THE PASSING THROUGH RESONANCE OF SYNCHRONOUS MACHINE ON ELASTIC PLATFORM

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Abstract
In the present paper it is proved that for synchronous machine on elastic platform with asynchronous actuation, the Sommerfeld effect is lacking for any passing through resonance.

Key words
synchronous machine, resonance, Sommerfeld effect

1 Introduction

The present paper was stimulated by the work [Blekhman, Indeitsev and Fradkov, 2007], where in the neighborhood of resonance a ”hanging” of oscillation frequency of machine on elastic platform was observed. This ”hanging” is realized in the form of random oscillations [Blekhman, Indeitsev and Fradkov, 2007] and is often called a Sommerfeld effect [Blekhman, Indeitsev and Fradkov, 2007; Blekhman, 1953; Blekhman, 1971; Blekhman, 2000; Fidlin, 2006; Panovko, Gubanova, 1979; Kononenko, 1964].

In studying a Sommerfeld effect a conventional mathematical model of machine was the following equation [Blekhman, Indeitsev and Fradkov, 2007; Blekhman, 1953; Blekhman, 1971; Blekhman, 2000; Fidlin, 2006; Panovko, Gubanova, 1979; Kononenko, 1964]

$$I\ddot{\varphi} = G(\dot{\varphi}) \tag{1.1}$$

Here $\varphi(t)$ is a phase, $\dot{\varphi}(t)$ is instantaneous frequency of rotation of motor armature, $G(x)$ is so-called ”machine characteristic minus load”, I is a moment of inertia of rotor.

In the present work it will be shown that under natural assumptions for synchronous machine with asynchronous actuation a Sommerfeld effect does not occur for any passing of resonance. This fact is based on a joint consideration of motions of elastic platform and mathematical models of synchronous machine, which are more complicated than (1.1). In this case it turns out that synchronous motors have internal stabilizing properties, preventing the occurrence of Sommerfeld effect.

2 The boundedness of solutions of equations of synchronous motor on elastic platform

Recall the classical equations of ”machine–elastic platform” system in the case of model (1.1) [Blekhman, Indeitsev and Fradkov, 2007; Blekhman, 1971; Panovko, Gubanova, 1979] (Fig. 1):

$$\begin{aligned} &I\ddot{\varphi} = G(\dot{\varphi}) + m\varepsilon\ddot{z}\sin\varphi \\ &M\ddot{z} + \beta\dot{z} + cz = -m\varepsilon(\cos\varphi)^{\bullet\bullet} \end{aligned} \tag{2.1}$$

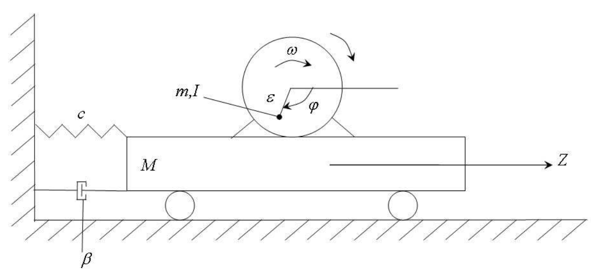

Figure 1.

Here φ is a phase of rotor rotation, z is a deviation of platform from equilibrium, M is a mass of platform, m is a mass of rotor, I is a moment of inertia of rotor, c and β are coefficients of elasticity and viscous friction, respectively, ε is an eccentricity.

As a mathematical model of synchronous machine we regard the following equation

$$I\ddot{\theta} = -\alpha\dot{\theta} - \sin\theta.$$

Then for the system: "synchronous motor – elastic platform" we obtain in a similar way the equation

$$\begin{aligned} &I\ddot{\theta} = -\alpha\dot{\theta} - \sin\theta + m\varepsilon\ddot{z}\sin(\omega t + \theta) \\ &M\ddot{z} + \beta\dot{z} + cz = -m\varepsilon(\cos(\omega t + \theta))^{\bullet\bullet} \end{aligned} \quad (2.2)$$

Here ω is a frequency of current in stator winding, θ is a phase difference of a rotating magnetic field and a rotor [Leonov, 2006; Leonov, 2006], α is a coefficient of damper windings. When the motor is asynchronously actuated, we have the relation $z(0) = \dot{z}(0) = \dot{\theta}(0) = 0$.

