ROTATIONAL SOLUTIONS FOR ELLIPTICALLY EXCITED PENDULUM

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Abstract
The author considers the planar rotational motion of the mathematical pendulum with its pivot oscillating both vertically and horizontally, so the trajectory of the pivot is an ellipse close to a circle. The analysis is based on the exact rotational solutions in the case of circular pivot trajectory and zero gravity. The conditions for existence and stability of such solutions are derived. Assuming that the amplitudes of excitations are not small while the pivot trajectory has small ellipticity the approximate solutions are found both for high and small linear damping. Comparison between approximate and numerical solutions is made for different values of the damping parameter.

Key words
elliptically excited pendulum; parametric pendulum; unbalanced rotor; asymptotic methods

1 Introduction

Elliptically excited pendulum (EEP) is a mathematical pendulum in the vertical plane whose pivot oscillates not only vertically but also horizontally with $\pi/2$ phase shift, so that the pivot has elliptical trajectory, see Fig. 1. EEP is a natural generalization of pendulum with vertically vibrating pivot that is one of the most studied classical systems with parametric excitation, so it is often referred to simply as parametric pendulum, see for example [Lenci, Pavlovskaia, Rega and Wiercigroch, 2008; Xu and Wiercigroch, 2007; Bogolyubov and Mitropol'skii, 1961; Seyranian, Yabuno and Tsumoto, 2005] and references therein.

Dynamics of EEP has been studied numerically and analytically in [Horton B, Sieber J, Thompson and Wiercigroch, 2008]. Approximate oscillatory and rotational solutions for EEP are the common examples in literature [Blekhman, 1954; Blekhman, 1979; Blekhman, 2000; Akulenko, 2001] on asymptotic methods. Sometimes EEP is presented in a slightly more general model of unbalanced rotor [Blekhman, 1954; Blekhman, 1979; Blekhman, 2000], where the phase shift between vertical and horizontal oscillations of the pivot can differ from $\pi/2$. EEP is also a special case of generally excited pendulum in [Trueba, Baltanáas and Sanjuáan, 2003].

The usual assumption for approximate solution in the literature is the smallness of dimensionless damping and pivot oscillation amplitudes in the EEP's equation of motion. The author could find only one paper [Fidlin and Thomsen, 2008], where oscillations of EEP with high damping and yet small relative excitation were studied.

In the present paper we study rotations of EEP with not small excitation amplitudes and with both small and not small linear damping. Our analysis uses the exact solutions for EEP with the absence of gravity and with equal excitation amplitudes, when elliptical trajectory of the pivot becomes circular. [1]

The paper is organized as follows. In Section 2 the dimensionless equation of EEP motion is derived. In Section 3 the exact rotational solutions and their stability conditions are obtained in the case, with no gravity and the circular trajectory of the pivot. In Section 4 first and second order approximate solutions are obtained by multiple scale method [Nayfeh, 1973] for the close to circle trajectory of the pivot and high damping, where we assume that gravity is small or the frequency of excitation is high. In Section 5 for the same excitation and small damping second order approximate solutions are obtained with the use of averaging method [Bogolyubov and Mitropol'skii, 1961; Volosov and Morgunov, 1971]. In Section 6 both solutions in Sections 4 and 5 are compared with the numerical solutions for different values of the damping parameter.

2 Main relations

Equation of EEP's motion can be derived with the use of angular momentum alteration theorem (e. g. [Hor-

[1] When there is no gravity the model of EEP coincides with that of hula-hoop, see [Belyakov and Seyranian, 2010] and references therein.

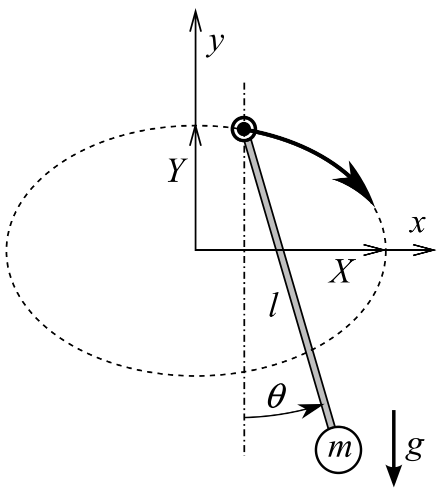

Figure 1. Scheme of the elliptically excited mathematical pendulum of length l. The pivot of the pendulum moves along the elliptic trajectory (dashed line) with semiaxis X and Y.

ton B, Sieber J, Thompson and Wiercigroch, 2008])

$$m\,l^2\frac{d^2\theta}{dt^2}+c\frac{d\theta}{dt}+m\,l\left(g-\frac{d^2y(t)}{dt^2}\right)\sin(\theta) \\ -m\,l\,\frac{d^2x(t)}{dt^2}\cos(\theta)=0, \tag{1}$$

where l is the distance between the pivot and the concentrated mass m; c is the viscous damping coefficient; θ is the angle of the pendulum deviation from the vertical position; t is time; g is gravitational acceleration.

