ON SOME NORMS OF LINEAR DISCRETE-TIME PERIODIC SYSTEMS

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Abstract: This paper focuses on the peak to peak norm and the generalized H_2 norm of linear discrete-time periodic systems. First, sufficient conditions based on matrix inequalities are given. Then, based on them, the disturbance attenuation problem is taken as an example to show the application of the derived results.

Keywords: linear discrete-time periodic systems, robustness, peak to peak norm, generalized H_2 norm, matrix inequality.

1. INTRODUCTION

Periodic system theory has been under continuous development in the last decades. While different norms have been proposed for characterizing the robustness of linear time-invariant systems (Scherer *et al.*, 1997), in the literature on periodic systems the most attention has been focused on the H_2 norm and the H_∞ norm. In the frequency domain, the parametric transfer function (Lampe and Rossenwasser, 2004; Lampe *et al.*, 2005), the harmonic analysis (Zhou and Hagiwara, 2002) and the lifting based methods (Varga, 2006) have been proposed. In the time domain, Riccati equation, Riccati inequality (Xie and de Souza, 1991; Colaneri, 2000) and linear matrix inequality (LMI) based approaches (Bittanti and Cuzzola, 2001*a*; Bittanti and Cuzzola, 2001*b*) have been presented.

As it is often difficult to get an exact model of the system, model uncertainties present one of the challenges in control and estimation problems. In (Xie and de Souza, 1991), norm-bounded uncertainty has been considered in the H_∞ filtering of the linear continuous-time periodic systems. Both norm-bounded uncertainty and polytopic uncertainty have been discussed by (Souza and Trofino, 2000) regarding the stabilization problem of the linear discrete-time periodic (LDP) systems. More recently, the H_2 norm of the LDP systems with polytopic uncertainties has been investigated in (Farges *et al.*, 2007).

In this paper, we shall focus on the **peak to peak norm** and the **generalized H_2 norm** of the LDP systems. We shall also consider the influence of norm-bounded and polytopic model uncertainties on these norms. To the authors' knowledge, the generalized H_2 norm of the LDP systems has not yet been considered. The peak to peak norm has been handled by (Voulgaris, 1996; Aubrecht and Voulgaris, 2001). In (Voulgaris, 1996), an estimator that minimizes the peak to peak norm from the disturbances to the estimation errors is designed using the lifting and model matching technique. A state feedback controller is synthesized in (Aubrecht and Voulgaris, 2001) to minimize the peak to peak norm of the closed-loop system by constructing a periodic controlled invariance kernel. In comparison, the solutions presented in this paper employ the matrix inequality technique that can be easily implemented with the help of the standard MATLAB toolbox. This work is motivated by our recent study on networked control systems (NCS), in which the theory of periodic systems can be applied from different viewpoints (Ding *et al.*, 2006; Zhang and Ding, 2006).

This paper is organized as follows. After the definition of the norms in Section 2, LMI characterizations of the norms will be given in Section 3. We include here also the extended LMI characterizations obtained by using the relaxation variable technique introduced in (de Oliveira *et al.*, 1999). Then, norm-bounded uncertainty and polytopic uncertainty will be considered in Section 4 and 5, respectively. Finally, in Section 6 we shall consider the control and estimation problem to meet the specifications on robustness.

2. DEFINITIONS

In this paper, we consider the LDP system Σ described by

$$\begin{aligned} x(k+1) &= A_k x(k) + B_k d(k) \\ y(k) &= C_k x(k) + D_k d(k) \end{aligned} \quad (1)$$

where $x \in \mathbf{R}^n$ denotes the state vector, $d \in \mathbf{R}^{n_d}$ the input vector, $y \in \mathbf{R}^m$ the output vector, A_k, B_k, C_k, D_k are known real periodic matrices of period T, i.e., $\forall k$,

$$\begin{bmatrix} A_{k+T} & B_{k+T} \\ C_{k+T} & D_{k+T} \end{bmatrix} = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$$

The peak to peak norm of the LDP system (1) is the induced norm with both the input signal and the output signal measured by the maximal amplitude, i.e.,

$$\|\Sigma\|_{peak} = \sup_{d \in l_\infty, d \neq 0} \frac{\|y\|_\infty}{\|d\|_\infty} \quad (2)$$

where $\|y\|_\infty = \sup_k \sqrt{y'(k)y(k)}$ and $\|d\|_\infty = \sup_k \sqrt{d'(k)d(k)}$ denote the l_∞-norm of the input and output signals, respectively.