Having performed the change of variable

$$z = u + \varkappa\cos(\theta + \omega t), \quad \varkappa = -\frac{m\varepsilon}{M},$$

from the second equation of system (2.2) we obtain

$$\begin{aligned} M\ddot{u} + \beta\dot{u} + cu = &\frac{\beta m\varepsilon}{M}(\cos(\theta + \omega t))^{\bullet} + \\ &+ \frac{cm\varepsilon}{M}(\cos(\theta + \omega t)) \end{aligned} \quad (2.3)$$

Relation $z(0) = 0$ implies the estimate

$$|u(0)| \le \frac{m\varepsilon}{M} \quad (2.4)$$

and from the relations

$$\dot{z}(0) = \dot{\theta}(0) = 0$$

we have

$$|\dot{u}(0)| \le |\varkappa\omega| = \frac{m\varepsilon\omega}{M} \quad (2.5)$$

From equation (2.3) and estimates (2.4), (2.5) we obtain the following inequality

$$|u(t)| \le Q\varepsilon, \quad \forall\, t \ge 0$$

Here Q is a number, depending on the parameters β, c, m, M, ω. This inequality implies the estimate

$$|z(t)| \le D\varepsilon, \quad \forall\, t \ge 0 \quad (2.6)$$

where D is a number, depending on the parameters β, c, m, M, ω.

Rewrite system (2.2) in the following way

$$\begin{aligned} &I\ddot{\theta} - m\varepsilon\ddot{z}\sin(\omega t + \theta) = -\alpha\dot{\theta} - \sin\theta \\ &M\ddot{z} - m\varepsilon\ddot{\theta}\sin(\omega t + \theta) = -\beta\dot{z} - cz + \\ &+ m\varepsilon(\cos(\omega t + \theta))(\omega + \dot{\theta})^2. \end{aligned} \quad (2.7)$$

System (2.7) is equivalent to the system

$$\begin{aligned} &\left(I - \frac{(m\varepsilon)^2}{M}(\sin(\omega t + \theta))^2\right)\ddot{\theta} = -\alpha\dot{\theta} - \\ &- \sin\theta + \frac{m\varepsilon}{M}(\sin(\omega t + \theta))(-\beta\dot{z} - cz + \\ &+ m\varepsilon(\cos(\omega t + \theta))(\omega + \dot{\theta})^2) \end{aligned} \quad (2.8)$$

$$\begin{aligned} &\left(M - \frac{(m\varepsilon)^2}{I}(\sin(\omega t + \theta))^2\right)\ddot{z} = -\beta\dot{z} - \\ &- cz + m\varepsilon((\dot{\theta} + \omega)^2\cos(\omega t + \theta)) + \\ &+ \frac{m\varepsilon}{I}\sin(\omega t + \theta)(-\alpha\dot{\theta} - \sin\theta) \end{aligned} \quad (2.9)$$

Consider further the function $V = \dot{\theta}^2 + \dot{z}^2$ and the values of $\dot{\theta}$ and $\dot{z}$ such that

$$V = R \quad (2.10)$$

It is clear that (2.10) yields the inequalities

$$|\dot{\theta}| \le \sqrt{R} \quad (2.11)$$

$$|\dot{z}| \le \sqrt{R} \quad (2.12)$$

For a derivative of the function V along trajectories of system (2.8), (2.9) we obtain the relation

$$\begin{aligned} \frac{1}{2}\dot{V} = &\frac{\dot{\theta}}{I(1 - \frac{(m\varepsilon)^2}{IM}\sin(\omega t + \theta)^2)}[-\alpha\dot{\theta} - \sin\theta + \\ &+ \frac{m\varepsilon}{M}\sin(\omega t + \theta)(-\beta\dot{z} - cz + \\ &+ m\varepsilon(\cos(\omega t + \theta))(\omega + \dot{\theta})^2] + \\ &+ \frac{\dot{z}}{M(1 - \frac{(m\varepsilon)^2}{IM}\sin(\omega t + \theta)^2)}[-\beta\dot{z} - cz + \\ &+ m\varepsilon((\dot{\theta} + \omega)^2\cos(\omega t + \theta) + \\ &+ \frac{m\varepsilon}{I}\sin(\omega t + \theta)(-\alpha\dot{\theta} - \sin\theta)] \end{aligned}$$

From estimates (2.6), (2.11), (2.12), and relation (2.10) it follows that for sufficiently large R and small ε the following estimate holds

$$\dot{V} \le -\delta R + E\varepsilon R^2$$

Here δ and E are certain positive numbers, depending on the parameters $\beta, c, m, M, \omega, I, \alpha$.

