Nonlinear Dynamics of Phase Control System Modeling by Modified Circle Map

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***Abstract*—Digital phase control system, which has a mathematical model of modified circle map, is considered from the viewpoint of chaotic generator, that can be used in secure ultrawideband communication networks. The diagram of dynamical modes, representing in parameter space the regions of different regular and chaotic attractors and its coexistence as well as Lyapunov exponent diagram are investigated.**

I. Introduction

The modern information and communication systems have to meet the requirements of transmitted information security with respect to unauthorized access [1]. One of the perspective ways to provide the confidentiality in telecommunications is the using of chaotic signals formed by nonlinear dynamical systems. Information sent with the help of random-like chaotic oscillations cannot be recovered by standard methods of signal processing and at absence of a priori knowledge's about the dynamical system.

An important element in such systems is the generator of chaotic signals. Phase control systems are paid an especial attention in literature, because of their demonstrate chaos in non-power parameters of quasi-sinusoidal signal, i.e. in its phase and frequency [2]. Examples of such systems are phase-locked loop (PLL), high-frequency amplifier (HFA) with phase control (APC), injection locked oscillators (ILO) with PLL, frequency-locked loop (FLL) and so on. All these systems have been originally developed for stabilization, synchronization, etc. Chaotic generators based on systems of this class have a number of advantages, in contrast with other generators, the most important amongst which is the possibility of direct generation of chaotic signals at high-frequency band.

Chaotic generators, modeled by discrete maps possess of interesting and various dynamical properties. They can be realized on basis of digital or impulse devices, operating in discrete time. Digital chaotic oscillator implementation would permit to provide the precision of chaotic signal generation and processing quality, stability of parameters, flexibility and reliability of operation.

The very perspective type of such oscillators is digital phase control system, the well-known variant of which is digital PLL [3-5]. It can be modeled by standard circle map, which is investigated in many works. Technically, schematic of DPLL, proposed in publications, includes the voltage-controlled oscillator (VCO), which forms at its output the quasi-periodical signal with period T_s, driven by control signal g, produced in feedback loop of the system. The standard circle map looks as: $x_{k+1} = x_k + a - \frac{b_T}{2\pi}\sin 2\pi x_k \mod 1$, where a is a ratio of VCO natural oscillations period and period of input signal, b_T means normalized maximum corrective phase detuning of VCO.

Standard circle map chaotic dynamics investigations show, that this variant of generator would have drawbacks from the practical point of view. Some of them are the pronounced multistability of dynamical modes structure in supercritical region of parameter space (i.e. for maximum corrective detuning $b_T > 1$), high value of parameter b_T, needed for chaotic regime with stable characteristics (preferably $b_T > 3$), etc. The last is especially important; because of it is very difficult to provide the high value of corrective phase detuning in the high-frequency band. So, these circumstances require to research for other variants of digital phase control (DPC) systems.

In given work we consider a DPC system variant, where a control signal g manages the frequency $\omega=1/T$ of VCO output signal, that is seems to be easier from the implementation point of view then period management in high-frequency band. Such system has the mathematical model in the form of so-called modified circle map. DPC system is considered in the general form, where a controlled elements (CE) is used, which can be represented as VCO, HFA, ILO and so on. This dynamical system, as we know, practically has not been investigated in public literature.

II. DPC Modeling by Modified Circle Map

A. DPC system block diagram and mathematical model

The DPC system block diagram is shown in Fig. 1. It consists of three main parts: impulse phase discriminator (IPD) of "sampling-memory" type, controlled element (CE) and control circuit (CC). In the case of using of HFA or ILO as CE the input signal has to be applied at the input of CE, that is not shown in given figure. CC consists of a passive filter with transmission function $W_f(p)$, where $p\equiv d/dt$ is a differential operator. IPD has two main parts: "Sampling" and "Memory".

