ENOC-2008, Saint Petersburg, Russia, June, 30–July, 4 2008

FORCE TRANSMISSIBILITY OF A NONLINEAR VIBRATION ISOLATOR WITH HIGH-STATIC-LOW-DYNAMIC-STIFFNESS

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Abstract

A problem that affects many engineering applications is the need for vibration isolation to reduce the level of vibration transmitted from a source to a receiver. A lower natural frequency would benefit the isolation performance of a vibration isolator because it would provide a wider frequency isolation region. However, if a linear mount is used, this approach is limited by the static displacement that derives from a soft spring. One possible solution is to employ nonlinear mounts with high-static-low-dynamic-stiffness (HSLDS) whose dynamics can often be described by the Duffing equation. Although the response of the Duffing oscillator to a harmonic force applied to the mass has been extensively studied, simple analytical expressions for the transmissibility of these systems seems not to be available yet. In this paper a simple expression for the maximum transmissibility is proposed. Furthermore, the transmissibility of an HSLDS isolator is compared with that of an equivalent linear model to show the improved performance.

1 Introduction

The use of passive isolators is ubiquitous in engineering systems (Hartog, 1985; Rivin, 2001). In the simplest case when the isolator is linear, a low natural frequency, which is desirable, can only be achieved by having a large static deflection, which is undesirable. This disadvantage can be overcome by employing isolation mounts with a nonlinear characteristic. For examples, Platus (Platus, 1999) and Plaut (Plaut *et al.*, 2005) exploited the buckling of structures under axial load in a specific configuration to achieve low dynamic stiffness without compromising on the static displacement. Others have achieved the same by connecting linear springs with positive stiffness in parallel with mechanical elements of negative stiffness (Alabuzhev *et al.*, 1989; Carrella *et al.*, 2007) or using magnets as a source of negative stiffness (Carrella *et al.*, 2008). When sets of elements with positive and negative stiffness act in parallel it is possible to achieve High-Static-Low-Dynamic-Stiffness (HSLDS). More specifically, this type of systems can be optimally tuned so that at the static equilibrium position only the positive stiffness mechanism exerts a restoring force (and therefore it has the same static displacement as a standard or equivalent linear system). However, for oscillations about the static equilibrium position, the effect of the negative stiffness components is to reduce the dynamic stiffness which implies a lower natural frequency than the linear model and, as a consequence, a greater frequency range over which there is vibration isolation (Rivin, 2001). Most generally, the load-deflection curve of a HSLDS mount can be described by a polynomial function of n-th degree. However, if it is possible to reduce the polynomial to a symmetric cubic, a relatively easy analytical formulation can be obtained. In most cases, this can be done with a good degree of approximation and allows to write the equation of motion in the form of the Duffing equation which has been extensively studied (Nayfeh and Mook, 1995; Jordan and Smith, 1999). In the literature analysis of the Duffing oscillator is usually confined to the study of the system response to a harmonic force applied to the mass. Only few papers investigate the transmissibility of this type of system and the reference work on the subject is (Ravindra and Mallik, 1994). The analysis presented therein does not offer analytical expressions for the characterisation of the transmissibility. Peleg (Peleg, 1979) has given a more analytical description of the transmissibility of a nonlinear system with cubic restoring force but also lacks a final explicit expression. In this article, an analytical formulation of the isolation performance of a nonlinear isolation mounts in terms of the system transmissibility is proposed and a simple, explicit formula for the peak transmissibility of a vibration isolator with cubic nonlinearity is provided. Furthermore, a numerical comparison between the transmissibilities of the HSLDS and its equivalent linear mount shows the advantages offered by the nonlinear mount.

2 Response to a harmonic force

Fig. 1 is a schematic representation of a single-degree-of-freedom system with an HSLDS mount. A mass m is suspended on a dashpot c and a nonlinear mount with HSLDS characteristic, k_{HSLDS}. When an element with