It is obvious that for sufficiently small $\varepsilon \in [0, \varepsilon_0], \varepsilon_0 = \varepsilon_0(R)$ this estimate results in the inequality

$$\dot{V} < 0$$

Then from relation (2.10) it follows that for the considered solution of system (2.2) with the initial data $z(0) = 0, \dot{z}(0) = 0, \dot{\theta}(0) = 0$ the inequality

$$\dot{z}(t)^2 + \dot{\theta}(t)^2 \leq R, \quad \forall\, t \geq 0$$

is satisfied for sufficiently large R (with respect to the parameters $I, M, m, \beta, c, \alpha, \omega$) and small $\varepsilon \in [0, \varepsilon_0(R)]$.

Thus, we have the following

Theorem 1. For sufficiently small $\varepsilon > 0$ the solution $\theta(t), z(t)$ with the initial data $\dot{\theta}(0) = z(0) = \dot{z}(0) = 0$ satisfies the estimate

$$\begin{aligned} &|\dot{\theta}(t)| \leq L, \quad |z(t)| \leq L, \\ &|\dot{z}(t)| \leq L,\ |\ddot{z}(t)| \leq L,\ \forall\, t \geq 0 \end{aligned}$$

for sufficiently large L.

3 Asymptotical estimates of solutions of pendulum type equation for small nonstationary disturbances

Consider now the equation

$$\ddot{x} + a\dot{x} + \sin x = p(t) \tag{3.1}$$

where a is a positive number, $p(t)$ is a continuous function, satisfying the following condition

$$|p(t)| \leq \varepsilon, \quad \forall\, t \in R^1 \tag{3.2}$$

Here ε is a certain positive number, which, by assumption, is small with respect to a and 1: $\varepsilon \ll a$, $\varepsilon \ll 1$.

Theorem 2. For any solution of equation (3.1) there exists an integer number k such that the inequality

$$\overline{\lim_{t \to +\infty}} |x(t) + k\pi| \leq C\varepsilon \tag{3.3}$$

is valid.

Here C satisfies the relations

$$\begin{aligned} &C > 1 \quad \text{for} \quad a \geq 2 \\ &C > \frac{1+P}{1-P},\ P = \exp\left(-\frac{a\pi}{\sqrt{4-a^2}}\right) \quad \text{for} \quad a < 2 \end{aligned}$$

Proof. Consider a system, which is equivalent to equation (3.1),

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_2 - \sin x_1 + p(t) \end{aligned} \tag{3.4}$$

and the so-called systems of comparison [Leonov, 2006]

$$\begin{aligned} \dot{x}_2 &= x_2 \\ \dot{x}_2 &= -ax_2 - \sin x_1 - \varepsilon \end{aligned} \tag{3.5}$$

$$\begin{aligned} \dot{x}_2 &= x_2 \\ \dot{x}_2 &= -ax_2 - \sin x_1 + \varepsilon \end{aligned} \tag{3.6}$$

It is well known [12] that for sufficiently small ε all solutions of systems (3.5) and (3.6) are bounded on the time interval $(0, +\infty)$. Therefore [Leonov, 2006] there exist the solutions $F_k(\sigma)$ and $G_k(\sigma)$ of the equations

$$F'F + aF + \sin\sigma = -\varepsilon \tag{3.7}$$

$$G'G + aG + \sin\sigma = \varepsilon \tag{3.8}$$

which satisfy the relations (Fig. 2)

$$\begin{aligned} &F_k(\sigma_1 + 2\pi k) = 0 \\ &F_k(\sigma) < 0,\ \forall\,\sigma > \sigma_1 + 2k\pi \\ &F_k(\sigma) > 0,\ \forall\,\sigma < \sigma_1 + 2k\pi \\ &\lim_{\sigma\to\infty} |F_k(\sigma)| = +\infty \end{aligned} \tag{3.9}$$

$$\begin{aligned} &G_k(\sigma_2 + 2k\pi) = 0 \\ &G_k(\sigma) < 0,\ \forall\,\sigma > \sigma_2 + 2k\pi \\ &G_k(\sigma) > 0,\ \forall\,\sigma < \sigma_2 + 2k\pi \\ &\lim_{\sigma\to\infty} |G_k(\sigma)| = +\infty \end{aligned} \tag{3.10}$$

Here σ_1 and σ_2 are zeros, of the functions $\sin\sigma + \varepsilon$ and $\sin\sigma - \varepsilon$, respectively, on the set $[0, 2\pi)$, such that $\cos\sigma_1 < 0$ and $\cos\sigma_2 < 0$.