The principle of DPC system operation shortly is the following. Let's consider that an output signal of CE $v(t)$ has rectangular form (generally, the type of CE output signal is not

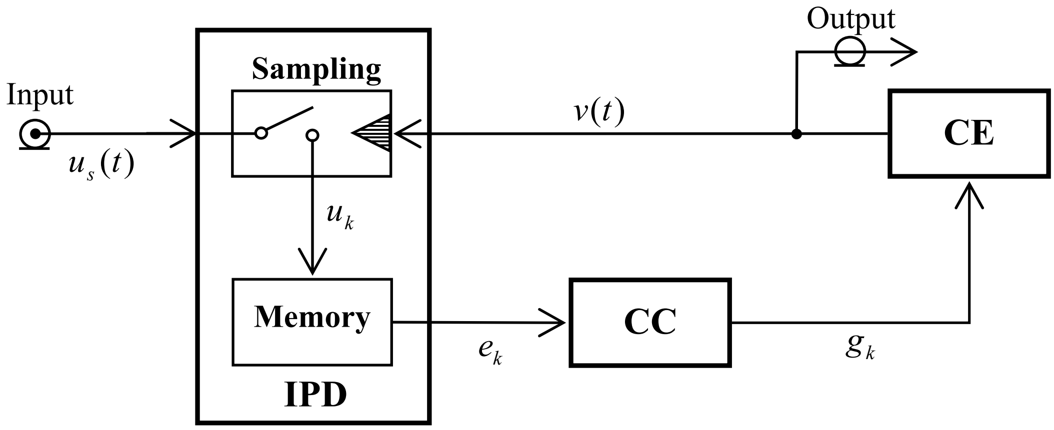


Fig. 1. Block diagram of digital phase control system

very important, it can be sinusoidal, triangular or other). The signal rise-up portion is used in sampling circuit of IPD for sampling of input signal instant values u_k at the time moment t_k, where k is a discrete time index, k=0,1,…,K, K is an observation interval. The memory circuit saves (in the case of ideal memory) this value until the next sampling moment t_{k+1}, i.e. during the whole time interval $T_{k+1}=t_{k+1}-t_k$. In such a way, the error signal: $e_k \equiv u_k$.

Let's an input signal $u_s(t)$ being a 2π-periodical sinusoidal function with constant amplitude U, frequency $\omega_s=2\pi/T_s$, where T_s is a period, and initial phase θ_0

$$u_s(t) = U\sin(\omega_s t+\theta_0) = U\sin\varphi_s(t), \quad (1)$$

where $\varphi_s(t)=\omega_s t+\theta_0$ is a full phase of $u_s(t)$. Generally, the input signal can has arbitrary form, including triangular, rectangular or other (the type of input signal is important, because of it affects on the system nonlinearity, see below). Sampling values at moments of time t_k are

$$u_k \equiv u(t_k) = U\sin\varphi_s(t_k) = U\sin\varphi_k, \quad (2)$$

where $\varphi_k=\omega_s t_k+\theta_0$ is a discrete phase ($0\le\varphi_k\le 2\pi$). We consider the case of $W_f(p)\equiv 1$, then a control signal: $g_k \equiv e_k$.

The voltage g_k is used in CE for the sampling time moment t_{k+1} management. In general case, it can be realized by two different ways. The first is a management of output signal period, when g_k controls the interval T_{k+1} with the linear function: $T_{k+1} \equiv t_{k+1}-t_k = T_{\mathrm{CE}} - S_T g_k$, where T_{CE} is a CE natural oscillations period, S_T is a modulation characteristics tangent of period controller, included in CE. In that way, we have DPC system modeling by standard circle map [3-5].

The second principle is the management of output signal frequency, when g_k controls the frequency $\omega_{k+1}=2\pi/T_{k+1}$ by the following way

$$\omega_{k+1} \equiv \frac{2\pi}{T_{k+1}} \equiv \frac{2\pi}{t_{k+1}-t_k} = \omega_{\mathrm{CE}} - S_\omega g_k, \quad (3)$$

where ω_{CE} is a CE natural oscillations frequency ($\omega_{\mathrm{CE}}=2\pi/T_{\mathrm{CE}}$), S_ω is a modulation characteristics tangent of frequency controller, including in CE.

Now, we can derive the model of DPC in the form of discrete map. Due to $g_k=e_k=u_k$ after substitution of (2) into (3) we obtain

$$t_{k+1} = t_k + \frac{2\pi}{\omega_{\mathrm{CE}} - \omega_M \sin\varphi_k}, \quad (4)$$

where $\omega_M=S_\omega U$ is a maximum corrective frequency detuning.

