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ZEROS OF DELAYED SAMPLED SYSTEMS

Vladimir Bondarko
Department of Mathematics and Mechanics
St. Petersburg State University
Russia
vbondarko@gmail.com

Abstract
Linear SISO time-invariant continuous systems with time delay are considered. A sufficiently rapid zero-order hold sampling of such a system leads to the discrete-time model with two subsets of zeros, namely so-called intrinsic zeros and sampling (or limiting) ones. Intrinsic zeros depend on original zeros almost exponentially, while sampling zeros are asymptotically close to the zeros of Euler polynomials. More accurately, they converge to the zeros of Euler polynomials in the case of zero time delay. This well known result is extended here to the case of positive time delay. It is shown that limiting zeros depend on the relative degree and on an additional parameter eps which is equal to the fractional part of the quotient of the time delay and the sampling period. Polynomials having those zeros are called here generalized Euler polynomials. They coincide with ordinary Euler polynomials if eps=0. It is shown that all zeros of generalized Euler polynomials are negative and simple. They monotonically vary between the neighboring zeros of the corresponding ordinary Euler polynomial when eps grows from 0 to 1. Since zeros of the ordinary Euler polynomial are pairwise mutually inverse, we obtain a criterion for a sampled system to be stably invertible.

Key words
Sampling zeros, generalized Euler polynomials.

1 Introduction

Let single-input/single-output (SISO) continuous-time system be influenced by a zero-order hold (ZOH) input with a sampling period h. Then samples of input and output for this system satisfy a certain difference equation, which describes the so-called ZOH discrete model of original continuous-time system, or prototype. It is well known from [Åström, Hagander and Sternby, 1980] and [Åström, Hagander and Sternby, 1984] that zeros of a discrete model may be divided into two groups. The first group consists of so-called *intrinsic* zeros; they have the form $e^{h\mu_j(h)}$, where $\mu_j(h) \to \mu_j$ as $h \to 0$, and μ_j are zeros of the prototype system. The second group is composed of *sampling* zeros, whose limits depend on the relative degree of the prototype only. It was shown in [Weller, Moran, Ninness and Pollington, 2001] that these limits are the roots of the Euler (or Euler-Frobenius) polynomials. Using the known properties of these polynomials (see, for example, [Sobolev 1977]), one may conclude that these limits are real negative, simple, and pairwise mutually inverse. Thus a discrete model cannot be stably invertible if it has more then one sampling zero and the sampling period h is sufficiently small. One should note here that the quantity of sampling zeros is smaller by 1 than the relative degree of the prototype for all sufficiently small h.

The object of this article is to extend the results mentioned above to continuous-time prototypes with time delay. In this case sampling zeros also exist (even more by 1,i.e. their quantity is equal to the relative degree). Their limits depend not only on the relative degree but on ϵ too, where ϵ is the fractional part of the quotient of the time delay by the sampling period. Polynomials having these zeros are called here *generalized Euler polynomials*. They coincide with ordinary Euler polynomials if $\epsilon = 0$. Their zeros are not mutually inverse, but they are real, negative, and monotonically move between neighboring roots of the ordinary Euler polynomials while ϵ varies from 0 to 1. This fact allows to obtain sufficient and almost necessary conditions of stable invertibility of the discrete model.

2 ZOH Models and Impulse Invariance Transform

Let us consider continuous systems

$$a(p)y(t) = b(p)u(t-\theta),\ t \in [0,\infty), \qquad (1)$$

where t denotes continuous time, $u(t)$ is input, $y(t)$ is output, $p^j y(t) = d^j y(t)/dt^j$, θ is time delay,

$$a(\lambda) = \prod_{j=1}^{n}(\lambda - \lambda_j),\ b(\lambda) = b_0 \prod_{j=1}^{m}(\lambda - \mu_j), \quad (2)$$

$m < n,\ b_0 \neq 0.$

Let the input have a ZOH form:

$$u(kh + \varepsilon) = u_k,\ \varepsilon \in [0, h),\ k = 0, 1, \ldots, \quad (3)$$

where h is a positive constant sampling period. Define a natural number ϑ such that

$$\theta = h[(\vartheta - 1) + \varepsilon],\ \varepsilon \in [0, 1). \quad (4)$$

Any input like (3) yields that the difference equation

$$\alpha(\nabla)y_k = \beta(\nabla)u_{k-\vartheta},\ k = 0, 1, \ldots, \quad (5)$$

is valid for input samples u_k and output samples $y_k = y(kh)$, where ∇ means forward shift operator, $\nabla^i y_k = y_{k+i}$,

$$\begin{aligned} \alpha(\lambda) &= \prod_{j=1}^{n}(\lambda - e^{h\lambda_j}), \\ \beta(\lambda) &= \beta_0\lambda^n + \beta_1\lambda^{n-1} + \ldots + \beta_n, \end{aligned} \quad (6)$$

and polynomial $\beta(\lambda)$ should be established because namely its roots are zeros of ZOH-sampled system (1).