It is assumed that the pivot of the pendulum moves according to the periodic law

$$x=X\sin(\Omega t),\quad y=Y\cos(\Omega t), \tag{2}$$

where X, Y, and Ω are the amplitudes and frequency of the excitation.

We introduce the following dimensionless parameters and new time

$$\varepsilon=\frac{Y-X}{2\,l},\quad \mu=\frac{Y+X}{2\,l}>0, \\ \omega=\frac{1}{\Omega}\sqrt{\frac{g}{l}},\quad \beta=\frac{c}{m\,l^2\,\Omega},\quad \tau=\Omega t, \tag{3}$$

in which equation (1) with substituted (2) in it takes the following form

$$\ddot{\theta}+\beta\dot{\theta}+\mu\sin(\tau+\theta)=\varepsilon\sin(\tau-\theta)-\omega^2\sin(\theta), \tag{4}$$

were we use formula $Y\cos(\Omega\,t)\sin(\theta)+X\sin(\Omega\,t)\cos(\theta)=\frac{Y+X}{2}\sin(\Omega\,t+\theta)-\frac{Y-X}{2}\sin(\Omega\,t-\theta)$. Here the upper dot denotes differentiation with respect to new time τ.

3 Exact rotational solution when $\varepsilon=0$ and $\omega=0$

Conditions $\varepsilon=\omega=0$ mean that we find the mode of rotation for the circular excitation $X=Y$ with absence of gravity $g=0$. In this case, we call equation (4) the *unperturbed equation*

$$\ddot{\theta}+\beta\dot{\theta}+\mu\sin(\tau+\theta)=0 \tag{5}$$

which has exact solutions

$$\theta=\theta_0-\tau, \tag{6}$$

where constants θ_0 are defined by the following equality

$$\sin(\theta_0)=\frac{\beta}{\mu}, \tag{7}$$

provided that $|\beta|\le\mu$.

To investigate the stability of these solutions we present the angle θ as $\theta=\theta_0-\tau+\eta$, where $\eta=\eta(\tau)$ is a small addition, and substitute it in equation (5). Then linearizing in (5) and using equality (7), we obtain the linear equation

$$\ddot{\eta}+\beta\dot{\eta}+\mu\cos(\theta_0)\eta=0, \tag{8}$$

According to the Lyapunov stability theorem, solution (6) is asymptotically stable according to the linear approximation if all eigenvalues of linearized equation (8) have negative real parts. Which happens when the following inequalities are satisfied

$$\beta>0,\quad \mu\cos(\theta_0)>0, \tag{9}$$

obtained from the Routh–Hurwitz conditions. From conditions (9), assumption $\mu>0$ in (3), and equality (7), it follows for $\beta>0$ that the solutions

$$\theta=\theta_0-\tau,\quad \theta_0=\arcsin\left(\frac{\beta}{\mu}\right)+2\pi k \tag{10}$$

are asymptotically stable, while the solutions

$$\theta=\theta_0-\tau,\quad \theta_0=\pi-\arcsin\left(\frac{\beta}{\mu}\right)+2\pi k \tag{11}$$

are unstable, where k is any integer number. For negative damping, $\beta<0$, both these solutions are unstable. From now on we will assume that the following conditions are satisfied

$$0<\beta<\mu, \tag{12}$$

which ensure the existence of stable rotational solution (10) as it is seen from (7) and (9). Indeed, in order to guarantee asymptotic stability β should be not only positive, but also strictly less than μ because of the second condition in (9), which can be transformed to inequality $\mu\cos(\theta_0) = \sqrt{\mu^2-\beta^2} > 0$ with the use of the positive root for $\mu\cos(\theta_0)$ from (7).

4 Approximate rotational solutions when $\varepsilon \approx 0$ and $\omega \sim \sqrt{\varepsilon}$

We assume that values of ε and ω^2 are small of the same order of smallness, i.e. $\varepsilon \sim \omega^2 \ll 1$, so we can introduce new parameter $w = \omega^2/\varepsilon$.

One can deduct from (3) and current assumptions that either gravity g is small or the frequency of excitation Ω is high with such damping c and mass m so that damping coefficient $\beta \sim 1$.