After multiplying of (4) by ω_s and adding θ_0 to each part of equation we obtain the modified circle map for normalized phase $x_k = \varphi_k/2\pi$, ($0\le x_k\le 1$)

$$x_{k+1} = x_k + \frac{a}{1-b_\omega \sin 2\pi x_k} \mod 1, \quad (5)$$

where $a = T_{\mathrm{CE}}/T_s \equiv \omega_s/\omega_{\mathrm{CE}}$, $b_\omega \equiv S_\omega U/\omega_{\mathrm{CE}}$ are two normalized parameters of the system. The operation "mod 1" means that values of x_k and $x_k \pm 1$ are identified.

B. *Modified circle map nonlinearity form*

Let's consider nonlinear map function features of system (5), for different parameters a and b_ω in interval (0,1). Fig. 2 shows a nonlinear map function changing tendency, when a is fixed and equals to 0.2, but b_ω is varied from 0.5 until 0.8.

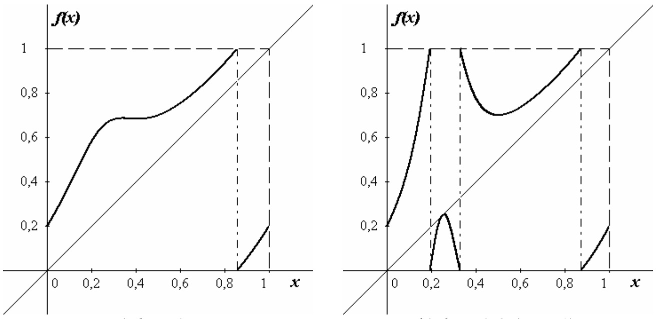
The case of a=0 doesn't have an interest because of the trivial form of map: $x_{k+1}=x_k$. If $b_\omega=0$, then map has also simple form: $x_{k+1}=(x_k+a) \mod 1$, that corresponds to opened feedback loop and asynchronous mode of DPC system. In other cases the increasing of any parameter from 0 to 1, if another is fixed, results in map function complication. The solution existence condition of equation $2\pi a b_\omega \cos 2\pi x_{\mathrm{inv}} = (1-b_\omega \sin 2\pi x_{\mathrm{inv}})^2$ with respect to x_{inv} gives us zones of parameter space, where the system (5) stays invertible. The stationary state equation

$$\left[\frac{a}{1-b_\omega \sin 2\pi x^*}\right] \mod 1 = 0 \quad (6)$$

gives the coordinates of fixed points x^*. The condition of its existence is

$$\frac{1-a/n_{\mathrm{fpb}}}{|b_\omega|} \le 1, \quad (7)$$

where n_{fpb}=0,1,2,… is the parameter, connected with the number of fixed points at bifurcation border. For instance, if n_{fpb}=1, then expression, defined a bifurcation border of 1st fixed point appearance, is $b_\omega = 1-a$ (Fig. 2,b); if n_{fpb}=2, then 2nd fixed point is occurred at the border $b_\omega = 1-a/2$ and so on. If we slightly increase the parameter b_ω more then its bifurcation value, then a fixed point will split into two different fixed points. Due to (7), if $b_\omega \to 1$, then maximum possible value of $n_{\mathrm{fpb}} \to \infty$ and the number of fixed points also tends to infinity.



a) $b_\omega = 0.5$ b) $b_\omega = 0.8$ (n_{fpb}=1)

Fig. 2. Nonlinearities of modified circle map; a=0.2

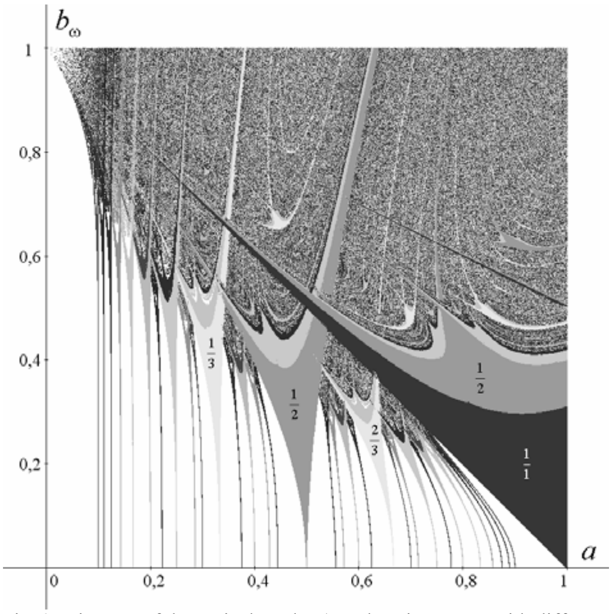


Fig. 3. Diagram of dynamical modes (synchronizm areas with different winding numbers)

