IS CHAOS A ROUTE TO COLLAPSE ?

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Abstract. *This paper deals with the dynamics of a single-degree-of-freedom unilateral damage oscillator. Using appropriate internal variables, the hysteretic dynamic system can be written as a non-smooth autonomous system. Free dynamics of such a non-linear system are simply reduced to periodic motion, eventually attractive trajectory and divergence motion. The natural frequency of this system depends on the stationary value of the damage internal variable. Nevertheless, the inelastic forced oscillator can exhibit very complex phenomena. When the damage parameter remains stationary, dynamics is similar to the one of an elastic oscillator with unsymmetrical stiffness. Dynamics appears to be controlled by the initial perturbations. Moreover, chaotic motions may appear in such a system, for severe damage values. Chaos can be understood as a route to collapse.*

Keywords: Damage oscillator, Non-linear dynamics, Seismic design, Concrete structures, Chaos, Unilateral effect.

1. INTRODUCTION

Dynamics of inelastic systems (plastic or damage systems) is a recent research field, essentially because these systems are non-smooth dynamical systems. Considering only single-degree-of-freedom system, most studies are devoted to plastic oscillator in the literature. Complex periodic motions have been found using numerical simulations of such plastic oscillator (Challamel, 2005; Challamel and Gilles, 2006). Limit cycles have been highlighted for the free undamped kinematic-hardening system (Pratap et al, 1994). The same oscillator solicited by a periodical (but not harmonic) pulsation shows very rich dynamical phenomena, and sometimes chaotic motion (Pratap and Holmes, 1995). Coupling of material and geometrical non-linearities can also lead to chaotic motion (Poddar et al, 1988). The contribution of damage in the dynamics response can also be predominant. Dynamics of concrete structures can be studied with a single-degree-of-freedom inelastic damage oscillator. The free dynamics of the softening damage oscillator has shown stationary periodic motion in a given perturbation domain (Challamel and Pijaudier-Cabot, 2004) (see also Challamel and Pijaudier-Cabot, 2006 for the softening plastic oscillator). Dynamics of the forced damage oscillator (without unilateral effect) is studied by DeSimone et al (2001). Chaotic phenomenon has been found in a fatigue-testing rig involving crack closure effect (Foong et al, 2003). The present study shows that dynamics of an unilateral damage oscillator may be chaotic, for severe damage values. Chaos can be understood as a route to collapse.

2. EQUATIONS OF MOTION

Consider the single-degree-of-freedom inelastic oscillator shown in Figure 1. A mass M is attached to a damage spring. The inelastic system is externally excited by a harmonic force.

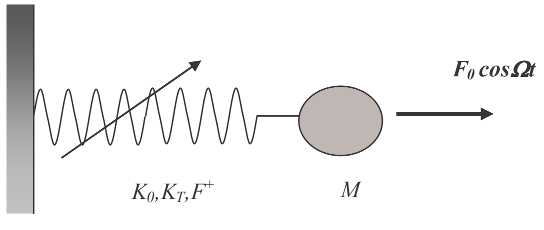

Figure 1 – The physical system

This oscillator is characterised by the displacement U, the displacement rate $\dot{U}$ and an additional internal variable for the inelastic damage process, namely the damage variable. This variable, classically denoted by D, characterises the induced microcracking of the oscillator in tension. It varies between 0 (initial virgin state) and 1 (at failure). The damage incrementally law is given on Figure 2.

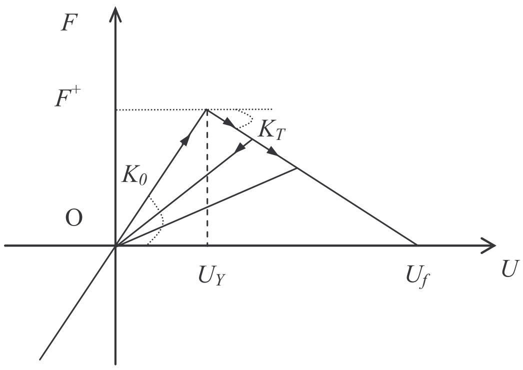

Figure 2 – Damage incremental law for the inelastic spring

A linear softening is assumed. This law depends on three parameters: the initial stiffness K_0, the tangent stiffness K_T which rules the damage evolution, and the maximum force F^+. In the case of softening process considered in the paper, the tangent stiffness is negative. Concrete material has essentially unsymmetrical behaviour in traction and compression. For this reason, it is assumed that no damage prevails in the compression zone and the model is clearly unilateral (the reader is reported to Mazars et al, 1990, or Challamel et al, 2005 for a wide description of this phenomenon). The materials parameters of the model may be easily expressed in terms of characteristic displacements: U_Y is the maximum displacement of the initial elastic domain, and U_f is the displacement at failure:

$$U_Y = \frac{F^+}{K_0} \; ; \; \frac{U_f}{U_Y} = 1 - \frac{K_0}{K_T} \tag{1}$$

The damage variable D can directly be expressed as a function of the memory variable V, defined by:

$$V(t) = \max_t U(t) \tag{2}$$

The relation between D and V is given by:

$$D = \left\langle 1 + \frac{K_T}{K_0}\left\langle -1 - \frac{K_0 - K_T}{K_T}\frac{U_Y}{V}\right\rangle\right\rangle \text{ with}$$

$$\langle x \rangle = \frac{x + |x|}{2} \tag{3}$$

It is easy to verify that the rate of damage is necessarily positive, and then, the classical thermodynamics inequality of such scalar damage model is verified (Mazars and Pijaudier-Cabot, 1996).

$$\dot{D} \geq 0 \tag{4}$$

Three dynamics can be distinguished. These three states correspond to a reversible state (or elastic state) in the tension domain $\hat{E}^+$, a reversible state in the compression domain $\hat{E}^-$ and a irreversible state $\hat{D}$ (necessarily in the tension domain) associating to damage evolution. Dynamics of the undamped system is then written as:

$$\left| \begin{array}{l} \hat{E}^+ : M\ddot{U} + K_0\big(1 - D(V)\big)U = F_0 \cos \Omega t; \; \dot{D} = 0 \\ \hat{E}^- : M\ddot{U} + K_0 U = F_0 \cos \Omega t; \; \dot{D} = 0 \\ \hat{D} : M\ddot{U} + \left\langle K_T\left(U - U_f\right)\right\rangle = F_0 \cos \Omega t; \; \dot{V} = \dot{U} \end{array} \right. \tag{5}$$

Each state is defined from a partition of the phase space:

$$\left| \begin{array}{l} \hat{E}^+ : \left(U > 0 \; or \; \left(U = 0 \; and \; \dot{U} \geq 0\right)\right) \; and \\ \left[\left(\dot{U} \leq 0\right) or \left(\dot{U} \geq 0 \; and \; U < V\right) or \left(V < U_Y\right)\right] \\ \hat{E}^- : \left(U < 0 \; or \; \left(U = 0 \; and \; \dot{U} \leq 0\right)\right) \\ \hat{D} : \left(\dot{U} > 0\right) and \left(U = V\right) and \left(V \geq U_Y\right) \end{array} \right. \tag{6}$$

One recognises in Eq. (5) and Eq. (6) a piecewise linear oscillator (see for instance Shaw and Holmes, 1983). The dimensionless phase variables are defined by:

$$(u, \dot{u}, v) = \left(\frac{U}{U_Y}, \frac{\dot{U}}{U_Y}, \frac{V}{U_Y}\right) \; ; \; v = \max_t u(t) \tag{7}$$

New temporal derivatives are written directly with respect to the dimensionless time parameter:

$$\tau = \frac{t}{t^*} \qquad \text{with} \qquad t^* = \sqrt{\frac{M}{K_0}} \tag{8}$$

The new dynamical system reads for dimensionless variables:

$$\left|\begin{array}{l}\hat{E}^{+}: \ddot{u}+(1-D(v))u = f_0 \cos\omega\tau;\ \dot{D}=0 \\ \hat{E}^{-}: \ddot{u}+u = f_0 \cos\omega\tau;\ \dot{D}=0 \\ \hat{D}: \ddot{u}+\left\langle \frac{K_T}{K_0}(u-1)+1 \right\rangle = f_0 \cos\omega\tau;\ \dot{v}=\dot{u}\end{array}\right.$$

$$\text{with } f_0 = \frac{F_0}{F^{+}} \text{ and } \omega = \Omega t^{*} \tag{9}$$

The damage function depends on the new memory dimensionless variable v:

$$D = \left\langle 1+\frac{K_T}{K_0}\left\langle -1-\frac{K_0-K_T}{K_T}\frac{1}{v}\right\rangle\right\rangle \tag{10}$$