Let us consider any matrix triple (A, B, C) such that pair (A, C) is observable, $\det(\lambda I - A) = a(\lambda)$, $C(\lambda I - A)^{-1}B = b(\lambda)/a(\lambda)$. For example, it may be

$$A = \begin{bmatrix} 0\ 0 \ldots & -a_n \\ 1\ 0 \ldots & -a_{n-1} \\ 0\ 1 \ldots & -a_{n-2} \\ \ddots & \vdots \\ 0\ 0 \ldots 1 & -a_1 \end{bmatrix}, B = \begin{bmatrix} b_n \\ b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{bmatrix}, \\ C = \begin{bmatrix} 0\ 0 \ldots 0\ 1 \end{bmatrix},$$

where a_i and b_j are the coefficients of the polynomials (2): $a(\lambda) = \lambda^n + \lambda^{n-1}a_1 + \ldots + a_n, b(\lambda) = \lambda^{n-1}b_1 + \ldots + b_n$. Now we are able to rewrite (1) in the state-space form:

$$\dot{x}(t) = Ax(t) + Bu(t - \Theta),\ y(t) = Cx(t). \quad (7)$$

Using Cauchy formula to solve this equation for time intervals $[(k+\varepsilon-1)h, (k+\varepsilon)h]$ and $[(k+\varepsilon-1)h, kh]$, we obtain

$$\begin{cases} x_{k+1} = P(h)x_k + Q(h)u_{k-\theta}, \\ y_k = R(h, \varepsilon)x_k + S(h, \varepsilon)u_{k-\theta}, \end{cases} \quad (8)$$

where $x_k = x(k + \theta)h$,

$$P(h) = e^{hA},\ Q(h) = \int_0^h e^{sA} ds B,$$

$$R(h, \varepsilon) = Ce^{h(1-\varepsilon)A},\ S(h, \varepsilon) = C \int_0^{h(1-\varepsilon)} e^{sA} ds B.$$

Since equation (8) describes the state-space form of the discrete model (5), the desired polynomial $\beta(\lambda)$ may be written as

$$\beta(\lambda) = \alpha(\lambda)\left[S(h, \varepsilon) + R(h, \varepsilon)(\lambda I - P(h))^{-1}Q(h)\right]. \quad (9)$$

The equation (9) gives us a possibility to obtain all required results about zeros of sampled system, but this straightforward method requires rather cumbersome computations. Another way seems to be more simple. It uses so-called impulse invariance-transform of an auxilliary continuouse-time prototype. Namely, we will consider the system

$$pa(p)y(t) = b(p)u(t - \theta)\ t \in [0, \infty), \quad (10)$$

and its state-space form

$$\dot{x}(t) = Ax(t) + Bu(t - \Theta),\ \dot{y}(t) = Cx(t),$$

or, equivalently,

$$\dot{z}(t) = \tilde{A}z(t) + \tilde{B}u(t - \Theta), y(t) = \tilde{C}z(t), \quad (11)$$

where $\tilde{C} = [0 \ldots 0\ 1]$,

$$\tilde{A} = \begin{bmatrix} A\ 0 \\ C\ 0 \end{bmatrix},\ \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix},\ z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

Next, we consider the discrete-time system

$$z_{k+1} = e^{h\tilde{A}}z_k + e^{(1-\varepsilon)h\tilde{A}}\tilde{B}u_{k-\vartheta},\ y_k = \tilde{C}z_k \quad (12)$$

with the transfer function $\lambda^{-\vartheta}\chi_h(\lambda)$, where

$$\chi_h(\lambda) = \tilde{C}(\lambda I - e^{h\tilde{A}})^{-1}e^{(1-\varepsilon)h\tilde{A}}\tilde{B}. \quad (13)$$

The discrete-time impulse response of the system (12) coincides with the continuous-time impulse response of the prototype at the moments $kh,\ k = 0, 1, \ldots$. For this reason the system (12) is called an impulse invariance-transform of the prototype (11). Zeros of this system are the roots of the polynomial $\tilde{\alpha}(\lambda)\chi_h(\lambda)$, where $\tilde{\alpha}(\lambda) = (\lambda - 1)\alpha(\lambda)$ is the characteristic polynomial of the matrix $e^{h\tilde{A}}$.

