Abstract: This paper focuses on the peak to peak norm and the generalized $H_2$ norm of linear discrete-time periodic systems. First, sufficient conditions based on matrix inequalities are given. Then, based on them, the disturbance attenuation problem is taken as an example to show the application of the derived results.

Keywords: linear discrete-time periodic systems, robustness, peak to peak norm, generalized $H_2$ norm, matrix inequality.

1. INTRODUCTION

Periodic system theory has been under continuous development in the last decades. While different norms have been proposed for characterizing the robustness of linear time-invariant systems (Scherer et al., 1997), in the literature on periodic systems the most attention has been focused on the $H_2$ norm and the $H_\infty$ norm. In the frequency domain, the parametric transfer function (Lampe and Rossenwasser, 2004; Lampe et al., 2005), the harmonic analysis (Zhou and Hagiwara, 2002) and the lifting based methods (Varga, 2006) have been proposed. In the time domain, Riccati equation, Riccati inequality (Xie and de Souza, 1991; Colaneri, 2000) and linear matrix inequality (LMI) based approaches (Bittanti and Cuzzola, 2001a; Bittanti and Cuzzola, 2001b) have been presented.

As it is often difficult to get an exact model of the system, model uncertainties present one of the challenges in control and estimation problems. In (Xie and de Souza, 1991), norm-bounded uncertainty has been considered in the $H_\infty$ filtering of the linear continuous-time periodic systems. Both norm-bounded uncertainty and polytopic uncertainty have been discussed by (Souza and Trofino, 2000) regarding the stabilization problem of the linear discrete-time periodic (LDP) systems. More recently, the $H_2$ norm of the LDP systems with polytopic uncertainties has been investigated in (Farges et al., 2007).

In this paper, we shall focus on the peak to peak norm and the generalized $H_2$ norm of the LDP systems. We shall also consider the influence of norm-bounded and polytopic model uncertainties on these norms. To the authors’ knowledge, the generalized $H_2$ norm of the LDP systems has not yet been considered. The peak to peak norm has been handled by (Voulgaris, 1996; Aubrecht and Voulgaris, 2001). In (Voulgaris, 1996), an estimator that minimizes the peak to peak norm from the disturbances to the estimation errors is designed using the lifting and model matching technique. A state feedback controller is synthesized in (Aubrecht and Voulgaris, 2001) to minimize the peak to peak norm of the closed-loop system by constructing a periodic controlled invariance kernel. In comparison, the solutions presented in this paper employ the matrix inequality technique that can be easily implemented with the help of the standard MATLAB toolbox. This work is mo-
tivated by our recent study on networked control systems (NCS), in which the theory of periodic systems can be applied from different viewpoints (Ding et al., 2006; Zhang and Ding, 2006).

This paper is organized as follows. After the definition of the norms in Section 2, LMI characterizations of the norms will be given in Section 3. We include here also the extended LMI characterizations obtained by using the relaxation variable technique introduced in (de Oliveira et al., 1999). Then, norm-bounded uncertainty and polytopic uncertainty will be considered in Section 4 and 5, respectively. Finally, in Section 6 we shall consider the control and estimation problem to meet the specifications on robustness.

2. DEFINITIONS

In this paper, we consider the LDP system $\Sigma$ described by

$$
\begin{align*}
    x(k+1) &= A_k x(k) + B_k d(k) \\
    y(k) &= C_k x(k) + D_k d(k)
\end{align*}
$$

which $x \in \mathbb{R}^n$ denotes the state vector, $d \in \mathbb{R}^{n_d}$ the input vector, $y \in \mathbb{R}^n$ the output vector, $A_k, B_k, C_k, D_k$ are known real periodic matrices of period $T$, i.e., $\forall k,

$$
\begin{bmatrix}
    A_{k+T} \\
    C_{k+T}
\end{bmatrix} =
\begin{bmatrix}
    A_k \\
    C_k
\end{bmatrix}
$$

The peak to peak norm of the LDP system (1) is the induced norm with both the input signal and the output signal measured by the maximal amplitude, i.e.,

$$
\|\Sigma\|_{peak} = \sup_{d \in \mathbb{R}^{n_d}, d \neq 0} \frac{\|y\|_{\infty}}{\|d\|_{\infty}}
$$

where $\|y\|_{\infty} = \sup_k \sqrt{y(k)^T y(k)}$ and $\|d\|_{\infty} = \sup_k \sqrt{d(k)^2}$ denote the $l_{\infty}$-norm of the input and output signals, respectively.