The three states are now governed by:

$$\left|\begin{array}{l}\hat{E}^{+}: (u>0 \ or\ (u=0 \ and\ \dot{u}\geq 0)) \\ and\ [(\dot{u}\leq 0)\ or\ (\dot{u}\geq 0 \ and\ u<v)\ or\ (v<1)] \\ \hat{E}^{-}: (u<0 \ or\ (u=0 \ and\ \dot{u}\leq 0)) \\ \hat{D}: (\dot{u}>0)\ and\ (u=v)\ and\ (v\geq 1)\end{array}\right. \tag{11}$$

For $f_0 = 0$ (free vibrations), the dynamics system is an autonomous system with a three-dimensional phase space associated to the co-ordinates $(u, \dot{u}, v)$. The periodically forced oscillator $(f_0 \neq 0)$ can be studied using an extended four-dimensional phase space with co-ordinates $(u, \dot{u}, v, \tau)$. Local solutions of Eq. (9) are known explicitly for each state (see for instance Challamel and Pijaudier-Cabot, 2007).

Piecing together these known solutions is not directly possible however, since the times of flight in each region (each state) cannot be found in closed form in the general case. The time which characterises the transition between each state, is computed from a Newton-Raphson procedure. Note that this solution is considerably more accurate than the usual numerical solutions of ordinary differential equations, the only approximations being made at the boundary of each state.

3. FREE VIBRATIONS

Dynamics of such a free inelastic system can be reduced to periodical regime (waiting for a certain time), attractive or divergent trajectories (see also Challamel and Pijaudier-Cabot, 2004-2006). These three cases are distinguished by the level of initial perturbation in the neighbouring of the origin point. For sufficient large perturbations, the motion diverges. On the opposite, for sufficiently small perturbations, the motion is described by both a circular (in the compression domain) and a elliptic (in the tension domain) periodic trajectory after a damage phase. The intermediate trajectory, represented on Figure 3, is an attractive trajectory. It asymptotically converges towards a fixed point. This attractive trajectory is structurally unstable. It is in fact the limit of the perturbations domain generating bounded evolutions and also the limit of the domain associated to stability of the origin (in the sense of Lyapunov). This domain is defined by:

$$u_0^{\,2}+\dot{u}_0^{\,2} \leq \frac{U_f}{U_Y} \text{ for } u_0 \leq v_0 \leq 1 \tag{12}$$

Moreover, when the motion is periodic, the damage reaches a stationary value denoted by $\overline{D}$. The global pulsation of this periodic motion can be obtained in closed-form solution from:

$$\overline{\omega} = \frac{2\sqrt{1-\overline{D}}}{1+\sqrt{1-\overline{D}}} \tag{13}$$

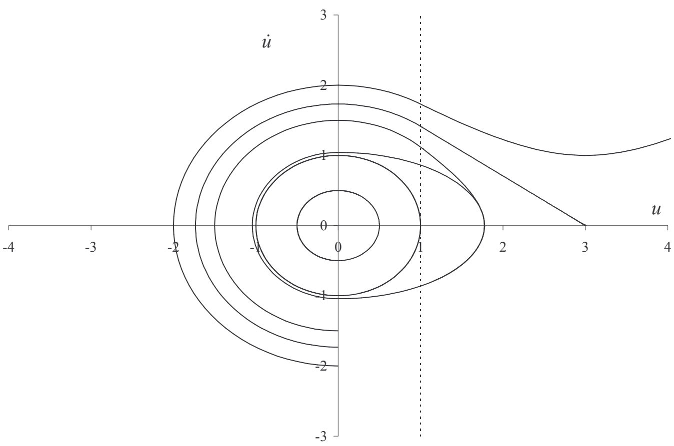

Figure 3 – Dynamics of the free damage system; $U_f / U_Y = 3$

4. FORCED VIBRATIONS

Numerical simulations show that two types of behaviour may be observed for such system, namely the shakedown phenomenon (damage shakedown means that $\dot{D}=0$ after a critical time), and the collapse characterised by a divergent evolution (in such a case, failure is reached and D

is equal to unity). The theoretical analysis consists in treating the bounded dynamics (in case of damage shakedown) as an equivalent elastic oscillator after a critical time. In this case, the extended four-dimensional phase space with co-ordinates $(u, \dot{u}, v, \tau)$ can be reduced to a three-dimensional phase space with co-ordinates $(u, \dot{u}, \tau)$. The new oscillator is an elastic oscillator with different stiffness in tension and compression. Results of Shaw and Holmes (1983), Thompson et al (1983) or Mahfouz and Badrakhan (1990) can be used for the dynamics of the oscillator studied in the three-dimensional phase space. The vector field defined by Eq. (5) is easily seen to be $2\pi/\omega$ periodic in τ. The Poincaré section is useful to investigate properties of the dynamics system. The value of v_0 (or in an equivalent way, the initial damage value D_0) has been varied, in order to investigate the damage effect. Periodic, quasiperiodic, chaotic and divergence behaviours have been observed (Figure 4).