Theorem 1. *The sets of zeros for the discrete ZOH-model (5) of the original prototype (1) and of the impulse invariance model (12) of auxilliary prototype (11) coincides.*

Proof. We compute the zeros of the system (12), i.e. the roots of the equation

$$(\lambda - 1)\alpha(\lambda)\chi_h(\lambda) = 0. \tag{14}$$

Consider the matrix function

$$G(t) = \begin{bmatrix} e^{tA} & 0 \\ C\int_0^t e^{sA}ds & 1 \end{bmatrix}.$$

Since $G(0) = I$ and $\dot{G}(t) = \tilde{A}G(t)$, $G(t) = e^{t\tilde{A}}$. Let z be a solution of the equation

$$[\lambda I - G(h)]z = G((1-\varepsilon)h)B.$$

Then

$$z = \begin{bmatrix} x \\ y \end{bmatrix},\ x = (\lambda I - e^{hA})^{-1}e^{(1-\varepsilon)hA}B,$$
$$(\lambda - 1)y = C\int_0^h e^{sA}ds\, x + C\int_0^{(1-\varepsilon)h} e^{sA}ds\, B.$$

Hence the equation (14) is equivalent to $\beta(\lambda) = 0$, where $\beta(\lambda)$ is the polynomial (9). The theorem is proved.

Thus we can consider roots of the polynomial $\tilde{\alpha}(\lambda)\chi_h(\lambda)$ as zeros of sampled system (5).

3 Asymptotics of Intrinsic and Sampling Zeros

First we estimate the number of zeros. The relative degree of the system (1) is $r = n - m$. Then

$$\tilde{C}\tilde{B} = 0,\ \tilde{C}\tilde{A}^j\tilde{B} = CA^{j-1}B = 0,\ j = 1,\dots,r-1,$$
$$\tilde{C}\tilde{A}^r\tilde{B} = CA^{r-1}B = b_0 \neq 0.$$

Hence the relative degree of the transfer function (13) is equal to 1 for all sufficiently small h.

Indeed, consider the leading term of the polynomial (9) as an analytic function of the complex argument h:

$$\beta_0 = \beta_0(h) = \tilde{C}e^{(1-\varepsilon)h\tilde{A}}\tilde{B} =$$
$$= \tilde{C}\sum_{j=0}^{\infty}\frac{[(1-\varepsilon)h]^j}{j!}\tilde{A}^j\tilde{B} = \sum_{j=0}^{\infty}\frac{[(1-\varepsilon)h]^{j+1}}{(j+1)!}CA^jB. \tag{15}$$

Assume that there exist $h_j \to 0$ such that $\beta_0(h_j) = 0$, $j = 0, 1, \dots$. Then $\beta_0(h_j) \equiv 0$ and $CA^jB = 0$ for all j in opposition that $b_0 = CA^{r-1}B \neq 0$. Hence $\beta_0(h) \neq 0$ for all sufficiently small h, q.e.d.

Thus the sampled system (5) has n zeros.

Lemma 1. *The sampled system (5) has m intrinsic zeros $e^{h\mu_j(h)}$, where $\mu_j(h) \to \mu_j$, $j = 1, 2, \dots, m$.*

Proof is almost the same as in [Bondarko, 1984]. Fix μ_i. Denote the multiplicity of this root of $b(\lambda)$ as q_i. Consider the disk $c_i(r) = \{\lambda : |\lambda - \mu_j| < r\}$ and its boundary $\partial c_i(r)$. Let r be small enough so that $c_i(r)$ does not containe other roots of $b(\lambda)$ except μ_j. We compare two analytic functions: $\zeta_h(\lambda) = \lambda a(\lambda)\chi_h(e^{h\lambda})$ and $b(\lambda) = \lambda a(\lambda)\tilde{C}(\lambda I - \tilde{A})^{-1}\tilde{B}$. Since