The generalized H_2 norm of the LDP system (1) is the induced norm with the input signal measured by the energy and the output signal measured by the maximal amplitude, i.e.,

$$\|\Sigma\|_g = \sup_{d \in l_2, d \neq 0} \frac{\|y\|_\infty}{\|d\|_2} \quad (3)$$

where $\|d\|_2 = \sqrt{\sum_{k=0}^{\infty} y'(k)y(k)}$ denotes the l_2-norm of the input signal. Note that $\|G\|_g$ is finite only if $D_k = 0$.

3. LMI CHARACTERIZATIONS

Theorem 1 (Peak to peak norm) Given the LDP system Σ described by (1) with zero initial conditions and a real number $\beta > 0$, then Σ is stable and $\|\Sigma\|_{peak} < \beta$, if there exist a T-periodic matrix $P_k > 0$ and T-periodic real numbers $\lambda_k > 0$ and μ_k, such that

$$\begin{bmatrix} -P_k + \lambda_k P_k & O & A_k' P_{k+1} \\ O & -\mu_k I & B_k' P_{k+1} \\ P_{k+1} A_k & P_{k+1} B_k & -P_{k+1} \end{bmatrix} < 0 \quad (4)$$

$$\begin{bmatrix} \lambda_k P_k & 0 & C_k' \\ 0 & (\beta - \mu_k) I & D_k' \\ C_k & D_k & \beta I \end{bmatrix} > 0 \quad (5)$$

Proof: Define a periodic Lyapunov function for the LDP system (1) as

$$V(x(k)) = x'(k) P_k x(k) \quad (6)$$

where $P_{k+T} = P_k > 0$. From Schur lemma, (4) is equivalent to

$$\Psi_k = \begin{bmatrix} A_k' P_{k+1} A_k - P_k + \lambda_k P_k & A_k' P_{k+1} B_k \\ B_k' P_{k+1} A_k & B_k' P_{k+1} B_k - \mu_k I \end{bmatrix} < 0$$

From $P_k > 0$ and $A_k' P_{k+1} A_k - P_k < 0$, it is clear that Σ is stable. Moreover,

$$\begin{aligned} &V(x(k+1)) - V(x(k)) + \lambda_k V(x(k)) - \mu_k d'(k) d(k) \\ &= \begin{bmatrix} x'(k) & d'(k) \end{bmatrix} \Psi_k \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} < 0, \quad \forall x, d \end{aligned}$$

which means that

$$\lambda_k V(x(k)) - \mu_k d'(k) d(k) < 0, \quad \forall x, d$$

As (5) is equivalent to

$$\begin{bmatrix} \lambda_k P_k & 0 \\ 0 & (\beta - \mu_k) I \end{bmatrix} > \beta^{-1} \begin{bmatrix} C_k' \\ D_k' \end{bmatrix} \begin{bmatrix} C_k & D_k \end{bmatrix}$$

there is

$$\begin{aligned} &y'(k)y(k) \\ &< \beta \begin{bmatrix} x'(k) & d'(k) \end{bmatrix} \begin{bmatrix} \lambda_k P_k & 0 \\ 0 & (\beta - \mu_k) I \end{bmatrix} \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} \\ &= \beta^2 d'(k) d(k) + \beta \left(\lambda_k V(x(k)) - \mu_k d'(k) d(k) \right) \\ &< \beta^2 d'(k) d(k) \end{aligned}$$

Thus, $\|G\|_{peak} < \beta$. ■

Theorem 2 (Generalized H_2 norm) Given the LDP system Σ described by (1) with $D_k = 0$ and zero initial conditions and a real number $\gamma > 0$, then Σ is stable and $\|\Sigma\|_g < \gamma$, if and only if there exist a T-periodic matrix $P_k > 0$ such that

$$\begin{bmatrix} -P_k & O & A_k' P_{k+1} \\ O & -I & B_k' P_{k+1} \\ P_{k+1} A_k & P_{k+1} B_k & -P_{k+1} \end{bmatrix} < 0 \quad (7)$$

$$\begin{bmatrix} P_k & C_k' \\ C_k & \gamma^2 I \end{bmatrix} > 0 \quad (8)$$

Proof: Assume that (6) is a periodic Lyapunov function of the LDP system (1). Note that (7) is equivalent to

$$\Psi_k = \begin{bmatrix} A_k' P_{k+1} A_k - P_k & A_k' P_{k+1} B_k \\ B_k' P_{k+1} A_k & B_k' P_{k+1} B_k - I \end{bmatrix} < 0$$