The generalized $H_2$ norm of the LDP system (1) is the induced norm with the input signal measured by the energy and the output signal measured by the maximal amplitude, i.e.,

$$
\|\Sigma\|_g = \sup_{d \in \mathbb{R}^{n_d}, d \neq 0} \frac{\|y\|_{\infty}}{\|d\|_2}
$$

where $\|d\|_2 = \sqrt{\sum_{k=0}^{T} g'(k) y(k)}$ denotes the $l_2$-norm of the input signal. Note that $\|G\|_g$ is finite only if $D_k = 0$.

3. LMI CHARACTERIZATIONS

**Theorem 1** (Peak to peak norm) Given the LDP system $\Sigma$ described by (1) with zero initial conditions and a real number $\beta > 0$, then $\Sigma$ is stable and $\|\Sigma\|_{peak} < \beta$, if there exist a $T$-periodic matrix $P_k > 0$ and $T$-periodic real numbers $\lambda_k > 0$ and $\mu_k$, such that

$$
\begin{bmatrix}
    -P_k + \lambda_k P_k & O \\
    O & -\mu_k I
\end{bmatrix}
\begin{bmatrix}
    A_k' P_{k+1} \\
    B_k' P_{k+1}
\end{bmatrix}
\begin{bmatrix}
    P_{k+1} A_k \\
    P_{k+1} B_k
\end{bmatrix} = 0
$$

**Proof:** Define a periodic Lyapunov function for the LDP system (1) as

$$
V(x(k)) = \langle 0, \psi_k \rangle
$$

where $P_{k+T} = P_k > 0$. From Schur lemma, (4) is equivalent to

$$
\psi_k = \begin{bmatrix}
    A_k' P_{k+1} A_k - \mu_k P_k & A_k' P_{k+1} B_k \\
    B_k' P_{k+1} A_k & B_k' P_{k+1} B_k - \mu_k I
\end{bmatrix}
$$

From $P_k > 0$ and $A_k' P_{k+1} A_k - \mu_k P_k < 0$, it is clear that $\Sigma$ is stable. Moreover,

$$
V(x(k+1)) - V(x(k)) + \lambda_k V(x(k)) - \mu_k d'(k) d(k)
$$

is finite only if $\lambda_k V(x(k)) - \mu_k d'(k) d(k) < 0$, $\forall x, d$

which means that

$$
\lambda_k V(x(k)) - \mu_k d'(k) d(k) < 0, \quad \forall x, d
$$

As (5) is equivalent to

$$
\begin{bmatrix}
    \lambda_k P_k & 0 \\
    0 & (\beta - \mu_k) I
\end{bmatrix}
\begin{bmatrix}
    C_k & D_k
\end{bmatrix}
$$

there is

$$
\begin{bmatrix}
    \lambda_k P_k & 0 \\
    0 & (\beta - \mu_k) I
\end{bmatrix}
\begin{bmatrix}
    x(k) \\
    d(k)
\end{bmatrix}
$$

Thus, $\|G\|_{peak} < \beta$.

**Theorem 2** (Generalized $H_2$ norm) Given the LDP system $\Sigma$ described by (1) with $D_k = 0$ and zero initial conditions and a real number $\gamma > 0$, then $\Sigma$ is stable and $\|\Sigma\|_g < \gamma$, if and only if there exist a $T$-periodic matrix $P_k > 0$ such that

$$
\begin{bmatrix}
    -P_k + \lambda_k P_k & O \\
    O & -I
\end{bmatrix}
\begin{bmatrix}
    A_k' P_{k+1} \\
    B_k' P_{k+1}
\end{bmatrix}
\begin{bmatrix}
    P_{k+1} A_k \\
    P_{k+1} B_k
\end{bmatrix} = 0
$$