$$\zeta_h(\lambda) - b(\lambda) = \lambda a(\lambda)\tilde{C}\Big[(\lambda I - e^{h\tilde{A}})^{-1}e^{(1-\varepsilon)h\tilde{A}} -$$
$$-(\lambda I - \tilde{A})^{-1}\Big]\tilde{B} \underset{h\to 0}{\to} 0$$

uniformly in $\lambda \in \partial c_i(r)$, we have $|\zeta_h(\lambda) - b(\lambda)| < |b(\lambda)|$ for all sufficiently small h. Using Rouchet theorem, we obtain that $\zeta_h(\lambda)$ and $b(\lambda)$ have the same number of zeros in $c_i(r)$, i.e. q_i. Since r may be chosen to be arbitrarily small, these zeros of $\zeta_h(\lambda)$ tend to μ_i as $h \to 0$. Thus the lemma is proved, because $\sum_i q_i = m$.

Introduce $(d \times d)$-matrices A_d and $E_d(\rho)$, $(d \times 1)$-matrices B_d and $(1 \times d)$-matrices C_d:

$$A_d = \begin{bmatrix} 0\,0\dots & 0 \\ 1\,0\dots & 0 \\ 0\,1\dots & 0 \\ \ddots & \\ 0\,0\dots 1 & 0 \end{bmatrix},\ B_d = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$
$$C_d = \begin{bmatrix} 0\,0\dots 0\,1 \end{bmatrix},\ E_d(\rho) = e^{\rho A_d},$$

for all $d = 1, 2, \dots$, $\rho \in [0, 1]$. Define *generalized Euler polynomials*

$$\xi_{d,\epsilon}(\lambda) = (d-1)!\det\begin{bmatrix} \lambda I - E(1) & -E(\epsilon)B_d \\ C_d & 0 \end{bmatrix}$$
$$d = 1, 2, \dots . \tag{16}$$

Polynomials $\xi_{d,0}(\lambda)$ coincide [Weller, Moran, Ninness and Pollington, 2001] with the well known Euler (or Euler-Frobenius) polynomials.

Let us denote by $\nu_{d,j}(\varepsilon)$ all the roots of $\xi_{d,\varepsilon}(\lambda)$, $j = 1, 2, \dots, d-1$.

Lemma 2. *Sampled system (5) has $r = n - m$ sampling zeros $\varkappa_j(h)$, where $\varkappa_j(h) \to \nu_{r+1,j}(1-\varepsilon)$ as $h \to 0$, $j = 1, 2, \dots, r$.*

Proof. Modify the continuous time scale: $\tau = t/h$. Denote $y_h(\tau) = y(h\tau)$, $u_h(\tau) = h^r u(h\tau)$. Then the equation (1) yields

$$a_h(d/d\tau)y_h(\tau) = b_h(d/d\tau)u_h(\tau - \theta/h),\ \tau \in [0, \infty), \tag{17}$$

where

$$a_h(\lambda) = \lambda^n + ha_1\lambda^{n-1} + \ldots + h^n a_N,$$
$$b_h(\lambda) = b_r\lambda^m + b_{r+1}h\lambda^{m-1} + \ldots + h^m b_n.$$

The discrete-time model of (17) with sampling period 1 obviously has the same zeros as (5) and (12). On the other hand, the roots are equal to the roots of the polynomial

$$\beta_h(\lambda) = \det\begin{bmatrix} \lambda I - e^{\tilde{A}_h} & -e^{(1-\varepsilon)\tilde{A}_h}\tilde{B}(h) \\ C_r & 0 \end{bmatrix}, \quad (18)$$

where

$$\tilde{A}_h = \begin{bmatrix} 0\,0\ldots & -h^n a_n & 0 \\ 1\,0\ldots & -h^{n-1}a_{n-1} & 0 \\ 0\,1\ldots & -h^{n-2}a_{n-2} & 0 \\ \ddots & \vdots & \\ 0\,0\ldots 1 & -ha_1 & 0 \\ 0\,0\ldots 0 & 1 & 0 \end{bmatrix}, \tilde{B}_h = \begin{bmatrix} h^m b_n \\ \vdots \\ b_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Obviously

$$\tilde{A}_h \to A_{n+1} = \begin{bmatrix} A_m & 0_{m\times m} \\ A_{21} & A_{r+1} \end{bmatrix}, \tilde{B}_h \to b_r \begin{bmatrix} 0_m \\ B_{r+1} \end{bmatrix}$$

as $h \to 0$, where $0_{m\times m}$ and 0_m are zero matrices of corresponding dimensions,

$$A_{21} = \begin{bmatrix} 0\ldots 0\,1 \\ 0\ldots 0\,0 \\ \vdots \quad \vdots\,\vdots \\ 0\ldots 0\,0 \end{bmatrix}.$$