As $P_k > 0$ and $A'_k P_{k+1} A_k - P_k < 0$, it implies that Σ is stable. Moreover, (7) holds if and only if

$$\begin{aligned} & V(x(k+1)) - V(x(k)) - d'(k)d(k) \\ &= \begin{bmatrix} x'(k) & d'(k) \end{bmatrix} \Psi_k \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} < 0, \quad \forall x, d \end{aligned}$$

which means that

$$V(x(k)) < \sum_{j=0}^{k-1} d'(j)d(j), \quad \forall x, d$$

Taking (8) into account, we have

$$y'(k)y(k) < \gamma^2 V(x(k)) < \gamma^2 \sum_{j=0}^{k-1} d'(j)d(j)$$

i.e., $\|G\|_g < \gamma$. ■

If we introduce the so-called relaxation variables (de Oliveira *et al.*, 1999) to decouple the Lyapunov variable from the system matrices to provide later more design freedom, then we obtain theorems 3-4 as follows.

Theorem 3 (Peak to peak norm) Given the LDP system Σ described by (1) with zero initial conditions and a real number $\beta > 0$, then Σ is stable and $\|\Sigma\|_{peak} < \beta$, if there exist T-periodic matrices $P_k > 0, G_k$, and T-periodic real numbers $\lambda_k > 0, \mu_k$, so that (5) and the following matrix inequality hold

$$\begin{bmatrix} -P_k + \lambda_k P_k & O & A'_k G'_k \\ O & -\mu_k I & B'_k G'_k \\ G_k A_k & G_k B_k & P_{k+1} - G_k - G'_k \end{bmatrix} < 0 \tag{9}$$

Proof: Assume that (5) and (9) hold for some $P_k > 0, G_k, \lambda_k > 0, \mu_k$. Pre- and postmultiplying (9) by Γ_k and Γ'_k, respectively, where

$$\Gamma_k = \begin{bmatrix} I & O & A'_k \\ O & I & B'_k \end{bmatrix}$$

As Γ_k is a matrix of full row rank, we get (4), i.e., the same matrices $P_k > 0, \lambda_k > 0, \mu_k$ satisfy (4)-(5). Recalling Theorem 1, the LDP system (1) is stable and its peak to peak norm is smaller than β. Indeed, if (4)-(5) hold for some $P_k > 0, \lambda_k > 0$ and μ_k, then (5) and (9) are satisfied by the same $P_k > 0, \lambda_k > 0, \beta_k, \mu_k$ and $G_k = P_{k+1}$. That means, the conditions (5) and (9) are equivalent with (4)-(5). ■

Theorem 4 (Generalized H_2 norm) Given the LDP system Σ described by (1) with $D_k = 0$ and zero initial conditions and a real number $\gamma > 0$, then Σ is stable and $\|\Sigma\|_g < \gamma$, if and only if there exist T-periodic matrices $P_k > 0$ and G_k, so that (8) and the following LMI hold

$$\begin{bmatrix} -P_k & O & A'_k G'_k \\ O & -I & B'_k G'_k \\ G_k A_k & G_k B_k & P_{k+1} - G_k - G'_k \end{bmatrix} < 0 \tag{10}$$

The proof is similar with that of Theorem 3 and thus omitted here.

In the following, we give equivalent conditions for the peak-to-peak norm and the generalized H_2 norm, respectively. These conditions can be readily used for the design of state feedback controllers, while the conditions presented above are more convenient for the observer design.

Lemma 1 (Peak to peak norm) Given the LDP system Σ described by (1) with zero initial conditions and a real number $\beta > 0$, then Σ is stable and $\|\Sigma\|_{peak} < \beta$, if any of following conditions holds:

(i) there exist a T-periodic matrix $Q_k > 0$ and T-periodic real numbers $\lambda_k > 0, \mu_k$, such that

$$\begin{bmatrix} -Q_k + \lambda_k Q_k & O & Q_k A'_k \\ O & -\mu_k I & B'_k \\ A_k Q_k & B_k & -Q_{k+1} \end{bmatrix} < 0 \tag{11}$$

$$\begin{bmatrix} \lambda_k Q_k & 0 & Q_k C'_k \\ 0 & (\beta - \mu_k) I & D'_k \\ C_k Q_k & D_k & \beta I \end{bmatrix} > 0 \tag{12}$$