All small terms are in the right-hand side of equation (4). To solve equation (4) we will use multiple scale method [Nayfeh, 1973]. In this method general solution of equation (4) is assumed to be of the following form

$$\theta = -\tau + \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \dots \tag{13}$$

A series of time scales (independent variables), T_0, T_1, …, is introduced, where $T_0 = \tau$, $T_1 = \varepsilon\tau$, …. So that θ is a function of these time scales, $\theta(T_0, T_1, \dots)$. Using the chain rule, the time derivatives become

$$\frac{d}{d\tau} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \dots \tag{14}$$
$$\frac{d^2}{d\tau^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_0D_2 + D_1^2) + \dots \tag{15}$$

where $D_n^m = \partial^m/\partial T_n^m$. Next the general solution and the time derivatives are substituted into equation (4), where sines are expended into the Taylor series with respect to ε. By grouping together the terms with the same powers of ε and equating to zero, a set of differential equations is obtained,

$$D_0^2\theta_0 + \beta D_0(\theta_0 - T_0) + \mu\sin(\theta_0) = 0, \tag{16}$$
$$\begin{aligned} &D_0^2\theta_1 + \beta D_0\theta_1 + \mu\cos(\theta_0)\theta_1 \\ &= -(2D_0D_1 + \beta D_1)(\theta_0 - T_0) \\ &+ \sin(2T_0 - \theta_0) + w\sin(T_0 - \theta_0), \end{aligned} \tag{17}$$
$$\begin{aligned} &D_0^2\theta_2 + \beta D_0\theta_2 + \mu\cos(\theta_0)\theta_2 \\ &= \mu\sin(\theta_0)\theta_1^2/2 - (2D_0D_1 + \beta D_1)\theta_1 \\ &-(D_0D_2 + D_1^2 + \beta D_2)(\theta_0 - T_0) \\ &- (\cos(2T_0 - \theta_0) + w\cos(T_0 - \theta_0))\,\theta_1, \end{aligned} \tag{18}$$
$$\dots$$

where we denote $w = \omega^2/\varepsilon$. We have already found solution (6) for equation (16) in the previous section. Here we consider the same stable regular rotations 1:1 whose zero approximation is given by (10). Hence, θ_0 is a constant and consequently we have $(2D_0D_1 + \beta D_1)(\theta_0 - T_0) = 0$ and $(D_0D_2 + D_1^2 + \beta D_2)(\theta_0 - T_0) = 0$. Thus, equations (17) and (18) can be written in the following way

$$\begin{aligned} &D_0^2\theta_1 + \beta D_0\theta_1 + \sqrt{\mu^2-\beta^2}\theta_1 \\ &= \sin(2T_0 - \theta_0) + w\sin(T_0 - \theta_0) \end{aligned} \tag{19}$$
$$\begin{aligned} &D_0^2\theta_2 + \beta D_0\theta_2 + \sqrt{\mu^2-\beta^2}\theta_2 \\ &= \beta\theta_1^2/2 - (2D_0D_1 + \beta D_1)\,\theta_1 \\ &- (\cos(2T_0 - \theta_0) + w\cos(T_0 - \theta_0))\,\theta_1, \end{aligned} \tag{20}$$

where we denote $\mu\sin(\theta_0) = \beta$ and $\mu\cos(\theta_0) = \sqrt{\mu^2-\beta^2}$ with the use of relation (7) and the second condition in (9).

4.1 First order approximation

In consequence of conditions (12) non-homogeneous linear differential equation (19) can be presented in the following form

$$\begin{aligned} &D_0^2\theta_1 + \beta D_0\theta_1 + \sqrt{\mu^2-\beta^2}\theta_1 \\ &= A_1\cos(T_0) + B_1\sin(T_0) \\ &+ A_2\cos(2T_0) + B_2\sin(2T_0)\,, \end{aligned} \tag{21}$$

where $A_1 = -w\beta/\mu$, $B_1 = w\sqrt{1-\beta^2/\mu^2}$, $A_2 = -\beta/\mu$, $B_2 = \sqrt{1-\beta^2/\mu^2}$, lower index denotes harmonic number. Equation (21) has a unique periodic solution

$$\begin{aligned} \theta_1(T_0) &= a_1\cos(T_0) + b_1\sin(T_0) \\ &+ a_2\cos(2T_0) + b_2\sin(2T_0)\,, \end{aligned} \tag{22}$$

where

$$\begin{aligned} a_1 &= -\frac{(1-\sqrt{\mu^2-\beta^2})A_1 + \beta B_1}{\mu^2+1-2\sqrt{\mu^2-\beta^2}}, \\ b_1 &= \frac{-\beta A_1 + (1-\sqrt{\mu^2-\beta^2})B_1}{\mu^2+1-2\sqrt{\mu^2-\beta^2}}, \\ a_2 &= -\frac{(4-\sqrt{\mu^2-\beta^2})A_2 + 2\beta B_2}{3\beta^2+\mu^2+4(4-2\sqrt{\mu^2-\beta^2})}, \\ b_2 &= \frac{-2\beta A_2 + (4-\sqrt{\mu^2-\beta^2})B_2}{3\beta^2+\mu^2+4(4-2\sqrt{\mu^2-\beta^2})}. \end{aligned}$$