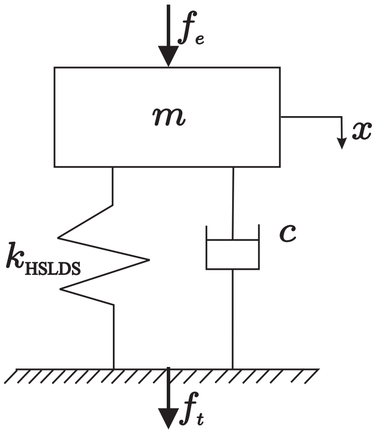

Figure 1. Single-degree-of-freedom system with an HSLDS mount with viscous damping: a mass m is suspended on a dashpot c in parallel with a nonlinear spring with HSLDS k_{HSLDS}. The excitation force acting on the mass is $f_e = F_e \cos\omega t$. f_t is the force transmitted to the base through the spring and the dashpot

constant positive stiffness is connected in parallel with a mechanism with nonlinear negative stiffness, e.g. the systems considered in (Carrella *et al.*, 2007; Carrella *et al.*, 2008), the restoring force can be expressed approximately as

$$f_{k_{\text{HSLDS}}} = k_1\, x + k_3\, x^3 \tag{1}$$

where k_1 and k_3 are the coefficients of the linear and nonlinear terms of the cubic restoring force respectively, and the sign of k_3 denotes the stiffness behaviour, hardening (+) or softening (-). It is important to recall that if the elements with negative stiffness (which confer the desired HSLDS characteristic) are removed then the isolator becomes a standard linear model, i.e. $k_3 = 0$, with constant stiffness, say, $k_l > k_1$. If the mass of the system shown in Fig. 1 is excited by a harmonic force $f_e = F_e\, \cos(\omega\, t)$ the

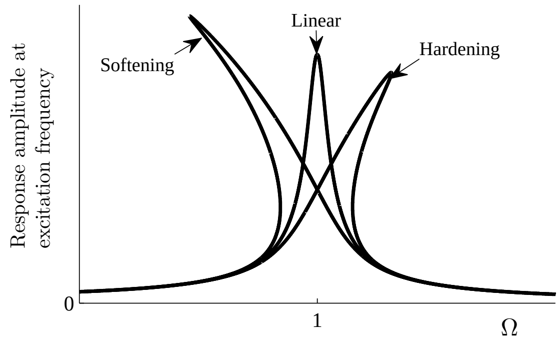

Figure 2. Frequency response function of the Duffing oscillator described by Eqn.(3). The sign of the cubic coefficient defines a softening (-) or hardening (+) behaviour. When $\alpha = 0$ the system becomes linear

equation of motion of the HSLDS model is

$$m\ddot{x} + c\dot{x} + k_1\, x + k_3\, x^3 = F_e\, \cos(\omega\, t) \tag{2}$$

It is helpful to nondimensionalise Eqn.(2), so that

$$\hat{x}'' + 2\,\zeta\,\hat{x}' + \hat{x} + \alpha\,\hat{x}^3 = \cos(\Omega\,\tau) \tag{3}$$

where:

$\zeta = \frac{c}{2m\omega_n}$ $\quad \omega_n^2 = \frac{k_1}{m}$ $\quad \alpha = \frac{k_3\, x_0^2}{k_1}$

$\Omega = \frac{\omega}{\omega_n}$ $\quad \tau = \omega_n\, t$

$\hat{x}'' = \frac{\ddot{x}}{\omega_n^2\, x_0}$ $\quad \hat{x}' = \frac{\dot{x}}{\omega_n\, x_0}$ $\quad \hat{x} = x/x_0$

with the symbol $'$ denoting differentiation with respect to the nondimensional time τ and

$$x_0 = \left.\frac{F_e}{k_1}\right|_{k_3=0,\,\omega=0} \tag{4}$$

It is noteworthy that ω_n *is not* the natural frequency of the HSLDS system but is a characteristic frequency which is the natural frequency of the linearised HSLDS isolator, i.e. when the amplitude of oscillations is small enough to make $\alpha\,\hat{x}^2 \ll 1$. Note also that α is a factor related to the type and degree of nonlinearity. Besides, α also takes into account the magnitude of the applied force because of its dependency on the displacement x_0. The role played by the sign of α is qualitatively shown in Fig. 2. When α is negative the FRF curve bends to the left, marking a softening behaviour. When $\alpha = 0$ the system becomes linear and the FRF assumes its standard shape with a peak at $\Omega = 1$ (when damping is small). Finally, as α is made positive, the plot leans over to the right because of its hardening characteristic.