(ii) there exist T-periodic matrices $Q_k > 0, G_k$, and T-periodic real numbers $\lambda_k > 0, \mu_k$, such that

$$\begin{bmatrix} (\lambda_k - 1)(G_k + G'_k - Q_k) & O & G'_k A'_k \\ O & -\mu_k I & B'_k \\ A_k G_k & B_k & -Q_{k+1} \end{bmatrix} < 0 \tag{13}$$

$$\begin{bmatrix} \lambda_k (G_k + G'_k - Q_k) & 0 & G'_k C'_k \\ 0 & (\beta - \mu_k) I & D'_k \\ C_k G_k & D_k & \beta I \end{bmatrix} > 0 \tag{14}$$

Proof: At first, we prove that condition (i) is sufficient for $\|\Sigma\|_{peak} < \beta$. Assume that (11)-(12) are satisfied by some $Q_k > 0, \lambda_k > 0, \mu_k$. Let

$$P_k = Q_k^{-1}, T_{1k} = \begin{bmatrix} P_k & O & O \\ O & I & O \\ O & O & P_{k+1} \end{bmatrix}, T_{2k} = \begin{bmatrix} P_k & O & O \\ O & I & O \\ O & O & I \end{bmatrix}$$

Pre- and postmultiplying (11) by T_{1k} and T'_{1k}, respectively, yields (4). Similarly, pre- and postmultiplying (12) by T_{2k} and T'_{2k} yields (5). According to Theorem 1, Σ is stable and $\|\Sigma\|_{peak} < \beta$.

In the next, we show that (13)-(14) are equivalent with (11)-(12). If (11)-(12) hold for some $Q_k > 0, \lambda_k > 0, \mu_k$, then (13)-(14) are satisfied by the same $Q_k > 0, \lambda_k > 0, \mu_k$ and $G_k = Q_k$. Assume now (13)-(14) hold for some $Q_k > 0, G_k, \lambda_k > 0, \beta_k, \mu_k$. Pre- and postmultiply (13) ((14)), respectively, by Γ_{1k} and Γ'_{1k} (Γ_{2k} and Γ'_{2k}), where

$$\Gamma_{1k} = \begin{bmatrix} O & -I & O \\ \dfrac{1}{\lambda_k - 1} A_k & O & -I \end{bmatrix}$$

$$\Gamma_{2k} = \begin{bmatrix} O & -I & O \\ \dfrac{1}{\lambda_k} C_k & O & -I \end{bmatrix}$$

As Γ_{1k} and Γ_{2k} are of full row rank, we get

$$\begin{bmatrix} -\mu_k I & B_k' \\ B_k & -Q_{k+1} - \dfrac{1}{\lambda_k - 1} A_k Q_k A_k' \end{bmatrix} > 0$$

$$\begin{bmatrix} (\beta_k - \mu_k) I & D_k' \\ D_k & \beta_k I - \dfrac{1}{\lambda_k} C_k Q_k C_k' \end{bmatrix} < 0$$

which are equivalent to (11)-(12) according to Schur lemma. Thus, condition (ii) is equivalent with condition (i) and is the sufficient condition for $\|\Sigma\|_{peak} < \beta$. ■

Lemma 2 (Generalized H_2 norm) Given the LDP system Σ described by (1) with $D_k = 0$ and zero initial conditions and a real number $\gamma > 0$, then the following conditions are equivalent:

(i) Σ is stable and $\|\Sigma\|_g < \gamma$,

(ii) there exist a T-periodic matrix $Q_k > 0$, s.t.

$$\begin{bmatrix} -Q_{k+1} & B_k & A_k Q_k \\ B_k' & -I & O \\ Q_k A_k' & O & -Q_k \end{bmatrix} < 0, \begin{bmatrix} Q_k & Q_k C_k' \\ C_k Q_k & \gamma^2 I \end{bmatrix} > 0$$

(iii) there exist T-periodic matrix $Q_k > 0$ and G_k such that

$$\begin{bmatrix} -Q_{k+1} & B_k & A_k G_k \\ B_k' & -I & O \\ G_k' A_k' & O & Q_k - G_k - G_k' \end{bmatrix} < 0 \quad (15)$$

$$\begin{bmatrix} G_k + G_k' - Q_k & G_k' C_k' \\ C_k G_k & \gamma^2 I \end{bmatrix} > 0 \quad (16)$$