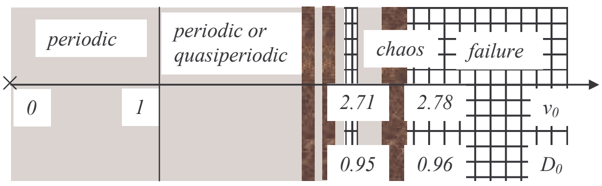

Figure 4 – Dynamics of the forced damage system; $U_f / U_Y = 3$; $f_0 = 0.05$; $\omega = 0.2$; $u_0 = 0$; $\dot{u}_0 = 0$

Quasiperiodic motions have been found for sufficiently weak damage system ($v_0 > 1$ or $D_0 \neq 0$) (see Figure 5).

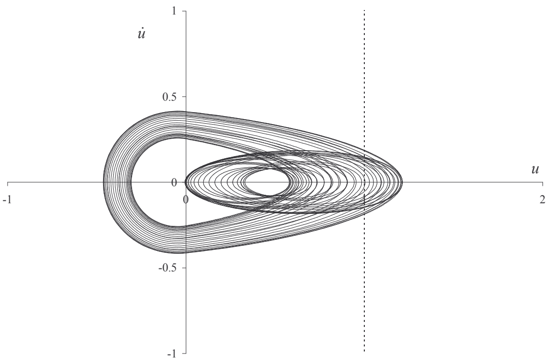

Figure 5 – Quasiperiodic motion; phase portrait; $v_0 = 2.70$ or $D_0 = 0.944$

The quasiperiodic nature of the motion is checked in the Poincaré map of Figure 6.

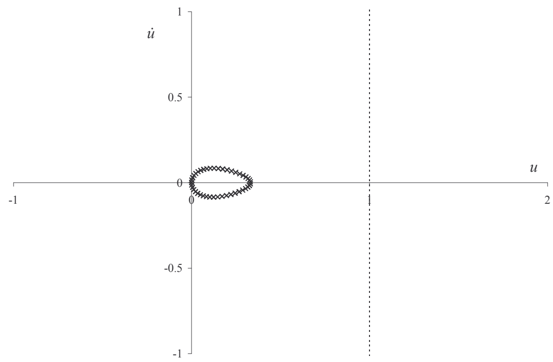

Figure 6 – Quasiperiodic motion; Poincaré map; $v_0 = 2.70$ or $D_0 = 0.944$

On simulation of Figure 7 ($v_0 = 2.65$ or $D_0 = 0.934$), the damage shakedown does not succeed and failure is reached after several cycles ($\overline{D} = 1$).

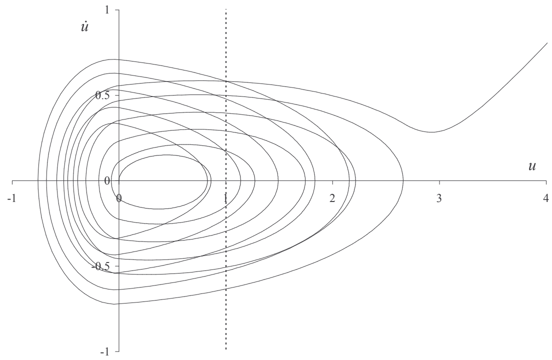

Figure 7 – Divergence motion; Phase portrait; $v_0 = 2.65$ or $D_0 = 0.934$

For $v_0 = 2.71$ ($D_0 = \overline{D} = 0.946$), chaotic vibrations can be seen in Figure 8, and a "strange attractor" is more specifically highlighted in the Poincaré map of Figure 9 (even if the system is undamped). Chaotic vibrations have been also observed for higher damage values ($v_0 \in [2.71; 2.78]$ or $D_0 \in [0.946; 0.960]$) or smaller damage values ($v_0 \in [2.26; 2.32]$ or $D_0 \in [0.836; 0.853]$; $v_0 \in [2.55; 2.58]$ or $D_0 \in [0.912; 0.919]$). These intermittent characteristic damage parameters are close to *1*, that is close to the failure value. The "strange attractor" possesses symmetry property in the Poincaré map, with respect to the *u*-axis.

