MATHEMATICAL ANALYSIS OF THE ENERGY CONCENTRATION IN WAVES TRAVELLING THROUGH A RECTANGULAR MATERIAL STRUCTURE IN SPACE-TIME

Konstantin Lurie
Mathematical Sciences
Worcester Polytechnic Institute
USA
klurie@wpi.edu

Daniel Onofrei
Mathematics Department
Rutgers University
USA
onofrei@math.rutgers.edu

Abstract

We consider propagation of waves through a spatio-temporal doubly periodic material structure with checkerboard microgeometry in one spatial dimension and time. Both spatial and temporal periods in this dynamic material are assumed to be the same order of magnitude. Mathematically the problem is governed by a standard wave equation $(\rho u_t)_t - (ku_z)_z = 0$ with variable coefficients. The rectangles in a space-time checkerboard are assumed filled with materials differing in the values of phase velocities $\sqrt{\frac{k}{\rho}}$ but having equal wave impedance $\sqrt{k\rho}$. Within certain parameter ranges, the existence of distinct and stable limiting characteristic paths, i.e., limit cycles, was observed in [Lurie, Weekes 2006]; such paths attract neighboring characteristics after a few time periods. The average speed of propagation along the limit cycles remains the same throughout certain ranges of structural parameters, and this was called in [Lurie, Weekes 2006] a plateau effect. Based on numerical evidence, it was conjectured in [Lurie, Weekes 2006] that a checkerboard structure is on a plateau if and only if it yields stable limit cycles and that there may be energy concentrations over certain time intervals depending on material parameters. In the present work we give a more detailed analytic characterization of these phenomena and provide a set of sufficient conditions for the energy concentration that was predicted numerically in [Lurie, Weekes 2006].

Key words

Dynamic materials, energy accumulation, limit cycles

1 Introduction

Dynamic composites are characterized as formations assembled from materials which are distributed on a microscale both in space and time. This material concept takes into consideration the inertial, elastic, electromagnetic and other material properties that affect the dynamic behavior of various mechanical, electrical and environmental systems. In static or non-smart applications the design variables such as material density and stiffness, yield force and other structural parameters are position dependent but invariant in time. When it comes to dynamic applications, we also need temporal variability in the material properties in order to adequately match the changing environment. To this end, in dynamic material design, dynamic materials will take up the role played by classical composites in static material design. Many applications of dynamic materials, including some unusual effects demonstrated by them, are described in [Lurie, 2007] and in reference therein. A dynamic disturbance on a scale much greater than the scale of a spatio-temporal microstructure may perceive this formation as a new material with its own effective properties; for example this is achieved by laminates in space-time (see [Lurie, 1997]). This is essentially true if the energy carried by the waves or the net momentum flux are bounded; on the other hand there are spatio-temporal assemblages that may not demonstrate this property. Specifically, in [Lurie, Weekes 2006] it was observed that the energy in a checkerboard may, for certain parameter ranges, exhibit exponential growth in time. It is certainly important to better specify conditions that lead up to such growth. Particularly the growth was observed in connection with the limit cycles that arise in a checkerboard within appropriate ranges of material parameters.

2 Analytic characterization of the limit cycles and plateau zones

In this section we will certify analytically the numerical observation made in [Lurie, Weekes 2006], section 4, regarding the formation of limit cycles and the existence of the so called plateau zones for problem (1) bellow. The units of space and time are so chosen that the periods of the assemblage along the z and t axes are

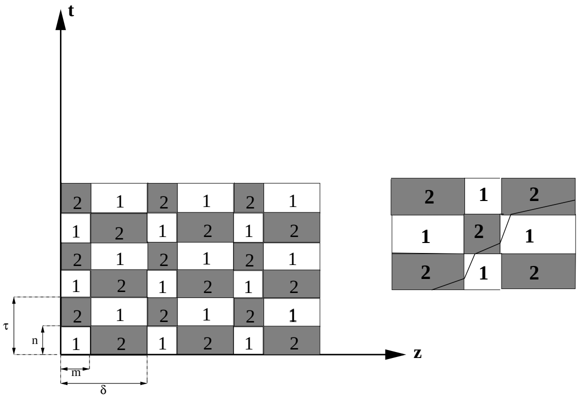

Figure 1. Space-time rectangular microstructure

dimensionless.