It is possible to obtain an approximate analytical solution to Eqn.(3) with different methods (Nayfeh and Mook, 1995). Amongst them, here the preferred method is the Harmonic Balance (HB) to a first order expansion, i.e. it is assumed that the response is harmonic at the excitation frequency (Hamdan and Burton, 1993; Friswell and Penny, 1994; Worden, 1996)

$$\hat{x} = \hat{X}\,\cos(\Omega\,\tau + \varphi) \tag{5}$$

where $\hat{X}$ is the amplitude and φ the phase of the response. The application of the HB leads to the frequency equation that relates the amplitude and frequency of the response and is given by (Magnus, 1965)

$$\Omega_1^2 = \left(1 + \frac{3}{4}\alpha\,\hat{X}^2 - 2\,\zeta^2\right) + - \frac{1}{\hat{X}}\sqrt{1 - 4\,\zeta^2\,\hat{X}^2\left(1 - \zeta^2 + \frac{3}{4}\alpha\,\hat{X}^2\right)} \tag{6a}$$

$$\Omega_2^2 = \left(1 + \frac{3}{4}\alpha\,\hat{X}^2 - 2\,\zeta^2\right) + + \frac{1}{\hat{X}}\sqrt{1 - 4\,\zeta^2\,\hat{X}^2\left(1 - \zeta^2 + \frac{3}{4}\alpha\,\hat{X}^2\right)} \tag{6b}$$

From equations (6a,b) it is possible to derive an expressions for the maximum amplitude of the response, i.e. when the known phenomenon of the jump-down takes place and for the frequency at which this occurs (Magnus, 1965; Carrella, 2008), both depicted in Fig.3. The maximum amplitude is

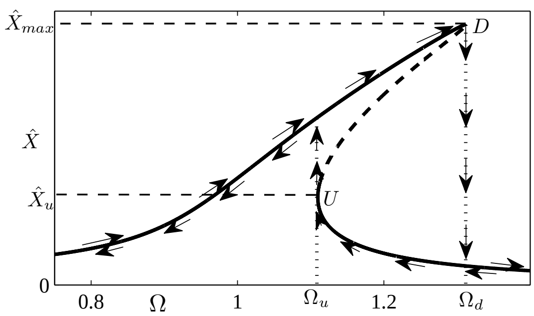

Figure 3. Plot of the frequency response of a hardening Duffing oscillator: as the frequency is increased the amplitude increases following the upper or resonant curve. At the frequency Ω_d, marked with the letter D, it suddenly drops to the lower or non-resonant branch. Similarly, decreasing the frequency, the response follows the non-resonant branch until the frequency Ω_u, marked with the letter U. A further decrease in frequency causes the response to jump up to the resonant branch.

$$\hat{X}_{max} \approx \sqrt{\frac{2}{3\,\alpha}\left[\sqrt{1 + \frac{3\,\alpha}{4\,\zeta^2}} - 1\right]} \tag{7}$$

and the jump-down frequency at which this occurs is

$$\Omega_d = \sqrt{\frac{3}{4}\alpha\,\hat{X}_{max}^2 + (1 - 2\,\zeta^2)} \tag{8}$$

By substituting Eqn.(7) into (8), and assuming that $\zeta \ll 1/\sqrt{2}$ ($\zeta \ll 0.7$) a simple expression for Ω_d can be found, which is

$$\Omega_d \approx \frac{1}{\sqrt{2}}\sqrt{1 + \sqrt{1 + \frac{3\,\alpha}{4\,\zeta^2}}} \tag{9}$$

From Eqn.(9) it can be seen that, when α is negative (softening system), for Ω_d to be real the following criteria has to hold

$$\alpha = \alpha_{max} \leq \frac{4}{3}\,\zeta^2 \tag{10}$$

Eqn.(10) expresses the fact that, when a softening system with a given damping ratio has too large a nonlinear coefficient, i.e. $|\alpha| > |\alpha|_{max}$, the two curves Ω_1 and Ω_2 never meet and the jump-down does not occur, (Hamdan and Burton, 1993). As it will be shown in the next section, Eqns.(7) and (9) will enable to characterise with simple analytical expressions the transmissibility of the nonlinear system.

3 Transmissibility of the HSDLS isolator

The quantity that is often used to evaluate the performance of an isolation mount is the absolute transmissibility which is non-dimensional and frequency-dependent. If the system is excited by a harmonic force applied to the mass the absolute transmissibility is the ratio between the magnitude of the transmitted force to a rigid foundation and the magnitude of the excitation force, in steady-state vibration and at a given excitation frequency. With reference to Fig. 1, the harmonic excitation force (source) acting on the mass is $f_e = F_e\,\cos(\omega\,t)$. The force transmitted to the base (receiver) is $f_t = F_t\,\cos(\omega\,t + \varphi_t)$. By definition the absolute transmissibility is

$$|T_a| = \frac{F_t}{F_e} \tag{11}$$

Expressing the equation of motion in nondimensional form as in Eqn.(3), the nondimensional transmitted force – through the spring and the dashpot – is