Consider now the solution $x_1(t), x_2(t)$, of system (3.4), which for the certain t satisfies the condition

$$x_2(t) = G_k(x_1(t)) > 0$$

Obviously,

$$\frac{\dot{x}_2(t)}{\dot{x}_1(t)} = \frac{-aG_k(x_1(t)) - \sin x_1(t) + p(t))}{G_k(x_1(t))} <$$

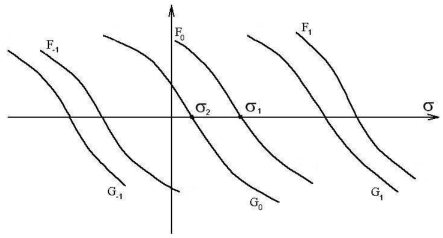

Figure 2.

$$< \frac{-aG_k(x_1(t)) - \sin x_1(t) + \varepsilon}{G_k(x_1(t))} = \left.\frac{dG_k(x)}{dx}\right|_{x=x_1(t)}$$

Hence the curve $x_2 = G_k(x_1),\ x_2 > 0$ is transversal with respect to a vector field of system (3.4) and the solution $x_1(t), x_2(t)$ traverses this curve "top-down" (Fig.3)

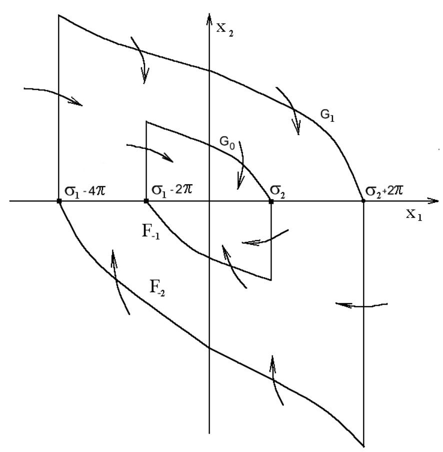

Figure 3.

We can prove similarly that the curve $x_2 = F_k(x_1),\ x_2 < 0$ is transversal with respect to the vector field of system (3.4) and the solution $x_1(t), x_2(t)$ transverses this curve "top-down" (Fig.3).

Thus, we have a family of closed transversal curves, which are shown in Fig. 3. This family depends on the integer parameter k and for any point of the space $\{x_1, x_2\}$ it can be found a closed transverse curve, containing inside this point. This implies at once the boundedness of any solution of system (3.4) in the time interval $(0, +\infty)$. Besides, from Fig. 3 and the relation $\dot{x}_1(t) = x_2(t)$ it follows that for any solution $x_1(t), x_2(t)$ either there exists a pair of numbers $\tau > 0$ and k such that

$$x_2(\tau) = 0, \quad x_1(\tau) \in (\sigma_2 + 2k\pi, \sigma_1 + 2(k+1)\pi)$$

or there exists k such that

$$\lim_{t\to+\infty} x_2(t) = 0,$$
$$\lim_{t\to+\infty} x_1(t) \in [\sigma_2 + 2k\pi, \sigma_1 + 2(k+1)\pi].$$

This implies that for any solution $x_1(t), x_2(t)$ of system (3.4) there exists a number $T > 0$ such that it satisfies the following condition.

For $t \geq T$ the solution $x_1(t), x_2(t)$ belongs for the certain k to the set Φ_k (Fig. 4):

$$\Phi_k = \Omega_k \cup \Psi_k \cup \Omega_{k+1}$$
$$\Omega_k = \{x_1 \in [\sigma_1 + 2k\pi, \sigma_2 + 2(k+1)\pi],$$
$$F_k(x_1) \leq x_2 \leq G_{k+1}(x_1)\}$$
$$\Psi_k = \{x_1 \in [\sigma_2 + 2(k+1)\pi, \sigma_1 + 2(k+1)\pi],$$
$$\widetilde{F_{k+1}}(x_1) \leq x_2 \leq \widetilde{G_{k+1}}(x_1)\}.$$

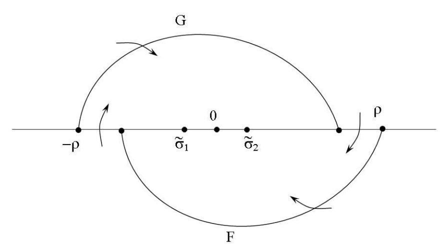

Figure 4.