2.1 Analysis of the limit cycles

As in [Lurie, Weekes 2006] we will consider a doubly-periodic distribution in the (z, t)-plane, i.e., in the rectangle $(0, \delta) \times (0, \tau)$ (the periodicity cell). Material 1 occupies the region formed by the rectangles $\{(0, m) \times (0, n)\} \cup \{(m, \delta) \times (n, \tau)\}$, with $0 < m < \delta$ and $0 < n < \tau$, the rest of the cell being occupied by material 2, see Fig.1. Material i is uniform with parameters ρ_i and k_i which play the role of the density and stiffness, respectively. In this structure we consider the wave motion governed in each material by the linear second order equation

$$(\rho u_t)_t - (k u_z)_z = 0, \tag{1}$$

with ρ, k taking the values ρ_i, k_i, within material i. The above equation can be equivalently reformulated as

$$\rho u_t = v_z, \qquad k u_z = v_t. \tag{2}$$

Continuity of u and v across the material boundaries is imposed. As commonly used in the literature, $\gamma_i = \sqrt{k_i \rho_i}$ and $a_i = \sqrt{\dfrac{k_i}{\rho_i}}$ will denote, respectively, the impedance and the phase speed. We will work in the same setting as in [Lurie, Weekes 2006] and assume that the two materials have the same value of their wave impedance, i.e., $\gamma_1 = \gamma_2 \doteq \gamma$. Hence the two constituent materials will differ in their phase speeds a_i alone. Without loss of generality, we will assume throughout the paper that $a_2 > a_1$. The system (2) will then be reduced to two independent first order equations

$$R_t + a R_z = 0, \quad L_t - a L_z = 0,$$

for the Riemann invariants $R = u - \dfrac{v}{\gamma}$, $L = u + \dfrac{v}{\gamma}$. Looking at a solution $L = 0$, $R = 2u$, we consider the waves travelling in a positive z-direction. Every such wave propagating through any material gives birth to only one secondary wave that travels into adjacent material after it enters it through a horizontal or vertical material interface. Without loss of generality, we will consider the R-waves alone, assuming that $L = 0$ throughout the structure.

Let the grid on the plane be formed by horizontal lines $t = i\tau$, or $t = n + i\tau$, with $i \in \mathbb{N}$, and by vertical lines $z = i\delta$, or $z = m + i\delta$, with $i \in \mathbb{N}$.

Let $i\delta \leq z_1 \leq (i+1)\delta$ for $i \in \mathbb{N}$ fixed. We denote by $\mathcal{C}_{z_1}$ the class of all characteristic lines (trajectories) for all possible material parameters, originating at points $(0, z_1)$ on the line $\{t = 0\}$ and having the property that they first intersect a vertical line of the grid, and after a vertical intersection there follows a horizontal intersection, and vice-versa. For a given characteristic line in the class $\mathcal{C}_{z_1}$ we denote by z_j the z coordinate of the point of intersection with the j-th horizontal line of the grid defined above. The first line of the grid is the line $t = 0$. In general, it is easy to observe that depending on j the j-th horizontal line of the grid is of the form $t = \left[\dfrac{j-1}{2}\right]\tau$ if j is odd or $t = \left[\dfrac{j-1}{2}\right]\tau + n$ if j is even, where here and below, $[\cdot]$ stands for the integer part associated to any real number. It is easy to observe, from its definition, that $\{z_j\}$ form an increasing sequence. Consider also w_j to be the euclidian distance from z_j to the closest node of the grid, located to the left of z_j.

Definition 2.1. *Let $i\delta \leq z_1 \leq (i+1)\delta$ for $i \in \mathbb{N}$ and let $q \in \mathbb{N}$ with $q > 1$. The average speed V_{av}^q over q material periods of a characteristic line belonging to the class $\mathcal{C}_{z_1}$ is defined by*

$$V_{av}^q \doteq \frac{\displaystyle\sum_{i=1}^{q} \frac{z_{2i+1} - z_1}{i\tau}}{2q} + \frac{\displaystyle\sum_{i=1}^{q} \frac{z_{2i+2} - z_2}{i\tau}}{2q}.$$

Next we will define the notion of a limit cycle.