Hence $\beta_h(\lambda)$ coefficient wise tends to

$$b_r\det\begin{bmatrix} \lambda I - e^{A_m} & & \\ 0 & \lambda I - e^{A_{r+1}} & e^{(1-\varepsilon)A_{r+1}}B_{r+1} \\ 0 & C_{r+1} & 0 \end{bmatrix} =$$
$$= b_r(\lambda-1)^m \xi_{r+1,(1-\varepsilon)}(\lambda)$$

as $h \to 0$. Thus m zeros of the sampled system tend to 1 (they are intrinsic zeros) and r zeros tend to the roots of $\xi_{r+1,(1-\varepsilon)}(\lambda)$, q.e.d.

Summing the lemmas 1,2 and the equation (15) we obtain

Theorem 2. *Let the continuous-time prototype (1) have m zeros μ_j, $j = 1, 2, \ldots, m$. Suppose that $\theta = h[(\vartheta - 1) + \varepsilon]$, where θ is the time delay, h is the sampling period, ϑ is natural number, $\varepsilon \in [0, 1)$. Then the transfer function of ZOH-sampled system (5) has the form*

$$\frac{h^r b_r(h)}{r!\lambda^\vartheta \alpha(\lambda)}\prod_{j=1}^{r}(\lambda - \nu_{r+1,j}(1-\varepsilon, h))\prod_{i=1}^{m}(\lambda - e^{h\mu_i(h)}),$$

where

$$b_r(h) \to b_r = b_0,\ \mu_i(h) \to \mu_i,$$
$$\nu_{r+1,j}(1-\varepsilon, h) \to \nu_{r+1,j}(1-\varepsilon)$$

as $h \to 0$, b_0 is the leading term of the polynomial $b(\lambda)$ in (2), $\nu_{r+1,j}(1-\varepsilon)$ denote the roots of the generalized Euler polynomials (16).

4 Disposition of the Roots of Generalized Euler Polynomials

At least since [Frobenius, 1910], it is known that all roots of $\xi_{d,0}(\lambda)$ are single and negative real for any d; thus we can assume that $\nu_{d,j}(0)$ are sorted in descending order. Besides it is known that these zeros are pairwise mutually inverse:

$$\nu_{d,j}(0) = 1/\nu_{d,d-1-j}(0). \quad (19)$$

Furthermore, the roots of $\xi_{d,0}(\lambda)$ interlace the roots of $\xi_{d+1,0}(\lambda)$ on the negative real axis:

$$0 > \nu_{d+1,j}(0) > \nu_{d,j}(0) > \nu_{d+1,j+1}(0), \quad j = 1, 2, \ldots, d-2,\ d = 3, 4, \ldots\,. \quad (20)$$

Now complete the set of $\nu_{d,j}(0)$ by $\nu_{d,0}(0) = 0$. It is easily seen that

$$\nu_{d,j}(1) = \nu_{d,j-1}(0),\ j = 1, 2, \ldots, d-2. \quad (21)$$

Indeed, the obvious identity $(\lambda I - E)^{-1}E = \lambda(\lambda I - E)^{-1} - I$ and Shur's lemma yield

$$\frac{\xi_{d,1}(\lambda)}{(d-1)!} = \det\begin{bmatrix} \lambda I - E(1) & -E(1)B_d \\ C_d & 0 \end{bmatrix} =$$
$$= (\lambda-1)^d C_d[\lambda I - E(1)]^{-1}E(1)B_d =$$
$$= (\lambda-1)^d C_d\left\{\lambda[\lambda I - E(1)]^{-1} - I\right\}B_d =$$
$$= \lambda\det\begin{bmatrix} \lambda I - E(1) & -B_d \\ C_d & 0 \end{bmatrix} = \lambda\frac{\xi_{d,0}(\lambda)}{(d-1)!}, \quad (22)$$

because $B_dC_d = 0$.