4. NORM-BOUNDED MODEL UNCERTAINTY

In this paper, two different kinds of model uncertainties will be studied. In this section, we shall consider the LDP system (1) with system matrices

$$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} = \begin{bmatrix} A_{ko} & B_{ko} \\ C_{ko} & D_{ko} \end{bmatrix} + \begin{bmatrix} \Delta A_k & \Delta B_k \\ \Delta C_k & \Delta D_k \end{bmatrix} \quad (17)$$

where the uncertainties are norm-bounded and described by

$$\begin{bmatrix} \Delta A_k & \Delta B_k \\ \Delta C_k & \Delta D_k \end{bmatrix} = \begin{bmatrix} E_k \\ F_k \end{bmatrix} \Omega_k \begin{bmatrix} M_k & N_k \end{bmatrix} \quad (18)$$

with $A_{ko}, B_{ko}, C_{ko}, D_{ko}, E_k, F_k, M_k, N_k$ known T-periodic matrices, Ω_k unknown but bounded, $\forall k$, $\Omega_k' \Omega_k \le I$. To cope with the model uncertainty, Lemma 3 is introduced (Cao and Lam, 2000).

Lemma 3 For any given constant matrices G_1, G_2, G_3, a positive definite matrix P, an uncertain matrix Ω of compatible dimensions, $\Omega\Omega' \le I$, and for any $\varepsilon > 0$ that satisfies $P^{-1} > \varepsilon G_3' G_3$, there is always

$$\begin{aligned} &(G_1 + G_2 \Omega G_3) P (G_1 + G_2 \Omega G_3)' \\ &\quad \le G_1 (P^{-1} - \varepsilon G_3' G_3)^{-1} G_1' + \frac{1}{\varepsilon} G_2 G_2' \quad (19) \end{aligned}$$

Theorem 5 (Peak to peak norm) Given the LDP system Σ described by (1) with norm-bounded model uncertainties (17)-(18). For a given real number $\beta > 0$, Σ is stable and $\|\Sigma\|_{peak} < \beta$, if any of the following conditions holds:

(i) there exist a T-periodic matrix $P_k > 0$ and T-periodic real numbers $\lambda_k > 0, \varepsilon_{1k} > 0, \varepsilon_{2k} > 0, \mu_k$, so that

$$\begin{bmatrix} -P_k + \lambda_k P_k & O & A_{ko}' P_{k+1} & O & M_k' \\ O & -\mu_k I & B_{ko}' P_{k+1} & O & N_k' \\ P_{k+1} A_{ko} & P_{k+1} B_{ko} & -P_{k+1} & P_{k+1} E_k & O \\ O & O & E_k' P_{k+1} & -\varepsilon_{1k}^{-1} I & O \\ M_k & N_k & O & O & -\varepsilon_{1k} I \end{bmatrix} < 0 \quad (20)$$

$$\begin{bmatrix} \lambda_k P_k & O & C_{ko}' & M_k' \\ O & (\beta - \mu_k) I & D_{ko}' & N_k' \\ C_{ko} & D_{ko} & \beta I - \varepsilon_{2k} F_k F_k' & O \\ M_k & N_k & O & \varepsilon_{2k} I \end{bmatrix} > 0 \quad (21)$$

(ii) there exist T-periodic matrices $P_k > 0, G_k$ and T-periodic real numbers $\lambda_k > 0, , \varepsilon_{1k} > 0, \varepsilon_{2k} > 0, \mu_k$, so that (21) and the following matrix inequality hold

$$\begin{bmatrix} \Lambda_{11} & O & A_{ko}' G_k' & O & M_k' \\ O & -\mu_k I & B_{ko}' G_k' & O & N_k' \\ G_k A_{ko} & G_k B_{ko} & \Lambda_{33} & G_k E_k & O \\ O & O & E_k' G_k' & -\varepsilon_{1k}^{-1} I & O \\ M_k & N_k & O & O & -\varepsilon_{1k} I \end{bmatrix} < 0$$

$$\Lambda_{11} = -P_k + \lambda_k P_k, \Lambda_{33} = P_{k+1} - G_k - G_k'$$

(iii) there exist T-periodic matrices $Q_k > 0, G_k$ and T-periodic real numbers $\lambda_k > 0, \varepsilon_{1k} > 0, \varepsilon_{2k} > 0, \mu_k$, such that