Here $\widetilde{F}_k(\sigma)$ and $\widetilde{G}_k(\sigma)$ are solutions of equations (3.7) and (3.8), respectively, satisfying the following properties:

$$\widetilde{F}_k(\sigma_1 + 2k\pi) = 0$$
$$\widetilde{F}_k(\sigma) < 0, \quad \forall \sigma \in (\sigma_2 + 2k\pi, \sigma_1 + 2k\pi)$$
$$\widetilde{G}_k(\sigma_2 + 2k\pi) = 0$$
$$\widetilde{G}_k(\sigma) > 0, \quad \forall \sigma \in (\sigma_2 + 2k\pi, \sigma_1 + 2k\pi).$$

From Fig. 4 we see that the sets Φ_k are positively invariant. In addition, either for all $t \geq T$ the solution $x_1(t), x_2(t)$ is situated in Ψ_k, either there exists $\tau \geq T$ such that for all $t \geq \tau$ this solution is situated in Ω_k or Ω_{k+1}.

In the first case we obtain the relation

$$\overline{\lim_{t\to+\infty}} \left|x_1(t) + \frac{\sigma_1 + \sigma_2}{2} + 2k\pi\right| \leq |\sigma_1 - \sigma_2|. \quad (3.11)$$

Since for small ε we have

$$\sigma_1 = \pi + \varepsilon + o(\varepsilon)$$
$$\sigma_2 = \pi - \varepsilon + o(\varepsilon),$$

inequality (3.11) yields relation (3.3) for $C > 1$.

Consider now the case when $x_1(t), x_2(t)$ is situated in Ω_k for all $t \geq \tau$.

Let us construct a continuum family of transversal curves.

Suppose that $\widetilde{\sigma_1}$ and $\widetilde{\sigma_2}$ are zeros, of the functions $\sin\sigma + \varepsilon$ and $\sin\sigma - \varepsilon$, respectively, on the set $[-\pi, \pi)$, such that $\cos\widetilde{\sigma_1} > 0$ and $\widetilde{\sigma_2} > 0$. Clearly, $\widetilde{\sigma_1} = -\varepsilon + o(\varepsilon)$, $\widetilde{\sigma_2} = \varepsilon + o(\varepsilon)$.

Without loss of generality, consider the case Ω_{-1}.

Introduce the parameter $\rho \geq \rho_0 \geq \widetilde{\sigma_2}$.

We choose the number ρ_0 in such a way that in the half-plane $\{F \leq 0\}$ the solution of equation (3.7) with the initial data $F(\rho) = 0$ has the following property

$$F(\sigma) < 0, \quad \forall\, \sigma \in (-\nu, \varrho), \;\; F(-\nu) = 0, \; -\nu > -\rho.$$

Similarly, in the half-plane $\{G \geq 0\}$ the solution of equation (3.8) with the initial data $G(-\rho) = 0$ has the property:

$$G(\sigma) > 0, \quad \forall\, \sigma \in (-\rho, \nu), \quad G(\nu) = 0, \quad \nu < \rho.$$

It is easily shown, as before, that the curves $x_2 = F(x_1)$, $x_2 = G(x_1)$ are transversal for system (2.3) (Fig. 5)

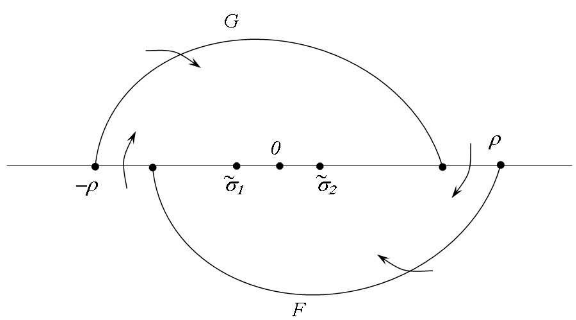

Figure 5.

