as well. The discontinuity in the controls of a bang-bang type controller are on Lebesgue negligible intervals so the control law is valid in the "almost everywhere" sense. On the relation of impulsive optimal control and the consequences for underactuated mechanical systems further research is done.

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