$$\begin{bmatrix} \Theta_{11} & O & G_k' A_{ko}' & G_k' M_k' \\ O & -\mu_k I & B_{ko}' & N_k' \\ A_{ko} G_k & B_{ko} & \Theta_{33} & O \\ M_k G_k & N_k & O & -\varepsilon_{1k} I \end{bmatrix} < 0$$

$$\begin{bmatrix} \Pi_{11} & O & G_k' C_{ko}' & G_k' M_k' \\ O & (\beta - \mu_k) I & D_{ko}' & N_k' \\ C_{ko} G_k & D_{ko} & \Pi_{33} & O \\ M_k G_k & N_k & O & \varepsilon_{2k} I \end{bmatrix} > 0$$

$$\begin{aligned} \Theta_{11} &= (\lambda_k - 1)(G_k + G_k' - Q_k) \\ \Theta_{33} &= -Q_{k+1} + \varepsilon_{1k} E_k E_k' \\ \Pi_{11} &= \lambda_k (G_k + G_k' - Q_k) \\ \Pi_{33} &= \beta I - \varepsilon_{2k} F_k F_k' \end{aligned}$$

Proof: At first, we show that $\|\Sigma\|_{peak} < \beta$ if condition (i) is satisfied. Let $G_{1k} = \begin{bmatrix} A_{ko} & B_{ko} \end{bmatrix}'$, $G_{2k} = \begin{bmatrix} M_k & N_k \end{bmatrix}'$. From Schur lemma, (20) holds if and only if $P_{k+1}^{-1} - \varepsilon_{1k} E_k E_k' > 0$ and

$$\begin{aligned} &\begin{bmatrix} -P_k + \lambda_k P_k & O \\ O & -\mu_k I \end{bmatrix} + \frac{1}{\varepsilon_{1k}} G_{2k} G_{2k}' \\ &\quad + G_{1k} \left(P_{k+1}^{-1} - \varepsilon_{1k} E_k E_k' \right)^{-1} G_{1k}' < 0 \quad (22) \end{aligned}$$

According to Lemma 3, if (22) holds, then for any Ω_k satisfying $\Omega_k'\Omega_k \leq I$ there is

$$\begin{bmatrix} -P_k+\lambda_k P_k & O \\ O & -\mu_k I \end{bmatrix} + (G_{1k}+G_{2k}\Omega_k' E_k')P(G_{1k}+G_{2k}\Omega_k' E_k')' < 0$$

i.e. (4) holds. It can similarly proven that (5) holds if (21) holds. Thus, if condition (i) is satisfied, then Σ is stable and $\|\Sigma\|_{peak} < \beta$. Note that condition (ii) and (iii) are equivalent to condition (i). The theorem is proven. ■

Theorem 6 (Generalized H_2 norm) Given the LDP system Σ described by (1) with norm-bounded model uncertainties (17)-(18). For a given real number $\gamma > 0$, Σ is stable and $\|\Sigma\|_g < \gamma$, if any of the following conditions holds:

(i) there exist a T-periodic matrix $P_k > 0$ and T-periodic real numbers $\varepsilon_{1k} > 0, \varepsilon_{2k} > 0$ such that

$$\begin{bmatrix} -P_k & O & A_{ko}'P_{k+1} & O & M_k' \\ O & -I & B_{ko}'P_{k+1} & O & N_k' \\ P_{k+1}A_{ko} & P_{k+1}B_{ko} & -P_{k+1} & P_{k+1}E_k & O \\ O & O & E_k'P_{k+1} & -\varepsilon_{1k}^{-1}I & O \\ M_k & N_k & O & O & -\varepsilon_{1k}I \end{bmatrix} < 0 \quad (23)$$

$$\begin{bmatrix} P_k & C_{ko}' & M_k' \\ C_{ko} & \gamma^2 I - \varepsilon_{2k}F_kF_k' & O \\ M_k & O & \varepsilon_{2k}I \end{bmatrix} > 0 \quad (24)$$

(ii) there exist T-periodic matrices $P_k > 0, G_k$ and T-periodic real numbers $\varepsilon_{1k} > 0, \varepsilon_{2k} > 0$, so that (24) and the following matrix inequality hold

$$\begin{bmatrix} -P_k & O & A_{ko}'G_k' & O & M_k' \\ O & -I & B_{ko}'G_k' & O & N_k' \\ G_kA_{ko} & G_kB_{ko} & P_{k+1}-G_k-G_k' & G_kE_k & O \\ O & O & E_k'G_k' & -\varepsilon_{1k}^{-1}I & O \\ M_k & N_k & O & O & -\varepsilon_{1k}I \end{bmatrix} < 0$$