Thus, the solution for (4) in the first approximation can be written as follows

$$\begin{aligned} \theta &= -\tau + \theta_0 \\ &- \varepsilon\,\frac{2\,\beta\cos(2T_0-\theta_0) + \left(4-\sqrt{\mu^2-\beta^2}\right)\sin(2T_0-\theta_0)}{3\beta^2+\mu^2+8(2-\sqrt{\mu^2-\beta^2})} \\ &- \omega^2\,\frac{\beta\cos(T_0-\theta_0) + \left(1-\sqrt{\mu^2-\beta^2}\right)\sin(T_0-\theta_0)}{\mu^2+1-2\sqrt{\mu^2-\beta^2}}, \end{aligned} \tag{23}$$

where constant θ_0 is defined in (10).

4.2 Second order approximation

Since (22) does not contain any constant of integration we set $(2D_0D_1 + \beta D_1)\theta_1 = 0$ in equation (20) and substitute in it an expression $\cos(2T_0 - \theta_0) + w\cos(T_0 - \theta_0) = B_1\cos(T_0) - A_1\sin(T_0) + B_2\cos(2T_0) - A_2\sin(2T_0)$ with coefficients defined in (21). Thus, equation (21) takes the following form

$$D_0^2\theta_2 + \beta D_0\theta_2 + \sqrt{\mu^2-\beta^2}\theta_2 = \frac{A'_0}{2} + \sum_{n=1}^{4}\left(A'_n\cos(nT_0) + B'_n\sin(nT_0)\right), \quad (24)$$

where coefficients in the right-hand side are the following

$$\begin{aligned}
A'_0 &= \left(b_2^2 + b_1^2 + a_2^2 + a_1^2\right)\beta \\
&\quad + (A_1b_1 + A_2b_2 - B_2a_2 - B_1a_1),\\
A'_1 &= (a_1a_2 + b_1b_2)\beta \\
&\quad + (A_1b_2 + A_2b_1 - B_1a_2 - B_2a_1)/2,\\
A'_2 &= \left(a_1^2 - b_1^2\right)\beta/2 - (A_1b_1 + B_1a_1)/2,\\
A'_3 &= (a_1a_2 - b_1b_2)\beta \\
&\quad - (A_2b_1 + A_1b_2 + B_1a_2 + B_2a_1)/2,\\
A'_4 &= \left(a_2^2 - b_2^2\right)\beta/2 - (A_2b_2 + B_2a_2)/2\\
B'_1 &= (a_1b_2 - b_1a_2)\beta \\
&\quad - (A_1a_2 - A_2a_1 + B_1b_2 - B_2b_1)/2,\\
B'_2 &= \beta\, a_1b_1 + (A_1a_1 - B_1b_1)/2,\\
B'_3 &= (a_1b_2 + b_1a_2)\beta \\
&\quad + (A_1a_2 + A_2a_1 - B_1b_2 - B_2b_1)/2,\\
B'_4 &= \beta\, a_2b_2 + (A_2a_2 - B_2b_2)/2.
\end{aligned}$$

Periodic solution for equation (24) has the following form which is obtained from (13) in the Appendix to [Belyakov, 2011]

$$\begin{aligned}
\theta_2(T_0) &= \frac{A'_0}{2\sqrt{\mu^2-\beta^2}} \qquad (25)\\
&- \sum_{n=1}^{4}\frac{(n^2-\sqrt{\mu^2-\beta^2})A'_n + n\beta B'_n}{(n^2-1)\beta^2+\mu^2+n^2(n^2-2\sqrt{\mu^2-\beta^2})}\cos(nT_0)\\
&- \sum_{n=1}^{4}\frac{-n\beta A'_n + (n^2-\sqrt{\mu^2-\beta^2})B'_n}{(n^2-1)\beta^2+\mu^2+n^2(n^2-2\sqrt{\mu^2-\beta^2})}\sin(nT_0),
\end{aligned}$$

where constant term is derived from (13) taking $A_0 = A'_0/2$. Thus, second order approximate solution can be shortly written in the following form

$$\theta = -\tau + \theta_0 + \varepsilon\theta_1(\tau) + \varepsilon^2\theta_2(\tau), \qquad (26)$$

where constant θ_0 is defined in (10), function θ_1 in (22), and function θ_2 in (25). In Fig. 2 it is shown how first and second order approximate solutions approach the numerical solution.