Definition 2.2. *Let $i\delta \leq z_1 \leq (i+1)\delta$ for $i \in \mathbb{N}$ fixed. A characteristic line in the class $\mathcal{C}_{z_1}$ is a limit cycle if there exist $p, q \in \mathbb{N}$ with $p \neq q$ and $p \equiv q \pmod 2$ such that*

$$w_p = w_q.$$

A limit cycle is called stable if it attracts neighboring trajectories, i.e., there exists $\epsilon > 0$ such that for any characteristic line L^ϵ in the class $\mathcal{C}_{z_1^\epsilon}$ with $|z_1^\epsilon - z_1| < \epsilon$ (if we denote by z_j^ϵ the intersection of the line L^ϵ with the j-th horizontal of the grid) we have

$$|w_j^\epsilon - w_j| \xrightarrow{j} 0.$$

where w_j, w_j^ϵ are defined as above. A limit cycle which is not stable will be called unstable.

It has been numerically observed in [Lurie, Weekes 2006] that, in the case of a square lattice, i.e., $\delta = \tau$, there will be exactly two limit cycles per material period, one stable and the other unstable. For example, for material parameters $m = 0.4, n = 0.5, a_1 = 0.6, a_2 = 1.1$, the origination points have the form $i + 0.4953$ for the stable limit cycles, and $i + 0.375$, with $i \in \mathbb{N}$ for the unstable limit cycles. We are able to completely characterize the formation of limit cycles in the class $\mathcal{C}_{z_1}$ with $i\delta \le z_1 \le (i+1)\delta$, $i \in \mathbb{N}$, their origin and their character, stable or unstable. Indeed, by using Definition 2.1, Definition 2.2, the definition of the class $\mathcal{C}_{z_1}$ and elementary arguments, we obtain

Proposition 2.3. *Recall that $a_2 > a_1$. Then*

(i) There exists a unique stable limit cycle per material period in the class $\mathcal{C}_{z_1}$, with the point of origination in material 2, i.e., $i\delta + m \le z_1 = w_1 + m + i\delta \le (i+1)\delta$ for $i \in \mathbb{N}$, and this cycle is characterized by the following conditions

$$\left.\begin{aligned} &w_1 = \frac{A_2 a_2^2}{a_2^2 - a_1^2} \\ &w_2 = a_1\left(n - \frac{\delta - m}{a_2}\right) + \frac{A_2 a_1 a_2}{a_2^2 - a_1^2} \\ &\delta - m - a_2 n \le w_1 \le \frac{a_2}{a_1} m - a_2 n + \delta - m, \\ &m - a_2(\tau - n) \le w_2 \\ &w_2 \le \frac{a_2}{a_1}(\delta - m) + m - a_2(\tau - n) \end{aligned}\right\} \tag{3}$$

where
$$A_2 = a_1\left(\tau - \frac{\delta}{a_2} + \left(\frac{a_1}{a_2} - 1\right) n + \frac{\delta - m}{a_1}\left(1 - \frac{a_1}{a_2}\right)\right).$$
Moreover, for any $j \ge 0$, w_j on the stable limit cycle are given by

$$\left.\begin{aligned} &w_{2j+1} = w_1 = \frac{A_2 a_2^2}{a_2^2 - a_1^2}, \\ &w_{2j+2} = w_2 = a_1\left(n - \frac{\delta - m}{a_2}\right) + \frac{A_2 a_1 a_2}{a_2^2 - a_1^2} \end{aligned}\right\} \tag{4}$$

Therefore the stable limit cycles originate at points $m + i\delta + \dfrac{A_2 a_2^2}{a_2^2 - a_1^2}$, for $i \in \mathbb{N}$.

(ii) There exists a unique unstable limit cycle per material period in the class $\mathcal{C}_{z_1}$, with the origination point in material 1, i.e., $i\delta \le z_1 = w_1 + i\delta \le i\delta + m$ for $i \in \mathbb{N}$, and this cycle is characterized by the following conditions

$$\left.\begin{aligned} &w_1 = \frac{A_1 a_1^2}{a_1^2 - a_2^2} \\ &w_2 = a_2\left(n - \frac{m}{a_1}\right) + \frac{A_1 a_1 a_2}{a_1^2 - a_2^2} \\ &m - a_1 n \le w_1 \le \frac{a_1}{a_2}(\delta - m) - a_1 n + m, \\ &\delta - m - a_1(\tau - n) \le w_2 \\ &w_2 \le \frac{a_1}{a_2} m + \delta - m - a_1(\tau - n) \end{aligned}\right\} \tag{5}$$

where
$$A_1 = a_2\left(\tau - \frac{\delta}{a_1} + \left(\frac{a_2}{a_1} - 1\right) n + \frac{m}{a_1}\left(1 - \frac{a_2}{a_1}\right)\right).$$
Moreover, for any $j \ge 0$, w_j on the unstable limit cycle are given by

$$\left.\begin{aligned} &w_{2j+1} = w_1 = \frac{A_1 a_1^2}{a_1^2 - a_2^2}, \\ &w_{2j+2} = w_2 = a_2\left(n - \frac{m}{a_1}\right) + \frac{A_1 a_1 a_2}{a_1^2 - a_2^2}, \end{aligned}\right\} \tag{6}$$

Therefore the unstable limit cycles originate at points $i\delta + \dfrac{A_1 a_1^2}{a_1^2 - a_2^2}$ for $i \in \mathbb{N}$.