**Proof:** Assume that (6) is a periodic Lyapunov function of the LDP system (1). Note that (7) is equivalent to

$$
\psi_k = \begin{bmatrix}
    A_k' P_{k+1} A_k - \mu_k P_k & A_k' P_{k+1} B_k \\
    B_k' P_{k+1} A_k & B_k' P_{k+1} B_k - I
\end{bmatrix}
$$


As $P_k > 0$ and $A'_k P_{k+1} A_k - P_k < 0$, it implies that $\Sigma$ is stable. Moreover, (7) holds if and only if
\[
V(x(k+1)) - V(x(k)) - d'(k)d(k) = [x'(k) \ d'(k)] \Psi_k \begin{bmatrix} x(k) \\ d(k) \end{bmatrix} < 0, \ \forall x, d
\]
which means that
\[
V(x(k)) < \sum_{j=0}^{k-1} d'(j)d(j), \ \forall x, d
\]
Taking (8) into account, we have
\[
y'(k)g(k) < \gamma^2 V(x(k)) < \gamma^2 \sum_{j=0}^{k-1} d'(j)d(j)
\]
i.e., $\|G\|_g < \gamma$.

If we introduce the so-called relaxation variables (de Oliveira et al., 1999) to decouple the Lyapunov variable from the system matrices to provide later more design freedom, then we obtain theorems 3-4 as follows.

**Theorem 3 (Peak to peak norm)** Given the LDP system $\Sigma$ described by (1) with zero initial conditions and a real number $\beta > 0$, then $\Sigma$ is stable and $\|\Sigma\|_{peak} < \beta$, if any of the following conditions hold:

(i) there exist a $T$-periodic matrix $Q_k > 0$ and $T$-periodic real numbers $\lambda_k > 0, \mu_k$, such that
\[
\begin{bmatrix}
-Q_k + \lambda_k Q_k & O & Q_k A'_k \\
A_k Q_k & -\mu_k I & B'_k \\
G_k A_k & -\mu_k I & B'_k G'_k \\
G_k B_k & P_{k+1} - G_k & -G_k - G'_k
\end{bmatrix} < 0
\]

(ii) there exist $T$-periodic matrices $Q_k > 0, G_k$, and $T$-periodic real numbers $\lambda_k > 0, \mu_k$, such that
\[
\begin{bmatrix}
-(\lambda_k - 1)(G_k + G'_k - Q_k) & O & G'_k A'_k \\
A_k G_k & -\mu_k I & B'_k \\
C_k G_k & 0 & (\beta - \mu_k) I & D'_k \\
0 & \beta I & &
\end{bmatrix} < 0
\]

Proof: Assume that (5) and (9) hold for some $P_k > 0, G_k, \lambda_k > 0, \mu_k$. Pre- and postmultiplying (9) by $\Gamma_k$ and $\Gamma'_k$, respectively, where
\[
\Gamma_k = \begin{bmatrix}
I & O & A'_k \\
O & I & B'_k
\end{bmatrix}
\]
As $\Gamma_k$ is a matrix of full row rank, we get (4), i.e., the same matrices $P_k > 0, \lambda_k > 0, \mu_k$ satisfy (4)-(5). Recalling Theorem 1, the LDP system (1) is stable and its peak to norm is smaller than $\beta$. Indeed, if (4)-(5) hold for some $P_k > 0, \lambda_k > 0$ and $\mu_k$, then (5) and (9) are satisfied by the same $P_k > 0, \lambda_k > 0, \beta, \mu_k$ and $G_k = P_{k+1}$. That means, the conditions (5) and (9) are equivalent with (4)-(5).

**Theorem 4 (Generalized $H_2$ norm)** Given the LDP system $\Sigma$ described by (1) with $D_k = 0$ and zero initial conditions and a real number $\gamma > 0$, then $\Sigma$ is stable and $\|\Sigma\|_g < \gamma$, if and only if there exist $T$-periodic matrices $P_k > 0$ and $G_k$, so that
\[
\begin{bmatrix}
-P_k & O & A'_k G'_k \\
O & -I & B'_k G'_k \\
G_k A_k & G_k B_k & P_{k+1} - G_k & -G_k - G'_k
\end{bmatrix} < 0
\]
\[
\text{The proof is similar with that of Theorem 3 and thus omitted here.}
\]

In the following, we give equivalent conditions for the peak-to-peak norm and the generalized $H_2$ norm, respectively. These conditions can be readily used for the design of state feedback controllers, while the conditions presented above are more convenient for the observer design.