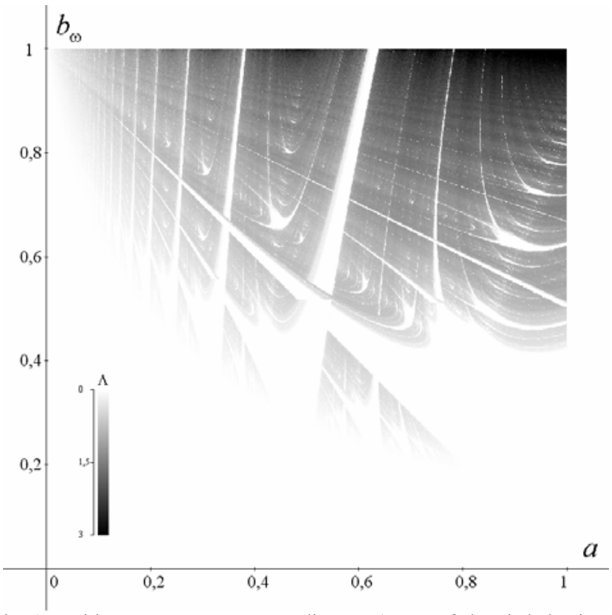


Fig. 4. Positive Lyapunov exponent diagram (zones of chaotic behaviour)

C. Diagram of Dynamical modes

It is possible to mark out regions of different dynamical behaviors of the system in parameter space (a,b_ω), i.e. to investigate the two-dimension diagram of dynamical modes. The analogous diagram, obtained for standard circle map in parameter interval (0,1), is known as "Arnold's tongues" [3].

The diagram of dynamical modes for modified circle map is shown in Fig. 3. It consists of the synchronism areas (they are expressed in diagram by different colors) with different "winding numbers", which are defined as

$$W = \lim_{k\to\infty} \frac{x_k - x_0}{k}. \qquad (8)$$

Winding number W represents the mean value of output signal phase shift by one iteration (sampling). It can be presented in the form $W=p/q$, where p and q are integers. Some synchronism areas concern the horizontal axis (b_ω=0) in points a, that values just equal to corresponding winding number W.

The quantity of different synchronism areas is infinite (there are only finite numbers of them in Fig. 3, which correspond to $q\leq10$, $p\leq10$, some values of W are pointed in the diagram). There is own area for any rational value W in the parameter space.

In some regions of diagram the synchronism areas overlap with each other, this corresponds to bifurcation of "fold" type. In these regions of fold's the system is multistable, i.e. demonstrates the behavior dependence on initial condition x_0. There are regions, where a lot of synchronism areas overlap and merge to zones of chaos.

Areas of chaotic modes are shown in Fig. 4, where only positive Lyapunov exponent (Λ>0) of system (5) is marked out by appropriate color. As one can see, there are several more or less solid regions of chaotic modes for no big values of maximum corrective frequency detuning in the diagram (b_ω>0.5).

III. CONCLUSIONS

Digital phase control system that includes the controlled element, generating the output signal with managed frequency by the control signal of feedback loop is considered. Discrete time mathematical model of such DPC system has been built in the form of modified circle map.

Synchronism areas, corresponded to different winding numbers of modified circle map, are calculated in the region of system parameters (0,1). Its fine structure and overlapping are revealed. By investigating the diagram of Lyapunov exponent, chaotic oscillations in the system have been detected. They arise here for relatively small values of parameters (even for b_ω~0.3) and have relatively wide size for b_ω>0.5.

DPC system, which is modeled by modified circle map, can be used as generator of chaotic signals for ultrawideband chaotic communication networks.

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