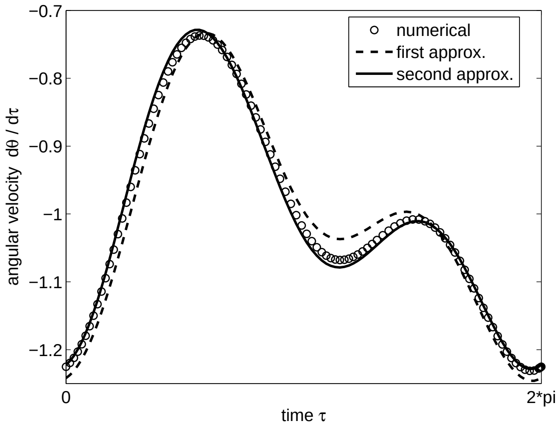

Figure 2. Angular velocities $\dot{\theta}$ calculated from the first order approximate solution (23), second order approximate solution (26), and results of numerical simulation, when damping coefficient β is not small. Parameters: $\delta = 0$, $\mu = 1$, $\omega = 0.3$, $\varepsilon = 0.2$, $\beta = 0.5$.

In this section we have used the fact that all solutions in each time scale must converge to corresponding unique periodic solutions because damping β is not small. If it is not the case the multiple scale analysis becomes more complicated. In the next section we will tackle the problem of small damping $\beta \sim \sqrt{\varepsilon}$ with the use of classical averaging technique.

5 Approximate rotational solutions when $\varepsilon \approx 0$, $\omega \sim \sqrt{\varepsilon}$, and $\beta \sim \sqrt{\varepsilon}$

One can see in (3) that assumptions $\omega \sim \beta \sim \sqrt{\varepsilon}$ are valid for the high frequency of excitation $\Omega \sim 1/\sqrt{\varepsilon}$ with other parameters being of order 1. Another option is small gravity $g \sim \varepsilon$ along with small ratio $c/m \sim \sqrt{\varepsilon}$.

After change of variable $\theta = -\tau + \sqrt{\varepsilon}\vartheta$ equation (4) takes the following form

$$\begin{aligned}
\ddot{\vartheta} + \mu\vartheta - \hat{\beta} &= \mu\left(\vartheta - \frac{\sin(\sqrt{\varepsilon}\vartheta)}{\sqrt{\varepsilon}}\right) - \sqrt{\varepsilon}\hat{\beta}\dot{\vartheta}\\
&+\sqrt{\varepsilon}\sin\left(2\tau - \sqrt{\varepsilon}\vartheta\right) + \sqrt{\varepsilon}w\sin(\tau - \sqrt{\varepsilon}\vartheta), \quad (27)
\end{aligned}$$

with small right-hand side, where we denote $\hat{\beta} = \beta/\sqrt{\varepsilon}$ and as in the previous section $w = \omega^2/\varepsilon$. With zero right-hand side equation (27) $\ddot{\vartheta} + \mu\vartheta - \hat{\beta} = 0$ would describe harmonic oscillations about $\hat{\beta}/\mu$ value with frequency $\sqrt{\mu}$. After Taylor's expansion of sines in the right-hand side of (27) about $\vartheta = 0$ we obtain the following equation

$$\begin{aligned}
\ddot{\vartheta} + (\mu + \varepsilon\cos(2\tau) + \varepsilon w\cos(\tau))\,\vartheta - \hat{\beta} &= -\sqrt{\varepsilon}\hat{\beta}\dot{\vartheta}\\
+\sqrt{\varepsilon}\sin(2\tau) + \sqrt{\varepsilon}w\sin(\tau) + \varepsilon\mu\frac{\vartheta^3}{6} + o(\varepsilon), \quad (28)
\end{aligned}$$

which describes oscillator with both basic and parametric excitations. To solve equation (28) we will use averaging method [Bogolyubov and Mitropol'skii, 1961; Volosov and Morgunov, 1971]. For that purpose we will write (28) in the *standard form* of first order differential equations with small right-hand sides. First, we use *Poincaré variables* q and ψ defined via the following solution of *generating system* $\ddot{\vartheta} + \mu\vartheta - \hat{\beta} = 0$ which is (28) with $\varepsilon = 0$

$$\vartheta = \frac{\hat{\beta}}{\mu} + q\cos(\psi), \quad \dot{\vartheta} = -\sqrt{\mu}q\sin(\psi). \tag{29}$$