Therefore, the disposition of $\nu_{d,j}(\epsilon)$ is clear for $\epsilon = 0$ and for $\epsilon = 1$; yet what about intermediate values of ϵ? The answer is given by

Theorem 3. *The roots of $\xi_{d,\epsilon}(\lambda)$ interlace the roots of $\xi_{d+1,\epsilon}(\lambda)$ on the negative real axis:*

$$0 > \nu_{d+1,j}(\epsilon) > \nu_{d,j}(\epsilon) > \nu_{d+1,j+1}(\epsilon) \qquad (23)$$

for any $\epsilon \in [0,1]$, $j = 1,2,\ldots,d-2$, $d = 3,4,\ldots$. These roots are continuous increasing functions of ϵ:

$$\nu_{d,j}(\epsilon_1) < \nu_{d,j}(\epsilon_2) \qquad (24)$$

for $0 < \varepsilon_1 < \varepsilon_2 \le 1$, $j = 1,2,\ldots,d-2$, $d = 3,4,\ldots$.

The disposition of limiting zeros is shown in figure 1.

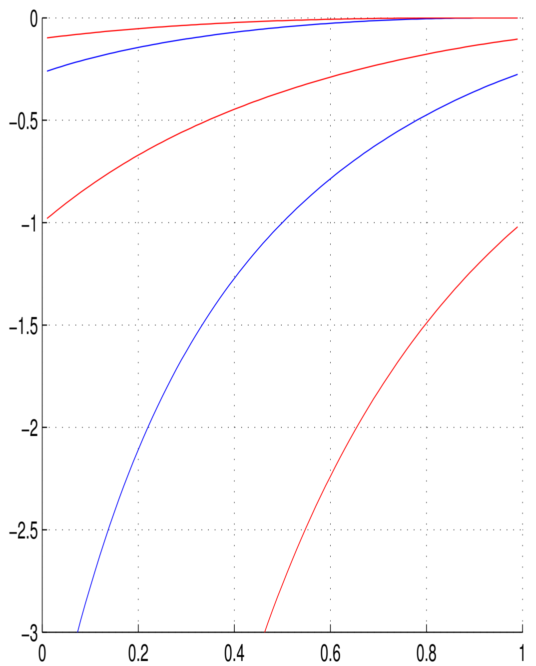

Figure 1. Zeros of generalized Euler polynomials $\xi_{d,\epsilon}(\lambda)$ (blue curves) and $\xi_{d+1,\epsilon}(\lambda)$ (red curves) as functions of $\epsilon \in [0,1]$.

Let us consider the functions

$$\zeta_{d,\epsilon}(\lambda) = \frac{B_{d,\epsilon}(\lambda)}{(d-1)!(\lambda-1)^d} =$$
$$= (\lambda-1)^d C_d[\lambda I - E(1)]^{-1}E(\epsilon)B_d =$$
$$= \begin{bmatrix}\zeta_{1,0}(\lambda) & \zeta_{2,0}(\lambda) & \ldots & \zeta_{d,0}(\lambda)\end{bmatrix}\begin{bmatrix}\frac{s^{d-1}}{(d-1)!}\\ \vdots \\ s \\ 1\end{bmatrix} =$$
$$= \sum_{i=1}^{d-1} s^{d-i}\zeta_{i,0}(\lambda)/(d-i)!\,. \qquad (25)$$

instead of polynomials (16). They have the same zeros $\nu_{d,j}(\epsilon)$.

Examine the first pair of $\zeta_{d,\epsilon}(\lambda)$ in order to apply induction:

$$\zeta_{2,\epsilon}(\lambda) = (\epsilon\lambda - \epsilon + 1)/(\lambda-1)^2,\ \nu_{2,1}(\epsilon) = (\epsilon-1)/\epsilon,$$
$$\zeta_{3,\epsilon}(\lambda) = \frac{1}{2(\lambda-1)^3}[\epsilon^2\lambda^2 + (1+2\epsilon-2\epsilon^2)\lambda + \epsilon^2 - \epsilon + 1],$$
$$\nu_{3,1}(\epsilon) = \frac{-2\,\epsilon + 2\,\epsilon^2 - 1 + \sqrt{4\,\epsilon + 1 - 4\,\epsilon^2}}{2\epsilon^2},$$
$$\nu_{3,2}(\epsilon) = \frac{-2\,\epsilon + 2\,\epsilon^2 - 1 - \sqrt{4\,\epsilon + 1 - 4\,\epsilon^2}}{2\epsilon^2}.$$

Thus the inductive assumptions (23),(24) are valid for $d = 2$. Now we make the inductive step from $d = k-1$ to $d = k$.