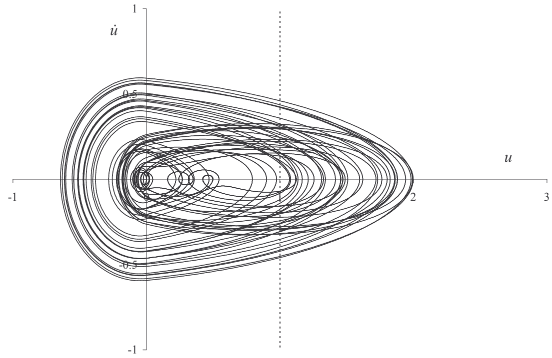

Figure 8 – « Chaotic » motion; Phase portrait;
$v_0 = 2.71$ or $D_0 = 0.946$

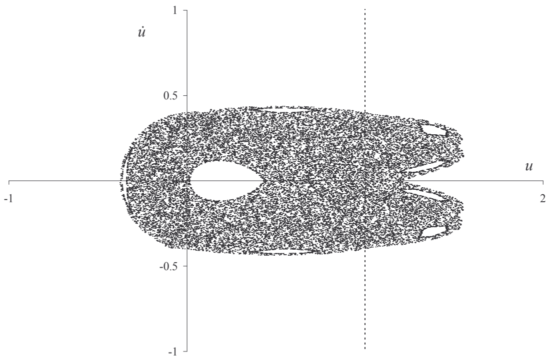

Figure 9 – « Chaotic » motion; Poincaré map;
$v_0 = 2.71$ or $D_0 = 0.946$

Mahfouz and Badrakhan (1990) also show that chaos can appear for large stiffness ratio. The asymptotic case is the obstacle case, where the stiffness ratio vanishing (case treated by Thompson et al, 1983 for instance). The main phenomena exhibited in this paper, may be also observed for a weakly damped system, whose damping ratio is denoted by ζ (Figure 10).

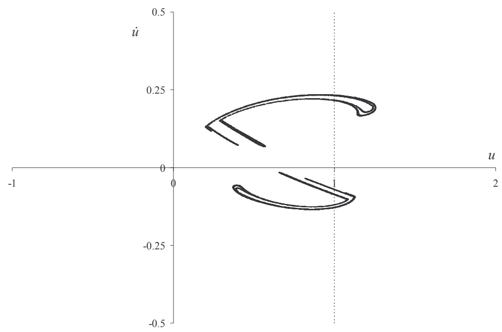

Figure 10 – « Chaotic » motion; Poincaré map;
$v_0 = 2.77$ or $D_0 = 0.958$; $\zeta = 0.01$

It is worth mentioning that the strange attractor associated to the damaged system is very analogous to the Hénon's attractor (see for instance Thompson, 1982). The bifurcation diagram can be presented in Figure 11.

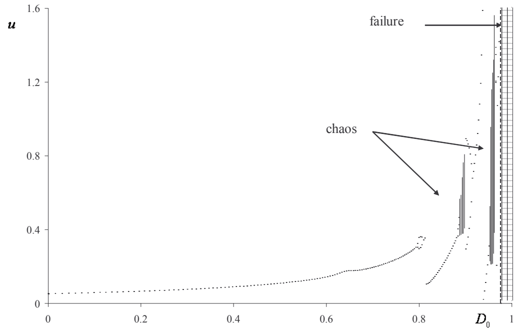

Figure 11 – Bifurcation diagram in the (D_0, u) space – Damped system - $\zeta = 0.01$

For the simulations considered, chaos has been found for large damage values: chaos can be considered as a route to collapse.

5. CONCLUSIONS

- This paper deals with the stability of a single-degree-of-freedom damage softening oscillator. For seismic design applications, a critical energy has been introduced (induced by seismic solicitation for instance) that the oscillator can support in order to remain stable.
- Periodic, quasi-periodic, chaotic, divergence motions have been numerically observed for the forced damage oscillator. Damage shakedown is firmly controlled by initial conditions. In this case, the stationary behavior of such inelastic oscillator is the same that for an elastic oscillator with different stiffness in tension and in compression. One of the specificity of the inelastic system considered in the paper is that chaotic behavior is strongly governed by the perturbations considered.
- These surprising results probably mean that the dynamics collapse of concrete structures can be controlled by chaotic phenomenon. This complex behavior is firmly linked to the breaking of symmetry of the constitutive law including unilateral effect. It is quite surprising at this stage that breaking of strength symmetry does not lead to the same

conclusion for an elastoplastic oscillator, where only periodic evolutions have been found (Challamel et al, 2007).

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