(iii) there exist T-periodic positive definite matrices $Q_k > 0, G_k$ and T-periodic real numbers $\varepsilon_{1k} > 0, \varepsilon_{2k} > 0$, such that

$$\begin{bmatrix} Q_k-G_k-G_k' & O & G_k'A_{ko}' & G_k'M_k' \\ O & -I & B_{ko}' & N_k' \\ A_{ko}G_k & B_{ko} & -Q_{k+1}+\varepsilon_{1k}E_kE_k' & O \\ M_kG_k & N_k & O & -\varepsilon_{1k}I \end{bmatrix} < 0 \quad (25)$$

$$\begin{bmatrix} G_k+G_k'-Q_k & G_k'C_{ko}' & G_k'M_k' \\ C_{ko}G_k & \gamma^2 I-\varepsilon_{2k}F_kF_k' & O \\ M_kG_k & O & \varepsilon_{2k}I \end{bmatrix} > 0 \quad (26)$$

5. POLYTOPIC MODEL UNCERTAINTY

In this section, we shall consider the LDP system (1) with polytopic model uncertainty, i.e.

$$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} = \sum_{i=1}^{p} \rho_i \begin{bmatrix} A_{ki} & B_{ki} \\ C_{ki} & D_{ki} \end{bmatrix} \quad (27)$$

where $A_{ki}, B_{ki}, C_{ki}, D_{ki}$, $i = 1, 2, \cdots, p_k$, are known T-periodic real matrices, $\rho_i, i = 1, 2, \cdots, p$, are unknown quantities but satisfy $\rho_i \geqslant 0$, $\sum_{i=1}^{p} \rho_i = 1$.

Theorem 7 (Peak to peak norm) Given the LDP system Σ described by (1) with polytopic model uncertainty (27). For a given real number $\beta > 0$, Σ is stable and $\|\Sigma\|_{peak} < \beta$, if any of the following conditions holds:

(i) there exist T-periodic matrices $P_k > 0$ and T-periodic real numbers $\lambda_k > 0, \mu_k$, such that

$$\begin{bmatrix} -P_k+\lambda_kP_k & O & A_{ki}'P_{k+1} \\ O & -\mu_k I & B_{ki}'P_{k+1} \\ P_{k+1}A_{ki} & P_{k+1}B_{ki} & -P_{k+1} \end{bmatrix} < 0$$

$$\begin{bmatrix} \lambda_kP_k & 0 & C_{ki}' \\ 0 & (\beta-\mu_k)I & D_{ki}' \\ C_{ki} & D_{ki} & \beta I \end{bmatrix} > 0, \ i = 1, \cdots, p$$

(ii) there exist T-periodic matrices G_k, $P_{ki} > 0$, $i = 1, 2, \cdots, p$, and T-periodic real numbers $\lambda_k > 0, \mu_k$, such that

$$\begin{bmatrix} -P_{ki}+\lambda_kP_{ki} & O & A_{ki}'G_k' \\ O & -\mu_k I & B_{ki}'G_k' \\ G_kA_{ki} & G_kB_{ki} & P_{k+1,i}-G_k-G_k' \end{bmatrix} < 0$$

$$\begin{bmatrix} \lambda_kP_{ki} & 0 & C_{ki}' \\ 0 & (\beta-\mu_k)I & D_{ki}' \\ C_{ki} & D_{ki} & \beta I \end{bmatrix} > 0, \ i = 1, 2, \cdots, p$$

(iii) there exist T-periodic matrices G_k, $Q_{ki} > 0$, $i = 1, 2, \cdots, p$, and T-periodic real numbers $\lambda_k > 0, \mu_k$, such that $\forall i = 1, \cdots, p$,

$$\begin{bmatrix} (\lambda_k-1)(G_k+G_k'-Q_{ki}) & O & G_k'A_{ki}' \\ O & -\mu_k I & B_{ki}' \\ A_{ki}G_k & B_{ki} & -Q_{k+1,i} \end{bmatrix} < 0$$

$$\begin{bmatrix} \lambda_k(G_k+G_k'-Q_{ki}) & 0 & G_k'C_{ki}' \\ 0 & (\beta-\mu_k)I & D_k' \\ C_{ki}G_k & D_k & \beta I \end{bmatrix} > 0$$

The proof of theorems 7 is easily obtained by noticing the linearity of (27) and thus omitted here. We would like to point out that comparing conditions (i) with (ii) and (iii) shows clearly the advantage of introducing the relaxation variable G_k, which allows the Lyapunov variable P_k and Q_k to be vertex dependent.