**Lemma 1 (Peak to peak norm)** Given the LDP system $\Sigma$ described by (1) with zero initial conditions and a real number $\beta > 0$, then $\Sigma$ is stable and $\|\Sigma\|_{peak} < \beta$, if any of the following conditions holds:

(i) there exist a $T$-periodic matrix $Q_k > 0$ and $T$-periodic real numbers $\lambda_k > 0, \mu_k$, such that
\[
\begin{bmatrix}
-Q_k + \lambda_k Q_k & O & Q_k A'_k \\
A_k Q_k & -\mu_k I & B'_k \\
G_k A_k & -\mu_k I & B'_k G'_k \\
G_k B_k & P_{k+1} - G_k & -G_k - G'_k
\end{bmatrix} < 0
\]

(ii) there exist $T$-periodic matrices $Q_k > 0, G_k$, and $T$-periodic real numbers $\lambda_k > 0, \mu_k$, such that
\[
\begin{bmatrix}
-(\lambda_k - 1)(G_k + G'_k - Q_k) & O & G'_k A'_k \\
A_k G_k & -\mu_k I & B'_k \\
C_k G_k & 0 & (\beta - \mu_k) I & D'_k \\
0 & \beta I & &
\end{bmatrix} < 0
\]

Proof: At first, we prove that condition (i) is sufficient for $\|\Sigma\|_{peak} < \beta$. Assume that (11)-(12) are satisfied by some $Q_k > 0, \lambda_k > 0, \mu_k$. Let
\[
P_k = Q_k^{-1}, \ T_{1k} = \begin{bmatrix} P_k & O & O \\
O & I & O \\
O & O & P_{k+1} \end{bmatrix}, \ T_{2k} = \begin{bmatrix} P_k & O & O \\
O & I & O \\
O & O & I \end{bmatrix}
\]
Pre- and postmultiplying (11) by $T_{1k}$ and $T'_{1k}$, respectively, yields (4). Similarly, pre- and postmultiplying (12) by $T_{2k}$ and $T'_{2k}$ yields (5). According to Theorem 1, $\Sigma$ is stable and $\|\Sigma\|_{peak} < \beta$.

In the next, we show that (13)-(14) are equivalent with (11)-(12). If (11)-(12) hold for some $Q_k > 0, \lambda_k > 0, \mu_k$, then (13)-(14) are satisfied by the same $Q_k > 0, \lambda_k > 0, \mu_k$ and $G_k = Q_k$. Assume now (13)-(14) hold for some $Q_k > 0, G_k, \lambda_k > 0, \beta, \mu_k$. Pre- and postmultiplying (13) (14), respectively, by $\Gamma_{1k}$ and $\Gamma'_{1k}$ ($\Gamma_{2k}$ and $\Gamma'_{2k}$), where
\[
\begin{bmatrix}
O & -I & A_k \\
\frac{1}{\lambda_k - I} & A_k & -I
\end{bmatrix}
\]
\[
\begin{bmatrix}
O & -I & O \\
\frac{1}{\lambda_k} & O & -I
\end{bmatrix}
\]
As $\Gamma_{1k}$ and $\Gamma_{2k}$ are of full row rank, we get
\[
\begin{bmatrix}
-\mu_k I \\
B_k \\
Q_{k+1} - \frac{B_k'}{\lambda_k - 1} A_k Q_k A_k' \\
\end{bmatrix} > 0
\]
which are equivalent to (11)-(12) according to Schur lemma. Thus, condition (ii) is equivalent with condition (i) and is the sufficient condition for $\|\Sigma\|_{\text{peak}} < \beta$. ■

**Lemma 2** (Generalized $H_2$ norm) Given the LDP system $\Sigma$ described by (1) with $D_k = 0$ and zero initial conditions and a real number $\gamma > 0$, then the following conditions are equivalent:

(i) $\Sigma$ is stable and $\|\Sigma\|_{\gamma} < \gamma$.