Thus, in the set Ω_{-1} we constructed a family of the transversal closed curves $\gamma(\rho)$ of the type

$$\begin{aligned} &x_2 = G(x_1), \quad x_1 \in (-\rho, \nu)\\ &x_2 = 0, \quad x_1 \in (\nu, \rho)\\ &x_2 = F(x_1), \quad x_1 \in (-\nu, \rho)\\ &x_2 = 0, \quad x_1 \in (-\rho, -\nu). \end{aligned}$$

It follows that for the solution $x_1(t), x_2(t)$, of system (3.4), from Ω_{-1} there exists a number τ such that for $t \geq \tau$ this solution is inside the curve $\gamma(\rho)$, $\rho > \rho_0$.

Determine now the number ρ_0, making use of the smallness of ε.

By the linearization of systems of comparison (3.5) and (3.6) equivalent to equations (3.7) and (3.8), respectively, for $a \geq 2$ we obtain at once the following relation

$$\rho_0 = \sigma_2. \tag{3.12}$$

For $a < 2$ for a linearized system we have the formula

$$\rho_0 = \frac{1+P}{1-P}\varepsilon. \tag{3.13}$$

Thus, for small $\varepsilon > 0$ the estimate

$$\overline{\lim_{t\to+\infty}} |x_1(t)| \leq \rho < \rho_0$$

is satisfied.

Then from (3.12) and (3.13) we obtain estimate (3.3).

Theorem 2 is proved.

Note that Theorem 2 permits the extensions for different nonautonomous nonlinear two-dimensional systems with cylindrical phase space in the spirit of the works [Leonov, 2006; Leonov, 2006; Leonov, 2001; Leonov, Burkin and Shepelyavyi, 1996; Leonov, Ponomarenko and Smirnova, 1996; Leonov, Reitmann, 1987].

Estimate (3.3) implies that for small ε the inequality

$$\overline{\lim_{t\to+\infty}} |\sin x(t)| \leq C\varepsilon$$

is valid.

Then from condition (3.2) we can prove the following

Theorem 3. For any solution of equation (3.1) the estimates

$$\overline{\lim_{t\to+\infty}} |\dot{x}(t)| \leq \frac{C+1}{a}\varepsilon \tag{3.14}$$

$$\overline{\lim_{t\to+\infty}} |\ddot{x}(t)| \leq 2(C+1)\varepsilon \tag{3.15}$$

are satisfied.

4 The proof of the lack of a Sommerfeld effect

We now apply Theorems 1–3 to system (2.2), assuming $t = \tau\sqrt{I}$, $p(\tau) = \frac{m\varepsilon}{I}\ddot{z}\sin(\sqrt{I}\omega\tau + \theta)$. From Theorem 1 it follows that for any solution of system (2.2) with the initial data $\dot{\theta}(0) = z(0) = \dot{z}(0) = 0$ for the certain k we have the relation

$$|p(t)| \leq \frac{m\varepsilon L}{I}, \quad \forall\, t \geq 0.$$

Then by Theorems 2 and 3 we obtain the estimates

$$\overline{\lim_{t\to+\infty}}|\theta(t)+k\pi|\le C\frac{m\varepsilon L}{I},\quad a=\frac{\alpha}{\sqrt{I}}$$
$$\overline{\lim_{t\to+\infty}}|\dot\theta(t)|\le\frac{(C+1)m\varepsilon L}{\alpha\sqrt{I}}$$
$$\overline{\lim_{t\to+\infty}}|\ddot\theta(t)|\le\frac{2(C+1)}{I}m\varepsilon L.$$

Applying this estimates and elementary trigonometric transformations to the second equation of system (2.7), we obtain the relation

$$M\ddot z+\beta\dot z+cz=m\varepsilon\omega^2\cos\omega t+O(\varepsilon^2),$$

which is satisfied for large t.

This implies that after the transient process in synchronous machine it is established an operating mode with the rotor speed $\omega+O(\varepsilon)$, in which case the vibrations of elastic platform are harmonic with the frequency $\omega+O(\varepsilon)$ and the amplitude

$$\frac{m\varepsilon\omega^2}{|M\omega^2-c-\beta i\omega|}+O(\varepsilon^2).$$

Thus, if the eigenfrequency of elastic platform is less than ω, then for the asynchronous actuation in a transient process the system: ”synchronous machine – elastic platform”, always jumps the resonance and puts on synchronous operation.

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