As a corollary we observe that

Corollary 2.4. *Let $i\delta \le z_1 \le (i+1)\delta$, $i \in \mathbb{N}$. A necessary and sufficient condition for a characteristic line in class $\mathcal{C}_{z_1}$ to be a limit cycle is that the average speed V_{av}^q introduced in Definition 2.1 satisfies*

$$V_{av}^q = \frac{\delta}{\tau} \text{ for all } q \in \mathbb{N}. \tag{7}$$

2.2 Characterization of plateaux zones

Another numerical observation made in [Lurie, Weekes 2006] was that if one considers the average speed associated with a trajectory in the composite with the two phase speeds a_1, a_2 fixed, then there exist intervals of n for which the average speed is constant for a given m value; these intervals were called "plateaux" and the associated structure was referred to as "being on a plateau". It was conjectured in [Lurie, Weekes 2006], that a structure is on a plateau if and only if the structure yields stable limit cycles. Using Proposition 2.3 we can analytically describe this behavior of trajectories in the class $\mathcal{C}_{z_1}$ for $i\delta \le z_1 \le (i+1)\delta$, $i \in \mathbb{N}$. We have

Proposition 2.5. *A structure yields two limit cycles, one stable and the other unstable if and only if the structure is on a plateau, i.e., the following two pairs* *of inequalities hold simultaneously:*

$$\left.\begin{array}{l} \dfrac{a_1\tau + \left(1 - \dfrac{a_1}{a_2}\right) m - \delta}{a_1 - a_2} \leq n \leq \dfrac{a_1\tau + \left(1 - \dfrac{a_2}{a_1}\right) m - \delta}{a_1 - a_2} \\ \dfrac{m - a_2\tau + \dfrac{a_2}{a_1}(\delta - m)}{a_1 - a_2} \leq n \leq \dfrac{m - a_2\tau + \dfrac{a_1}{a_2}(\delta - m)}{a_1 - a_2} \end{array}\right\}$$

3 Conditions on material parameters necessary and sufficient for energy accumulation

The phenomenon of energy accumulation in a time-space checkerboard microstructure originally observed in [Lurie, Weekes 2006], came up as a consequence of some special kinematics of characteristics, such that the energy is periodically pumped into the wave as it travels through the checkerboard. This happens each time as the characteristics enter the material with higher phase velocity from across the horizontal interface. The energy growth then appears to be exponential, with the exponent defined as the ratio of higher/ lower phase velocity. In the case when this ratio exceeds the unity only slightly, the characteristics may (in certain circumstances analyzed below) become close to the straight lines. To preserve the energy accumulation, it is then sufficient that these lines continue to enter the higher phase velocity material from across the horizontal interface. In this section, we will give bounds for a small parameter $\mu > 0$ such that a given microstructure with parameters $a_1^\mu = \dfrac{p}{q} - \mu$ and $a_2^\mu = \dfrac{p}{q} + \mu$ will exhibit a limit cycle in the class $\mathcal{C}_{z_1}$. The results obtained will help us understand for which values of $p, q > 0$, a straight line with slope $\dfrac{p}{q}$ can be viewed as asymptotic limit ($\mu \to 0$) of the above microstructure and prescribe a set of *necessary and sufficient* conditions on the material parameters for such situations. From Proposition 2.5 applied to a_1^μ and a_2^μ we obtain