$$\hat{f}_t = 2\,\zeta\,\hat{x}' + \hat{x} + \alpha\,\hat{x}^3 \tag{12}$$

where $\hat{f}_t = (c\,x + k_1\,x + k_3\,x^3)/(k_1\,x_0)$. Being interested in the response at the excitation frequency only, as expressed by Eqn.(5), the transmitted force can be written as

$$\begin{aligned}\hat{f}_t &= -2\,\zeta\,\Omega\,\hat{X}\,\sin\,\theta_t + \left(\hat{X} + \frac{3}{4}\,\alpha\,\hat{X}^3\right)\,\cos\,\theta_t \\ &= A\,\sin\,\theta_t + B\,\cos\,\theta_t\end{aligned} \tag{13}$$

where $\theta_t = \cos(\Omega\,\tau + \varphi_t)$. Thus, the magnitude of the transmitted force is, (Ravindra and Mallik, 1994)

$$\hat{F}_t = \sqrt{A^2 + B^2} \tag{14}$$

On the other hand, from Eqn.(2), the non-dimensional magnitude of the applied force is $\hat{F}_e = 1$. It follows that the magnitude of the force transmissibility is

$$|T_a| = \frac{\hat{F}_t}{\hat{F}_e} = \hat{X}\,\sqrt{\left(1 + \frac{3}{4}\,\alpha\,\hat{X}^2\right)^2 + 4\,\zeta^2\,\Omega^2} \tag{15}$$

The transmissibility of a hardening system with $\zeta = 0.01$ and $\alpha = 10^{-4}$ is plotted in Fig.4. The transmissibility of a system with softening nonlinearity with $\zeta = 0.01$ and $\alpha = -10^{-4}$ is instead shown in Fig.5. Note that the curves have been obtained by substituting Eqns.(6) into Eqn.(15) and letting $\hat{X}$ vary between 0 and $\hat{X}_{max}$. The dashed part of the curve denotes the unstable solution (Hamdan and Burton, 1993; Rand, 2005).

3.1 Peak transmissibility

It can be argued that there are two indices to measure the effectiveness of a vibration isolator: one is the bandwidth of the isolation region, which is the frequency region within which the transmitted force becomes smaller than the excitation force, that is when $|T_a| < 1$[1]; the other is the peak-transmissibility, which is the maximum amplitude of the transmitted force for a given amplitude of the input force.

In order to find an expression for the maximum transmissibility, the maximum amplitude response, $\hat{X}_{max}$, and the frequency at which this occurs, i.e. the jump-down frequency Ω_d, given by Eqns.(7) and (9) respectively, are substituted into Eqn.(15). The resulting expression, valid for small damping, $\zeta \ll 1$, is

$$|T_a|_{max} \approx \frac{1}{2\,\sqrt{2}\,\zeta}\,\sqrt{1 + \sqrt{1 + \frac{3\,|\alpha|}{4\,\zeta^2}}} \tag{16}$$

[1] For a linear system this begins at $\sqrt{2}$ time the natural frequency (Rivin, 2001; Harris, 1995)

Recall that for a linear system the peak-transmissibility is given by, (Rivin, 2001; Harris, 1995)

$$|T_a|_{max\,(linear)} \approx \frac{1}{2\,\zeta} \tag{17}$$

It should also be noted that the expression for the peak transmissibility of a nonlinear isolator with symmetric cubic restoring force given by Eqn.(16), reduces to that of a linear system Eqn.(17) when $\alpha = 0$. In Figures 4 and 5 the transmissibility of hardening and softening HSLDS isolators with $\zeta = 0.01$ is plotted and the maximum transmissibility calculated with Eqn.(16) shown.

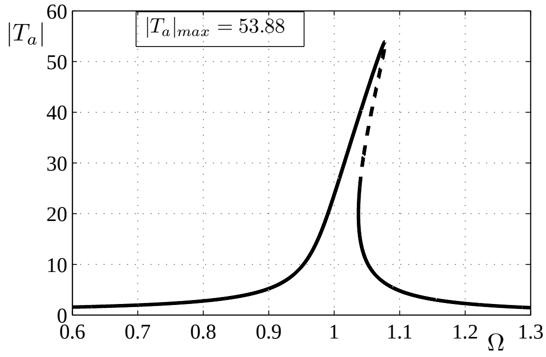

Figure 4. Absolute Transmissibility of an isolator with hardening HSLDS characteristic ($\alpha = 10^{-4}$) and $\zeta = 0.01$. The maximum transmissibility calculated with Eqn.(16) is also shown