Theorem 8 (Generalized H_2 norm) Given the LDP system Σ described by (1) with polytopic model uncertainty (27) and a real number $\gamma > 0$, then Σ is stable and $\|\Sigma\|_{peak} < \gamma$, if any of the following conditions hold:

(i) there exist T-periodic positive definite matrices $P_k > 0$, such that $\forall i = 1, 2, \cdots, p$,

$$\begin{bmatrix} -P_k & O & A_{ki}'P_{k+1} \\ O & -I & B_{ki}'P_{k+1} \\ P_{k+1}A_{ki} & P_{k+1}B_{ki} & -P_{k+1} \end{bmatrix} < 0, \begin{bmatrix} P_k & C_{ki}' \\ C_{ki} & \gamma^2 I \end{bmatrix} > 0$$

(ii) there exist T-periodic positive definite matrices G_k, $P_{ki} > 0$, $i = 0, 1, \cdots, p$, such that $i = 1, 2, \cdots, p$,

$$\begin{bmatrix} -P_{ki} & O & A'_{ki}G'_k \\ O & -I & B'_{ki}G'_k \\ G_k A_{ki} & G_k B_{ki} & \Lambda_{33} \end{bmatrix} < 0, \begin{bmatrix} P_{ki} & C'_{ki} \\ C_{ki} & \gamma^2 I \end{bmatrix} > 0$$
$$\Lambda_{33} = P_{k+1,i} - G_k - G'_k$$

6. CONTROLLER AND OBSERVER DESIGN

Based on the results presented in Section 3-5, it is straightforward to design controllers and observers for the LDP systems with and without model uncertainty. Due to the space limitation, in this section we only give an example of controller design.

Consider an LDP system described by

$$\begin{aligned} x(k+1) &= A_k x(k) + B^u_k u(k) + B^d_k d(k) \\ y(k) &= C_k x(k) + D^u_k u(k) + D^d_k d(k) \end{aligned} \quad (28)$$

where u is the control input vector and d the disturbance input vector. Assume that the state feedback control law

$$u(k) = K_k x(k) \quad (29)$$

is used, where K_k is a T-periodic gain matrix, for the purpose of disturbance attenuation. The closed-loop dynamics Σ_{cl} is governed by

$$\begin{aligned} x(k+1) &= (A_k + B^u_k K_k)x(k) + B^d_k d(k) \\ y(k) &= (C_k + D^u_k K_k)x(k) + D^d_k d(k) \end{aligned} \quad (30)$$

As an example, Theorem 9 shows how to design the feedback gain matrix K_k to satisfy the specifications on the peak to peak norm.

Theorem 9 (Controller design) Given the nominal LDP system (28), control law (29) and a real number $\beta > 0$. The closed-loop dynamics (30) is stable and $\|\Sigma_{cl}\|_{peak} < \beta$, if there exist T-periodic matrices $Q_k > 0, G_k, Y_k$ and T-periodic real numbers $\lambda_k > 0, \mu_k$, such that

$$\begin{bmatrix} (\lambda_k - 1)(G_k + G'_k - Q_k) & O & G'_k A'_k + Y'_k (B^u_k)' \\ O & -\mu_k I & (B^d_k)' \\ A_k G_k + B^u_k Y_k & B^d_k & -Q_{k+1} \end{bmatrix} < 0 \quad (31)$$

$$\begin{bmatrix} \lambda_k (G_k + G'_k - Q_k) & 0 & G'_k C'_k + Y'_k (D^u_k)' \\ 0 & (\beta - \mu_k) I & (D^d_k)' \\ C_k G_k + D^u_k Y_k & D^d_k & \beta I \end{bmatrix} > 0 \quad (32)$$

The corresponding controller gain can be set as $K_k = Y_k G_k^{-1}$.

The proof consists in applying Lemma 1 and using the variable substitution $K_k G_k = Y_k$. It is omitted here due to space limitation.

7. CONCLUSION

In this paper, the matrix inequality technique is applied to study the norms, in particular the peak to peak norm and the generalized H_2 norm, of linear discrete-time periodic systems. First, the specifications on norms are expressed by matrix inequalities. Then, based on them, the disturbance attenuation problem is taken as an example to show the application of the derived results.

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