Let d be odd (for definiteness only). Fix $\bar{\epsilon} \in (0,1)$ and a negative $\lambda_* > \nu_{d+1,1}(0)$, whose position is known because it is the first zero of the ordinary Euler polynomial $B_{d+1,0}(\lambda)$. By (22) the values of $\zeta_{d+1,0}(\lambda_*)$ and $\zeta_{d+1,1}(\lambda_*)$ have opposite signs ($+$ and $-$, respectively). The equation (25) yields

$$\frac{d\zeta_{d+1,\epsilon}(\lambda)}{d\epsilon} = \zeta_{d,\epsilon}(\lambda). \qquad (26)$$

By the inductive assumption, $\zeta_{d,\epsilon}(\lambda_*) = 0$ for an unique value of ϵ. Hence we could determine the function $\epsilon_{d,1}(\lambda)$ from the equation

$$\zeta_{d,\epsilon_{d,1}(\lambda)}(\lambda) = 0,$$

and this function would be continuous and increasing. Therefore, $\zeta_{d+1,\epsilon}(\lambda_*)$ decreases as a function of $\epsilon \in (0, \epsilon_{d,1}(\lambda_*))$, and then increases at $\epsilon \in (\epsilon_{d,1}(\lambda_*), 1)$, having the minimal value at $\epsilon = \epsilon_{d,1}(\lambda_*)$, and this value is negative:

$$\zeta_{d+1,\epsilon_{d,1}}(\lambda_*) \le \zeta_{d+1,1}(\lambda_*) < 0.$$

Hence there exists a unique $\epsilon_* \in (0, \epsilon_{d,1}(\lambda_*))$ such that $\zeta_{d+1,\epsilon_*}(\lambda_*) = 0$. Define the function $\epsilon_{d+1,1}(\cdot)$ by the equality $\epsilon_{d+1,1}(\lambda_*) = \epsilon_*$. The implicit function theorem guarantees that $\epsilon_{d+1,1}(\cdot)$ is a continuous function for $\lambda \in [\nu_{d+1,1}(0), \nu_{d+1,0}(0)]$. Continuous function should possess every intermediate value between boundary values $\epsilon_{d+1,1}(\nu_{d+1,1}(0)) = 0$ and $\epsilon_{d+1,1}(\nu_{d+1,0}(0)) = 1$. Hence there exists $\lambda_1(\bar{\epsilon}) \in (\nu_{d+1,1}(0), \nu_{d+1,0}(0))$ such that $\epsilon_{d+1}(\nu_{d+1,1}(0)) = \bar{\epsilon}$, i.e. $\zeta_{d+1,\bar{\epsilon}}(\lambda_1(\bar{\epsilon})) = 0$. Note that

$$\epsilon_{d+1,1}(\lambda) < \epsilon_{d,1}(\lambda) \qquad (27)$$

A similar argument for $\lambda_* \in (\nu_{d+1,2}(0), \nu_{d+1,1}(0))$ leads to the existence of

$$\epsilon_{d+1,2}(\lambda) > \epsilon_{d,2}(\lambda) \qquad (28)$$

and $\lambda_2(\bar{\epsilon}) \in (\nu_{d+1,2}(0), \nu_{d+1,1}(0))$ such that $\zeta_{d+1,\bar{\epsilon}}(\lambda_2(\bar{\epsilon})) = 0$, and so on down to $\lambda_d(\bar{\epsilon}) < \nu_{d+1,d}(0)$. The only difference between the steps consists of inequalities like (27) for odd j and (28) for even indices j in numbering $\epsilon_{d+1,j}(\lambda)$ and $\lambda_j(\bar{\epsilon})$.

The numerator of $\zeta_{d+1,\epsilon}(\lambda)$ is the polynomial $B_{d+1,\epsilon}(\lambda)$ of degree d. Thus there remain no possible zeros of $\zeta_{d+1,\bar{\epsilon}}(\lambda)$. This yields the uniqueness of every $\lambda_j(\bar{\epsilon}) \in (\nu_{d+1,j}(0), \nu_{d+1,j-1}(0))$. Thus every $\lambda_j(\epsilon)$ is the inverse function to the continuous function $\epsilon_{d+1,j}(\lambda)$ at $\lambda \in (\nu_{d+1,j}(0), \nu_{d+1,j-1}(0))$. The inverse function theorem guarantees that $\nu_{d+1,j}(\epsilon) = \lambda_j(\epsilon)$ is continuous, too, and both mutually inverse functions are monotonic. Taking into account their boundary values, we obtain that these functions are increasing.