(ii) there exist a $T$-periodic matrix $Q_k > 0$, s.t.
\[
\begin{bmatrix}
-\frac{B_k'}{\lambda_k - 1} A_k Q_k \\
B_k' - I & O \\
Q_k A_k' & O & -Q_k \\
\end{bmatrix} < 0,
\begin{bmatrix}
Q_k & Q_k C_k' & \gamma^2 I \\
C_k Q_k & G_k + G_k' - Q_k & G_k' C_k' \\
G_k C_k' & C_k G_k & \gamma^2 I \\
\end{bmatrix} > 0
\]

(iii) there exist T-periodic matrix $Q_k > 0$ and $G_k$ such that
\[
\begin{bmatrix}
-\frac{B_k'}{\lambda_k - 1} A_k G_k \\
B_k' - I & O \\
G_k A_k' & O & -Q_k - G_k - G_k' \\
\end{bmatrix} < 0
\]

\[\text{(15)}\]

\[\text{(16)}\]

\[\text{(17)}\]

\[\text{(18)}\]

4. NORM-BOUNDED MODEL UNCERTAINTY

In this paper, two different kinds of model uncertainties will be studied. In this section, we shall consider the LDP system (1) with system matrices
\[
\begin{bmatrix}
A_k & B_k \\
C_k & D_k \\
\end{bmatrix} = \begin{bmatrix} A_{ko} & B_{ko} \\
C_{ko} & D_{ko} \end{bmatrix} + \begin{bmatrix} \Delta A_k & \Delta B_k \\
\Delta C_k & \Delta D_k \end{bmatrix}
\]
where the uncertainties are norm-bounded and described by
\[
\begin{bmatrix}
\Delta A_k & \Delta B_k \\
\Delta C_k & \Delta D_k \\
\end{bmatrix} = \begin{bmatrix} E_k & F_k \\
\end{bmatrix} \Omega_k \begin{bmatrix} M_k & N_k \end{bmatrix}
\]
with $A_{ko}, B_{ko}, C_{ko}, D_{ko}, E_k, F_k, M_k, N_k$ known $T$-periodic matrices, $\Omega_k$ unknown but bounded, $\forall k$, $\Omega_k^T \Omega_k \leq I$. To cope with the model uncertainty, Lemma 3 is introduced (Cao and Lam, 2000).

**Lemma 3** For any given constant matrices $G_1, G_2, G_3$, a positive definite matrix $P$, an uncertain matrix $\Omega$ of compatible dimensions, $\Omega \Omega^T \leq I$, and for any $\varepsilon > 0$ that satisfies $P^{-1} > \varepsilon G_2 G_3$, there is always
\[
(G_1 + G_2 \Omega G_3)P(G_1 + G_2 \Omega G_3)' \leq G_1 (P^{-1} - \varepsilon G_2 G_3)^{-1} G_1' + \frac{1}{\varepsilon} G_2 G_2'
\]

**Theorem 5** (Peak to peak norm) Given the LDP system $\Sigma$ described by (1) with norm-bounded model uncertainties (17)-(18). For a given real number $\beta > 0$, $\Sigma$ is stable and $\|\Sigma\|_{\text{peak}} < \beta$, if any of the following conditions holds:

(i) there exist a $T$-periodic matrix $P_k > 0$ and $T$-periodic real numbers $\lambda_k > 0, \varepsilon_{1k} > 0, \varepsilon_{2k} > 0, \mu_k$, so that
\[
\begin{bmatrix}
-\frac{B_k'}{\lambda_k - 1} A_k & O & A_{ko} P_{k+1} & O & M_k' \\
O & -\mu_k I & B_{ko} P_{k+1} & O & N_k' \\
P_{k+1} A_{ko} & P_{k+1} B_{ko} & -\frac{B_k'}{\lambda_k - 1} A_k & P_{k+1} & E_k \\
O & O & E_k P_{k+1} & -\varepsilon_{1k} I & O \\
M_k & N_k & O & O & -\varepsilon_{1k} I \\
\end{bmatrix} < 0
\]

\[\text{(20)}\]