Theorem 3.1. *Suppose $\tau \neq 2n$. Let $\mu > 0$ be a small parameter. Consider the microstructure with phase velocities $a_1^\mu = \dfrac{p}{q} - \mu$ and $a_2^\mu = \dfrac{p}{q} + \mu$ respectively. Then there exists $\bar{\mu} > 0$ such that the microstructure will form a limit cycle in the class C_{z_1} for any $\mu \in (0, \bar{\mu}]$ if and only if the following two conditions are simultaneously satisfied*

$$1. \quad \frac{p}{q} = \frac{\delta}{\tau}\,, \tag{8}$$

$$\left.\begin{array}{l} 2. \quad \textit{if } \tau > 2n \;\; \textit{then } \dfrac{1}{2} < \dfrac{n}{\tau} + \dfrac{m}{\delta} \leq \dfrac{3}{2} \\ \quad\; \textit{if } \tau < 2n \;\; \textit{then } \dfrac{1}{2} \leq \dfrac{n}{\tau} + \dfrac{m}{\delta} < \dfrac{3}{2} \end{array}\right\} \tag{9}$$

If at least one of the above conditions 1., 2., is not satisfied, there exist two positive values, $0 < \mu_1 < \mu_2$, such that the microstructure will exhibit limit cycles in the class $\mathcal{C}_{z_1}$ for any $\mu \in [\mu_1, \mu_2]$.

Corollary 3.2. *If the above conditions are satisfied then the microstructure will form a limit cycle in the class C_{z_1} for all $\mu \in (0, \bar{\mu}]$ with $\bar{\mu}$ given by*

$$\bar{\mu} = \begin{cases} \min\left\{\delta\dfrac{\left(\frac{m}{\delta} + \frac{n}{\tau} - \frac{1}{2}\right)}{\frac{\tau}{2} - n}, \dfrac{\delta}{\tau}\right\} & \textit{if } \tau > 2n, \\ \min\left\{\delta\dfrac{\left(\frac{3}{2} - \frac{m}{\delta} - \frac{n}{\tau}\right)}{n - \frac{\tau}{2}}, \dfrac{\delta}{\tau}\right\} & \textit{if } \tau < 2n. \end{cases} \tag{10}$$

For the case when $\tau = 2n$ we prove

Remark 3.3. *In the case when $\tau = 2n$ no limit cycles are formed in the microstructure when $\mu \to 0$ if $C \neq 0$, where $C \doteq \dfrac{p}{q}\left(\delta - \dfrac{p}{q}\tau\right)$. On the other hand, if $C = 0$, then one always has limit cycles for arbitrarily small μ. That is, in this case, the limit cycles approach the line with slope $r = \frac{\delta}{\tau}$ as $\mu \to 0$.*

4 Numerical Verification

In this section, we provide numerical support for the theory developed herein (the results of this section belong to prof. S. L. Weekes). We use $\delta = 2,\ \tau = 3$. The first set of results investigates the checkerboard structure described by $\frac{n}{\tau} = 0.4$ and $\frac{m}{\delta} = 0.15$. Condition 2 of Theorem 3.1 is thus satisfied, and according to the Theorem and Corollary and (10), there is a critical value

$$\bar{\mu} = 0.5\,\delta/\tau = 1/3$$

such that, when $a_1 = 2/3 - \mu$ and $a_2 = 2/3 + \mu$, limit cycles with speed $\delta/\tau = 2/3$ for $\mu \in [0, \bar{\mu}]$ develop. The figures in this section show the behavior of paths of right-going information $R = u - v/\gamma$; these paths originated at uniform locations on the interval $[-2, 2]$.

Figures 2 and 3 show these paths in the cases when μ is chosen in the subcritical zone, taking values $0.4\,\delta/\tau$ and $0.499\,\delta/\tau$. It is clearly seen that the paths converge to limit cycles so that information propagates with an overall speed of $\delta/\tau = 2/3$ as predicted. When supercritical values of μ are considered, we find that no limit cycles form. Figures 4 and 5 illustrate this for $\mu = 0.51\,\delta/\tau$ and $\mu = 0.6\,\delta/\tau$.

5 Conclusion

In the above, we considered trajectories belonging with the class $\mathcal{C}_{z_1}$ introduced in subsection 2.1 . The specifics of this class is that every trajectory enters the higher phase velocity (hpv) material 2 from across the horizontal (temporal) interface, and leaves it through the vertical (static) interface (see Fig 1.). With special behavior of characteristics, the wave energy increases