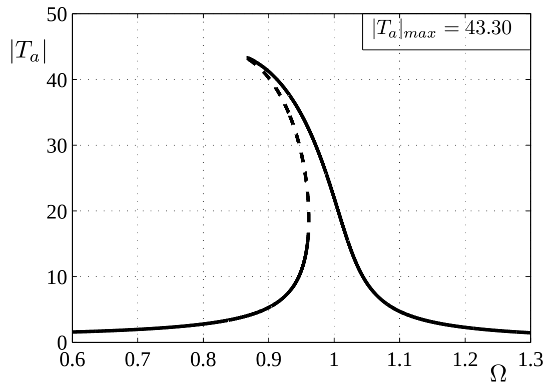

Figure 5. Absolute Transmissibility of an isolator with softening HSLDS characteristic ($\alpha = -10^{-4}$) and $\zeta = 0.01$. The maximum transmissibility calculated with Eqn.(16) is also shown

4 Comparison between the transmissibility of an HSLDS mechanism and an equivalent linear isolator

In order to assess the vibration isolation performance of an HSLDS mount, its transmissibility is now compared with that of an equivalent linear model. As said, one way of obtaining the HSLDS characteristic is to connect in parallel elements with constant positive with other with nonlinear negative stiffness. In this case the equivalent linear model is defined as the system deprived of the element with negative stiffness. If optimally tuned, that is if in the static equilibrium position the elements with negative stiffness are ineffective, the two systems have the same static stiffness. However, the insertion of elements with negative stiffness alters also the linear coefficient of the restoring force, Eqn.(1). Without loss of generality, it can be stated that

$$k_1 = \beta^2 \, k_l \tag{18}$$

where k_l is the stiffness of the equivalent linear isolator, and $0 < \beta^2 < 1$ depends on the type of negative stiffness mechanism. If there were no mechanism with negative stiffness then $\beta = 1$ and, of course, $k_1 = k_l$ and $k_3 = 0$. This observation is important when comparing the transmissibility curves of a linear and a HSLDS isolator. In fact, the dynamic properties of the HSLDS isolation mount (e.g. jump frequency, maximum transmissibility) have been derived in terms of the nondimensional parameters of Eqn.(3). A key parameter in the nondimensionalisation is the characteristic frequency $\omega_n^2 = k_1/m$ which is clearly different for a linear and a HSLDS system. In particular, for a linear system

$$\omega_l = \sqrt{\frac{k_l}{m}} \tag{19}$$

whilst for an HSLDS mount is

$$\omega_n = \sqrt{\frac{\beta^2 \, k_l}{m}} = \beta \, \omega_l \tag{20}$$

As a consequence, the value of the damping ratio also changes between a linear and a HSLDS isolator. The damping ratio of a linear system is given by

$$\zeta_l = \frac{c}{2 \, m \, \omega_l} \tag{21}$$

whereas for an HSLDS model is

$$\zeta = \frac{c}{2 \, m \, \omega_n} = \frac{\zeta_l}{\beta} \tag{22}$$

Eqns.(20) and (22) can be seen as 'scaling laws', by means of which the transmissibility curves of the HSLDS and linear isolator models can be plotted on the same graph.

For a linear system, the transmissibility is, (Harris, 1995)

$$|T_a|_{(linear)} = \sqrt{\frac{1 + 4 \, \zeta_l^2 \, \Omega_l^2}{\left(1 - \Omega_l^2\right)^2 + 4 \, \zeta_l^2 \, \Omega_l^2}} \tag{23}$$

where the nondimensional frequency ratio is $\Omega_l = \omega/\omega_l$.
On the other hand, the transmissibility of the HSLDS mount is given by Eqn.(15) where $\Omega = \omega/\omega_n$ or

$$\Omega_l = \Omega \, \beta \tag{24}$$

For the sake of consistency, when plotting the transmissibility of a HSLDS and a linear isolator on the same graph, the values on the frequency axis have to comply with Eqn.(24). To appreciate the benefit offered by the HSLDS isolator, the transmissibility of the linear and nonlinear isolator have been plotted on a decibel scale in Fig. 6. In the example shown the linear model has $\zeta_l = 0.005$ and for the hardening HSLDS isolator $\beta = 0.5$ and $\alpha = 10^{-4}$. Because the frequency ratio on the x-axis is $\Omega_l = \omega/\omega_l$, the value of Ω used to compute the transmissibility of the HSLDS system, given by Eqn.(15), had to be scaled according to Eqn.(24). As expected, the transmissibility of the linear system reaches its peak value of $1/2\zeta = 40$ dB at $\Omega_l = 1$. The wider range of the frequency isolation region and the reduction of the peak transmissibility are clearly visible. From the figure it can be seen that the HSLDS mount does indeed offer better isolation performance than its equivalent linear model. The peak value is smaller and the isolation region is extended. It is important to note that the 'bend' of the transmissibility