(ii) there exist $T$-periodic matrices $P_k > 0, G_{\gamma}$ and $T$-periodic real numbers $\lambda_k > 0, \varepsilon_{1k} > 0, \varepsilon_{2k} > 0, \mu_k$, so that (21) and the following matrix inequality hold
\[
\begin{bmatrix}
\lambda_k P_k & O & C_{ko} & M_k' \\
O & (\beta - \mu_k) I & D_{ko} & N_k' \\
C_{ko} & D_{ko} & \beta I - \varepsilon_{2k} F_k F_k' & O \\
M_k & N_k & O & \varepsilon_{2k} I \\
\end{bmatrix} > 0
\]

\[\text{(21)}\]

(iii) there exist $T$-periodic matrices $Q_k > 0, G_{\gamma}$ and $T$-periodic real numbers $\lambda_k > 0, \varepsilon_{1k} > 0, \varepsilon_{2k} > 0, \mu_k$, such that
\[
\begin{bmatrix}
A_{11} & O & A_{ko} G_{\gamma}' & O & M_k' \\
O & -\mu_k I & B_{ko} G_{\gamma}' & O & N_k' \\
A_{ko} G_{\gamma} & B_{ko} & \Theta_{33} & O \\
M_k G_{\gamma} & N_k & O & -\varepsilon_{1k} I \\
\end{bmatrix} < 0
\]

\[\text{(19)}\]

\[\text{(20)}\]

\[\text{(21)}\]

**Proof:** At first, we show that $\|\Sigma\|_{\text{peak}} < \beta$ if condition (i) is satisfied. Let $G_{1k} = [A_{ko} B_{ko}]'$, $G_{2k} = [M_k N_k]'$. From Schur lemma, (20) holds if and only if $P_k^{-1} > \varepsilon_{1k} E_k E_k' > 0$ and
\[
\begin{bmatrix}
-\frac{B_k'}{\lambda_k - 1} A_k & O & A_{ko} P_{k+1} & O & M_k' \\
O & -\mu_k I & B_{ko} P_{k+1} & O & N_k' \\
P_{k+1} A_{ko} & P_{k+1} B_{ko} & -\frac{B_k'}{\lambda_k - 1} A_k & P_{k+1} & E_k \\
O & O & E_k P_{k+1} & -\varepsilon_{1k} I & O \\
M_k & N_k & O & O & -\varepsilon_{1k} I \\
\end{bmatrix} < 0
\]

\[\text{(20)}\]
According to Lemma 3, if (22) holds, then for any \( \Omega_k \) satisfying \( \Omega_k^T \Omega_k \leq I \) there is
\[
\begin{bmatrix}
-P_k + G_k P_k & O \\
O & -\mu_k I \\
\end{bmatrix} + (G_{1k} + G_{2k} \Omega_k^T E_k^T) P (G_{1k} + G_{2k} \Omega_k E_k^T)^T < 0
\]
i.e. (4) holds. It can similarly proven that (5) holds if (21) holds. Thus, if condition (i) is satisfied, then \( \Sigma \) is stable and \( \|\Sigma\|_{peak} < \beta \). Note that condition (ii) and (iii) are equivalent to condition (i). The theorem is proven.

**Theorem 6** (Generalized \( H_\infty \) norm) Given the LDP system \( \Sigma \) described by (1) with polytopic model uncertainty (17)-(18). For a given real number \( \gamma > 0 \), \( \Sigma \) is stable and \( \|\Sigma\|_\infty < \gamma \), if any of the following conditions hold:

(i) there exist a \( T \)-periodic matrix \( P_k > 0 \) and \( T \)-periodic real numbers \( \varepsilon_{1k} > 0, \varepsilon_{2k} > 0 \) such that
\[
\begin{bmatrix}
-P_k & O & A'_{ko} P_{k+1} & O & M'_{ko} \\
O & -I & B'_{ko} P_{k+1} & O & N'_{ko} \\
A_{ko} & G_{ko} & P_{k+1} & G_{ko} & O \\
O & G_{ko} & E_k P_{k+1} & -\varepsilon_{1k} I & O \\
M_k & N_k & O & O & -\varepsilon_{1k} I
\end{bmatrix} < 0
\]