In Poincaré variables equation (28) becomes a system of first order differential equations

$$\dot{q} = -\frac{\sin\psi}{\sqrt{\mu}} f(\tau, q, \psi), \tag{30}$$

$$\dot{\psi} = \sqrt{\mu} - \frac{\cos\psi}{q\sqrt{\mu}} f(\tau, q, \psi), \tag{31}$$

where small function $f(\tau, q, \psi) = \sqrt{\varepsilon} f_1(\tau, q, \psi) + \varepsilon f_2(\tau, q, \psi) + o(\varepsilon)$ is the right hand side of (27), where

$$f_1(\tau, q, \psi) = \sin(2\tau) + w\sin(\tau) + \hat{\beta}q\sqrt{\mu}\sin(\psi), \tag{32}$$

$$f_2(\tau, q, \psi) = -(\cos(2\tau) + w\cos(\tau))\left(\frac{\hat{\beta}}{\mu} + q\cos(\psi)\right) + \frac{\mu}{6}\left(\frac{\hat{\beta}}{\mu} + q\cos(\psi)\right)^3, \tag{33}$$

meaning that $f(\tau, q, \psi) = O(\sqrt{\varepsilon})$. Our next assumption is that $\sqrt{\mu} - 1 \sim \sqrt{\varepsilon}$ which means that excitation frequency is close to the first resonant frequency of basic excitation component $\sin(\tau)$ and to the first resonant frequency of parametric excitation component $\cos(2\tau)$ in equation (28). Thus, system (30), (31) is transformed by $\psi = \zeta + \tau$ to the standard form

$$\dot{q} = -\frac{1}{\sqrt{\mu}}\sin(\zeta + \tau)\, f(\tau, q, \zeta + \tau), \tag{34}$$

$$\dot{\zeta} = \sqrt{\mu} - 1 - \frac{1}{q\sqrt{\mu}}\cos(\zeta + \tau)\, f(\tau, q, \zeta + \tau) \tag{35}$$

with small right-hand side, where new slow variable ζ is often referred to as *phase mismatch*.

5.1 First order approximation

In the first approximation so called *averaged equations* can be obtained by averaging the system (34), (35) over period 2π

$$\dot{Q} = -\frac{\sqrt{\varepsilon}}{2\pi\sqrt{\mu}}\int_0^{2\pi} \sin(Z + \tau)\, f_1(\tau, Q, Z + \tau)\, \mathrm{d}\tau + o(\sqrt{\varepsilon}), \tag{36}$$

$$\dot{Z} = \sqrt{\mu} - 1 - \frac{\sqrt{\varepsilon}}{2\pi\sqrt{\mu}Q}\int_0^{2\pi} \cos(Z + \tau)\, f_1(\tau, Q, Z + \tau)\, \mathrm{d}\tau + o(\sqrt{\varepsilon}), \tag{37}$$

where Q and Z are the *averaged variables* corresponding to q and ζ. After taking the integrals we have the following system

$$\dot{Q} = -\frac{\sqrt{\varepsilon}w}{2\sqrt{\mu}}\cos(Z) - \frac{\sqrt{\varepsilon}\hat{\beta}}{2}Q + o(\sqrt{\varepsilon}), \tag{38}$$

$$\dot{Z} = \sqrt{\mu} - 1 + \frac{\sqrt{\varepsilon}w}{2\sqrt{\mu}Q}\sin(Z) + o(\sqrt{\varepsilon}), \tag{39}$$

stationary solutions ($\dot{Q} = 0$, $\dot{Z} = 0$) of which are the following

$$Q^2 = \frac{\omega^2/\varepsilon}{\mu\,(4(\sqrt{\mu} - 1)^2 + \beta^2)} + o(1), \tag{40}$$

$$Z = \arctan\left(\frac{2(\mu - 1)}{\beta}\right) + 2\pi k + o(1), \tag{41}$$

where we have substituted back $w = \omega^2/\varepsilon$ and $\hat{\beta} = \beta/\sqrt{\varepsilon}$. Symbol $\arctan$ stands for the principal value of the function on the interval from 0 to π. Note that the phase Z is determined to within 2π rather than π, since the functions $\sin(Z)$ and $\cos(Z)$ obtained from equations (38) and (39) determine Z up to an additive term $2\pi k$. Solution of system (34-35) in the first approximation is $q = Q + o(1)$, $\zeta = Z + o(1)$ so the solution of (4) is the following

$$\theta = -\tau + \frac{\beta}{\mu} + \sqrt{\varepsilon}Q\cos(Z + \tau) + o(\sqrt{\varepsilon}), \tag{42}$$

which does not contain higher harmonics observed numerically. That is why we need to proceed to the second order approximation.