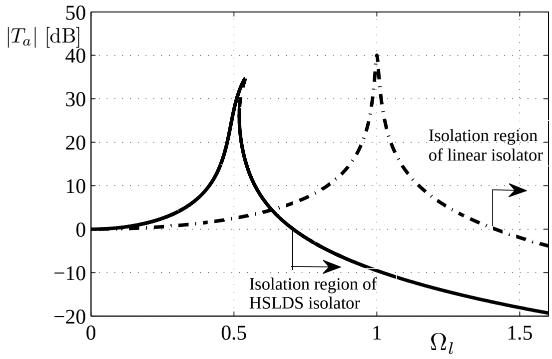

Figure 6. Comparison between the absolute transmissibility curves of a linear (-.) and a hardening HSLDS (–) mount. For the linear system $\zeta_l = 0.005$ and for the HSLDS mount $\zeta = 0.01$ ($\beta = 0.5$)

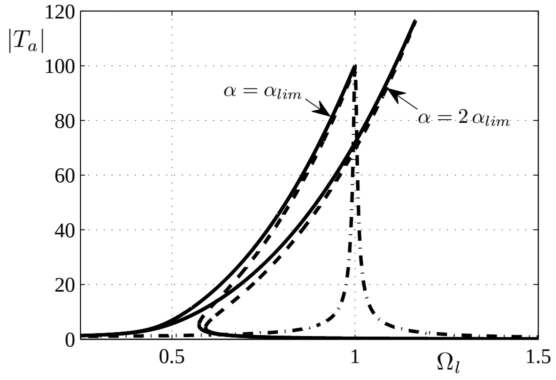

Figure 7. Transmissibility curves of a linear mount (-.) and of a two HSLDS (–) system with $\alpha = \alpha_{lim}$ and $\alpha = 2\,\alpha_{lim}$. For the linear system $\zeta_l = 0.005$ and for the HSLDS mount $\zeta = 0.01$ ($\beta = 0.5$)

curve depends on the coefficient of the nonlinear term α which, in turn, depends on the amplitude of the applied force and the coefficient of nonlinearity. If a system with a hardening HSLDS mount is subject to large amplitudes of excitation or has a strong nonlinearity, its transmissibility curve might intersect and even go beyond that of the linear mount[2]. In order to set a criterion for comparing the isolation performance, it can be argued that the benefits of a HSLDS mount cease when the jump-down frequency coincides with the natural frequency of its equivalent linear model. The limiting value of α can be thus found by imposing that

$$\beta\,\Omega_d = 1 \qquad (25)$$

If it is assumed that $\zeta \ll 1$, substituting Eqn.(9) in Eqn.(25) and solving for α yields

$$\alpha_{lim} = \frac{16}{3}\,\frac{\zeta^2\,(1-\beta^2)}{\beta^4} \qquad (26)$$

Fig. 7 shows the transmissibility curves of an HSLDS with $\beta = 0.5$ and $\zeta = 0.01$ when $\alpha = \alpha_{lim} = 0.0065$ and $\alpha = 2\,\alpha_{lim} = 0.013$. It can be seen that when $\alpha = \alpha_{lim}$ the jump-down frequency is equal to the natural frequency. The figure also shows that when $\alpha = \alpha_{lim}$ the linear and HSLDS isolator mounts have the same peak-transmissibility.

5 Conclusions

Nonlinear vibration isolators with high-static-low-dynamic-stiffness offer a solution to the problem of having to choose between a low natural frequency, desired for a wider frequency isolation bandwidth, and the consequent high static displacement that would result from using a linear softer mount. The dynamics of a mass suspended on a HSLDS spring can often be described by the Duffing equation. In this paper, the approximate solution to the Duffing equation has been used to determine simple analytical expressions for the maximum amplitude of the response and the jump-down frequency. These expressions have been used to derive an analytical expression for the transmissibility of the system which compares favourably against the equivalent linear isolator.

[2]For a softening system this issue does not arise.

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