(ii) there exist \( T \)-periodic matrices \( P_k > 0, G_k \) and \( T \)-periodic real numbers \( \varepsilon_{1k} > 0, \varepsilon_{2k} > 0 \), so that (24) and the following matrix inequality hold
\[
\begin{bmatrix}
P_k & C_{ko} & O \\
C_{ko} & \gamma^2 I - \varepsilon_{2k} P_k F_k' & O \\
M_k & O & \varepsilon_{2k} I
\end{bmatrix} > 0
\]

(iii) there exist \( T \)-periodic positive definite matrices \( Q_k > 0, G_k \) and \( T \)-periodic real numbers \( \varepsilon_{1k} > 0, \varepsilon_{2k} > 0 \), such that
\[
\begin{bmatrix}
Q_k - G_k - G_k' & O & G_k A_{ko}' & G_k M_{ko}' \\
O & -I & B_{ko}' & N_{ko}' \\
A_{ko} G_k & B_{ko} & Q_{k+1} & + \varepsilon_{1k} E_k E_k' & O \\
M_k G_k & N_k & O & -\varepsilon_{1k} I
\end{bmatrix} < 0
\]

\[
\begin{bmatrix}
G_k + G_k' - Q_k & G_k C_{ko} & G_k M_{ko}' \\
C_{ko} G_k & \gamma^2 I - \varepsilon_{2k} P_k F_k' & O \\
M_k G_k & O & \varepsilon_{2k} I
\end{bmatrix} > 0
\]

5. POLYTOPIC MODEL UNCERTAINTY

In this section, we shall consider the LDP system (1) with polytopic model uncertainty, i.e.

\[
\begin{bmatrix}
A_k & B_k \\
C_k & D_k
\end{bmatrix} = \sum_{i=1}^{p} \pi_i \begin{bmatrix}
A_{ki} & B_{ki} \\
C_{ki} & D_{ki}
\end{bmatrix}
\]

where \( A_{ki}, B_{ki}, C_{ki}, D_{ki}, i = 1, 2, \ldots, p \), are known \( T \)-periodic real matrices, \( \pi_i, i = 1, 2, \ldots, p \), are unknown quantities but satisfy \( \pi_i \geq 0 \), \( \sum_{i=1}^{p} \pi_i = 1 \).

**Theorem 7** (Peak to peak norm) Given the LDP system \( \Sigma \) described by (1) with polytopic model uncertainty (27). For a given real number \( \beta > 0 \), \( \Sigma \) is stable and \( \|\Sigma\|_{peak} < \beta \), if any of the following conditions holds:

(i) there exist \( T \)-periodic matrices \( P_k > 0 \) and \( T \)-periodic real numbers \( \lambda_k > 0, \mu_k \), such that
\[
\begin{bmatrix}
-P_k + \lambda_k P_k & O & A'_{ki} P_{k+1} \\
O & -\mu_k I & B'_{ki} P_{k+1} \\
\lambda_k A_{ki} & 0 & \beta I \\
\end{bmatrix} < 0
\]

(ii) there exist \( T \)-periodic matrices \( G_k, P_k \) and \( T \)-periodic real numbers \( \lambda_k > 0, \mu_k \), such that
\[
\begin{bmatrix}
P_k + \lambda_k P_k & O & A'_{ki} G_k' \\
O & -\mu_k I & B'_{ki} G_k' \\
\lambda_k A_{ki} G_k & 0 & \beta I \\
\end{bmatrix} > 0, \ i = 1, 2, \ldots, p
\]

(iii) there exist \( T \)-periodic matrices \( G_k, Q_k \) and \( T \)-periodic real numbers \( \lambda_k > 0, \mu_k \), such that \( \forall i = 1, 2, \ldots, p \),
\[
\begin{bmatrix}
(\lambda_k - 1) G_k + G'_k - Q_k & O & G'_k A'_{ki} \\
O & -\mu_k I & B'_{ki} \\
A_{ki} G_k & 0 & \beta I \\
\end{bmatrix} < 0
\]

The proof of theorems 7 is easily obtained by noticing the linearity of (27) and thus omitted here. We would like to point out that comparing conditions (i) with (ii) and (iii) shows clearly the advantage of introducing the relaxation variable \( G_k \), which allows the Lyapunov variable \( P_k \) and \( Q_k \) to be vertex dependent.