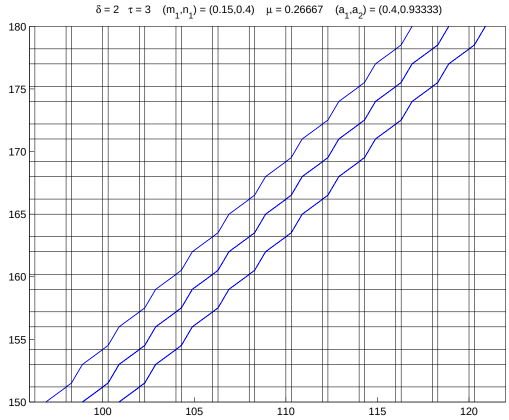

Figure 2. Characteristic paths when $\mu = 0.4\,\delta/\tau$

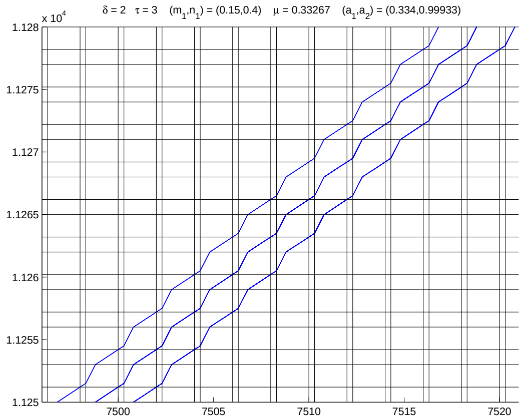

Figure 3. Characteristic paths when $\mu = 0.499\,\delta/\tau$

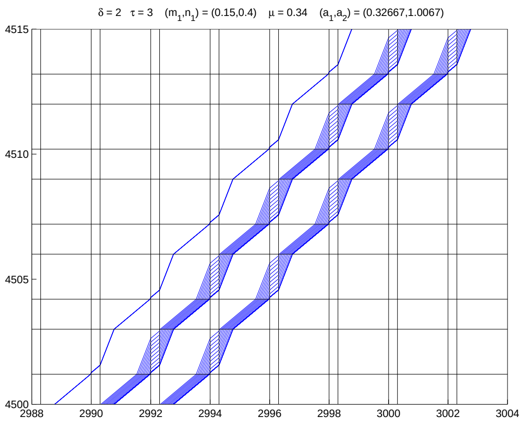

Figure 4. Characteristic paths when $\mu = 0.51\,\delta/\tau$

by the factor $\frac{a_2}{a_1}$ at each entrance into the hpv-material, and the energy flux remains continuous at each departure from this material. As a result, we create a non-stop wave energy accumulation by the factor $\left(\frac{a_2}{a_1}\right)^2$ per period. The advantage of such an arrangement is

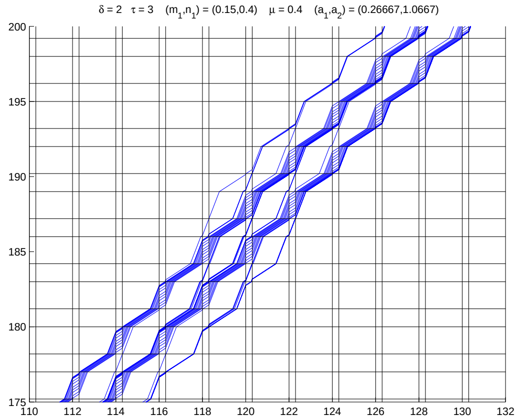

Figure 5. Characteristic paths when $\mu = 0.6\,\delta/\tau$

obvious: it avoids entrances of the characteristics into the lower phase velocity (lpv) material 1 from across the horizontal interface: every such entrance would cause the decrease of energy by the factor $\frac{a_1}{a_2}$. Instead, the characteristics enter the lpv-material through the vertical interface which does not affect the energy due to the continuity of the energy flux. The Corollary 3.2 in Section 3 establishes conditions necessary and sufficient for a microstructure with parameters $a_1^\mu = \frac{p}{q} - \mu$ and $a_2^\mu = \frac{p}{q} + \mu$ to exhibit limit cycles in the class $\mathcal{C}_{z_1}$ within the range $(0, \bar{\mu}]$ for μ. The closure of the range includes the point $\mu = 0$ which means that the line of slope $\frac{\delta}{\tau}$ may then be viewed as a limit of neighboring trajectories that approach it as $\mu \to 0$; at the same time the energy blows up for all such trajectories with $\mu \neq 0$, but in the limit (when $\mu \to 0$) the energy does not approach the value associated with the line $\frac{\delta}{\tau}$. This is another reason for which homogenization as classically understood is not possible for this problem. Instead, the study of the limit behavior of the characteristics appears to be the only instrument through which one can gain information about the wave propagation through the checkerboard structures.

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