5.2 Second order approximation

In the second approximation averaged equations can be obtained as follows

$$\dot{Q} = \left(-\sqrt{\varepsilon}\cos(Z) + \frac{\varepsilon\hat{\beta}}{4}\left(\frac{4}{\mu}-1\right)\sin(Z)\right)\frac{w}{2\sqrt{\mu}} + \left(-\frac{\sqrt{\varepsilon}\hat{\beta}}{2} + \frac{\varepsilon}{4\sqrt{\mu}}\sin(2Z)\right)Q + o(\varepsilon), \quad (43)$$

$$\dot{Z} = \sqrt{\mu} - 1 + \left(\sqrt{\varepsilon}\sin(Z) + \frac{\varepsilon\hat{\beta}}{4}\left(\frac{4}{\mu}-1\right)\cos(Z)\right)\frac{w}{2\sqrt{\mu}Q} - \frac{\varepsilon\hat{\beta}^2}{8}\left(\frac{2}{\mu\sqrt{\mu}}+1\right) + \frac{\varepsilon}{4\sqrt{\mu}}\cos(2Z) - \frac{\varepsilon\sqrt{\mu}}{16}Q^2 + o(\varepsilon), \quad (44)$$

stationary solutions ($\dot{Q} = 0$, $\dot{Z} = 0$) can be found numerically or with absence of gravity ($\omega = 0$) analytically. Solution of system (34-35) in the second approximation is the following

$$q = Q + \frac{\sqrt{\varepsilon}}{2\sqrt{\mu}}\left(-\sin(\tau - Z) + \frac{w}{2}\sin(2\tau + Z) + \frac{1}{3}\sin(3\tau + Z)\right) + \sqrt{\varepsilon}\frac{\hat{\beta}\,Q}{4}\sin\left(2\tau + 2Z\right) + o(\sqrt{\varepsilon}), \quad (45)$$

$$\zeta = Z + \frac{\sqrt{\varepsilon}}{2\sqrt{\mu}Q}\left(\cos(\tau - Z) + \frac{w}{2}\cos(2\tau + Z) + \frac{1}{3}\cos(3\tau + Z)\right) + \sqrt{\varepsilon}\frac{\hat{\beta}}{4}\cos\left(2\tau + 2Z\right) + o(\sqrt{\varepsilon}), \quad (46)$$

Substitution of these expressions into (29) yields the second order approximate solution of (27) in the following form

$$\vartheta = \frac{\hat{\beta}}{\mu} + Q\cos(Z + \tau) + \sqrt{\varepsilon}\frac{\hat{\beta}\,Q}{4}\sin(Z + \tau) + \frac{\sqrt{\varepsilon}w\sin(\tau)}{4\sqrt{\mu}} - \frac{\sqrt{\varepsilon}\sin(2\tau)}{3\sqrt{\mu}} + o(\sqrt{\varepsilon}) \quad (47)$$

which after changes of variable $\theta = -\tau + \sqrt{\varepsilon}\vartheta$ and parameters $w = \omega^2/\varepsilon$, $\hat{\beta} = \beta/\sqrt{\varepsilon}$ results in the approximate solution of the original equation (4)

$$\theta = -\tau + \frac{\beta}{\mu} + \sqrt{\varepsilon}Q\cos(Z + \tau) + \sqrt{\varepsilon}\frac{\beta\,Q}{4}\sin(Z + \tau) + \frac{\omega^2\sin(\tau)}{4\sqrt{\mu}} - \frac{\varepsilon\sin(2\tau)}{3\sqrt{\mu}} + o(\varepsilon). \quad (48)$$