**Theorem 8** (Generalized \( H_\infty \) norm) Given the LDP system \( \Sigma \) described by (1) with polytopic model uncertainty (27) and a real number \( \gamma > 0 \), then \( \Sigma \) is stable and \( \|\Sigma\|_{peak} < \gamma \), if any of the following conditions hold:

(i) there exist \( T \)-periodic positive definite matrices \( P_k > 0 \), such that \( \forall i = 1, 2, \ldots, p \),
\[
\begin{bmatrix}
-P_k & O & A'_{ki} P_{k+1} \\
O & -I & B'_{ki} P_{k+1} \\
\lambda_k A_{ki} & 0 & \beta I \\
\end{bmatrix} < 0
\]

\[
\begin{bmatrix}
P_k & C_{ki} \\
C_{ki} & \gamma^2 I
\end{bmatrix} > 0
\]
(ii) there exist $T$-periodic positive definite matrices $G_k, P_{ki} > 0$, $i = 0, 1, \cdots, p$, such that $i = 1, 2, \cdots, p,$
\[
\begin{bmatrix}
-P_{ki} & O & A_k' G_k' \\
O & -I & B_k' G_k' \\
G_k A_k & G_k B_k & A_{33}
\end{bmatrix} < 0, \quad \begin{bmatrix}
P_{ki} & G_k' \\
C_k & \gamma^2 I
\end{bmatrix} > 0
\]

\[A_{33} = P_{k+1,i} - G_k - G_k'\]

6. CONTROLLER AND OBSERVER DESIGN

Based on the results presented in Section 3-5, it is straightforward to design controllers and observers for the LDP systems with and without model uncertainty. Due to the space limitation, in this section we only give an example of controller design.

Consider an LDP system described by
\[
x(k+1) = A_k x(k) + B_k u(k) + B_d d(k) \\
y(k) = C_k x(k) + D_k u(k) + D_d d(k)
\]

where $u$ is the control input vector and $d$ the disturbance input vector. Assume that the state feedback control law
\[
u(k) = K_k x(k)
\]
is used, where $K_k$ is a $T$-periodic gain matrix, for the purpose of disturbance attenuation. The closed-loop dynamics $\Sigma_{cl}$ is governed by
\[
x(k+1) = (A_k + B_k' K_k)x(k) + B_d d(k) \\
y(k) = (C_k + D_k' K_k)x(k) + D_d d(k)
\]

As an example, Theorem 9 shows how to design the feedback gain matrix $K_k$ to satisfy the specifications on the peak to peak norm.

Theorem 9 (Controller design) Given the nominal LDP system (28), control law (29) and a real number $\beta > 0$, the closed-loop dynamics (30) is stable and $\|\Sigma_{cl}\|_{\text{peak}} < \beta$, if there exist $T$-periodic matrices $Q_k > 0, G_k, Y_k$ and $T$-periodic real numbers $\lambda_k > 0, \mu_k$, such that
\[
\begin{bmatrix}
(\lambda_k - 1)(G_k + G_k' - Q_k) & O & G_k' A_k' + Y_k' (B_k')' \\
O & -\mu_k I & (B_k')' \\
A_k G_k + B_k' Y_k & B_d G_k & -Q_{k+1}
\end{bmatrix} < 0
\]

\[\begin{bmatrix}
\lambda_k (G_k + G_k' - Q_k) & 0 & G_k' C_k' + Y_k' (D_k')' \\
0 & (\beta - \mu_k) I & (D_k')' \\
C_k G_k + D_k' Y_k & D_d G_k & \beta I
\end{bmatrix} > 0
\]

The corresponding controller gain can be set as $K_k = Y_k G_k^{-1}$. The proof consists in applying Lemma 1 and using the variable substitution $K_k G_k = Y_k$. It is omitted here due to space limitation.

7. CONCLUSION

In this paper, the matrix inequality technique is applied to study the norms, in particular the peak to peak norm and the generalized $H_2$ norm, of linear discrete-time periodic systems. First, the specifications on norms are expressed by matrix inequalities. Then, based on them, the disturbance attenuation problem is taken as an example to show the application of the derived results.

REFERENCES