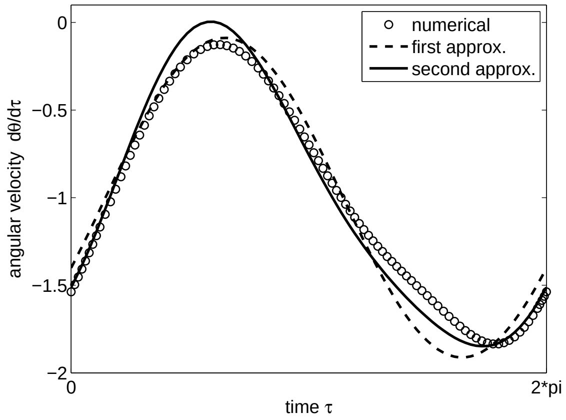

Figure 3. Angular velocity $\dot{\theta}$ calculated from the first order approximate solution (42) and second order approximate solution (48) compared with the results of numerical simulations in the case of small damping β. Parameters: $\delta = 0$, $\mu = 1$, $\omega = 0.3$, $\varepsilon = 0.2$, $\beta = 0.01$. Steady state averaged variables Q and Z are given by expressions (40) and (41) for the first approximation while for the second approximation they are obtained numerically ($Q = 2.0348$, $Z = 2.6838$) from the second order averaged equations (43), (44).

Agreement of solution (48) with the numerical experiment is shown in Fig. 3. We see that the amplitude of angular velocity oscillations is much higher than that for not small β in Fig. 2.

6 Domains of applicability

For asymptotic solutions in the previous two sections the parameter constraints are more strict than those for the existence of stable exact solution in (12). Thus, for our analysis in Section 4 to be valid we must exclude cases when $\beta = o(1)$ or/and $\sqrt{\mu^2 - \beta^2} = o(1)$. The case when $\beta = o(1)$ and $\mu = O(1)$ is studied in Section 4. Note that the case when both $\beta = o(1)$ and $\mu = o(1)$ meaning $\sqrt{\mu^2 - \beta^2} = o(1)$ has already been studied in the literature, for example in the more general model of unbalanced rotor in [Blekhman, 1954]. The case when $\beta = O(1)$ and $\mu = o(1)$ is not feasible for asymptotic rotational solution. Indeed, with such assumptions the generating system $\ddot{\theta} + \beta\dot{\theta} = 0$ has only constant solutions. These different cases are presented in the Table 1.

To show quantitatively the limits of applicability of the assumptions in Sections 4 and 5 we plot the absolute and relative angular velocity errors depending on parameter β while excitation was constant $\mu = 1$, see Fig. 4.

7 Conclusion

The exact rotational solutions in the case of equal excitation amplitudes and zero gravity are obtained. The conditions for existence and stability of such solutions

	small β	not small β
small μ	studied in the literature	no rotations
not small μ	studied in section 5	studied in section 4

Table 1. Model assumptions on smallness of dimensionless damping β and dimensionless semiaxes half-sum μ of the ellipse along which the pivot of the pendulum moves in the problem to find pendulum rotations. In all cases we assume that dimensionless half-difference ε of semiaxes is small as well as ω^2.

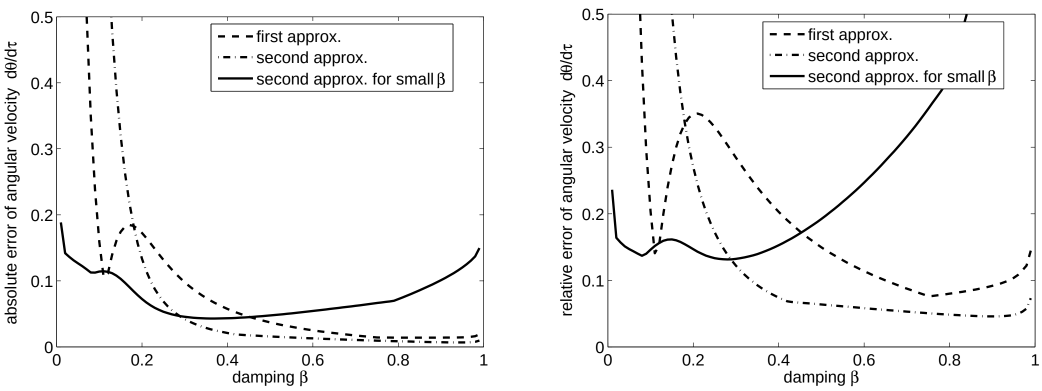

Figure 4. Maximal absolute (left) and relative (right) deviation of analytically obtained angular velocities $\dot{\theta}$ in (23), (26), and (48) from the numerically obtained $\dot{\theta}$. Solution (48) specially obtained for better approximation at small damping β. Error in the right graph is calculated relative to the oscillation amplitude of angular velocity $\dot{\theta}$. Parameters: $\delta = 0, \mu = 1, \omega = 0.3, \varepsilon = 0.2$.

are derived. Based on these exact solutions the approximate solutions are found both for high and small linear damping, assuming that the amplitudes of excitations are not small. Comparison between approximate and numerical solutions shows a good agrement for the damping values of the assumed order.

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