Denote by $G_{d,j}$ the set consisting of points $(\epsilon, \nu_{d,j}(\epsilon))$ in the plane (ϵ, λ). The inequalities of the sort (27) and (28) mean that the continuous curve $G_{d,j}$ lies between $G_{d+1,j}$ and $G_{d+1,j+1}$, see figure 1. This guarantees the interlacing inequalities (23). Thus the inductive step from d to $d+1$ is proved, as well as the theorem itself.

Corollary. If a prototype (1) has a zero μ_j with a positive real part, then ZOH-sampling for a sufficiently small sampling period h necessarily leads to a unstably invertible sampled system (1) due to corresponding intrinsic zero $e^{h\mu_j(h)}$. Suppose that the prototype (1) is minimal-phase (i.e. that real parts of all μ_j are negative) and h is sufficiently small. Then

1. If the relative degree (i.e. $n-m$) of the system (1) is equal to 1, the sampling zero of ZOH-sampled system (5) tends to the zero of $\xi_{2,\epsilon}(\lambda) = 1 - \epsilon + \epsilon\lambda$ as $h \to 0$. Thus a sampled system is stably invertible for $\epsilon < 1/2$ and isn't stably invertible for $\epsilon > 1/2$.
2. If $d = n - m > 2$, or $d = 2$ and $\epsilon \neq 0$, then sampling zeros of the ZOH-sampled system (5) are asymptotically close to the zeros of $\xi_{d+1,\epsilon}(\lambda)$, including unstable ones. Thus sampled system cannot be stably invertible.

The corollary doesn't treat the case $n - m = 2$ and $\epsilon = 0$ where the only one Euler polynomial zero $\nu_{3,1}(0) = -1$ lies in the border of stability. In this case the stable invertibility of the sampled plant depends on the sign of $\sum_{i=1}^{n} \lambda_i - \sum_{j=1}^{m} \mu_j$. This result is known as the Hagiwara criterion [Hagiwara, Yuasa and Araki, 1993], it was also published in [Bondarko, 1991]. In [Bondarko, 1996] this criterion was extended to the case of infinite-dimensional systems. Anyway, condition $\epsilon = 0$ is not robust relative to small deviations of the moments when control input changes its values and the moments when output is observed. By the virtue of Theorem 3, even a very small difference between these moments guarantees the existence of unstable limiting zero, if the relative degree $n - m$ is greater than 1.

5 Conclusion

A delayed sampled system has one more sampling zero then an undelayed one. All sampling zeros are asymptotically close to the roots of the newly introduced generalized Euler polynomials, if the sampling period h tends to zero. The properties of these roots are studied in detail similarly to the classical results on the ordinary Euler polynomials. In particular, this leads to the following conclusion: if the relative degree of the continuous prototype is equal to 1 and the fractional part of quotient of delay and sampling period is less then $1/2$, the additional zero of the delayed sampled system is stable for all sufficiently small h. Otherwise a delayed sampled system is not stably invertible for all sufficiently small h.

References

Åström, K.J, Hagander, P., and Sternby, J. (1980). Zeros of sampled systems. In *proc. IEEE CDC'80* Albuquerque, New Mexico, USA, December 10–12. pp. 1077–1081.

Åström K. J., Hagander P., and Sternby, J. (1984). Zeros of sampled systems. *Automatica*, **20**(1), pp. 21–38.

Bondarko, V.A. (1984). Discretization of continuous linear dynamic systems. Analysis of the methods. *Systems @ Control Letters,* **5**(1), pp. 97–101.

Bondarko, V.A. (1991). Suboptimal and adaptive control of continuous linear systems with delay, *Soviet Journal of Computer and Systems Sciences*, **29**(6), pp. 137–143.

Bondarko, V.A. (1996). Discretization of infinite-dimensional linear dynamical systems. *Differential Equations*, **32**(10), pp. 1309–1318.

Frobenius, G. (1910). On Bernoulli numbers and Euler polynomials. *Sitzungsberichte der Königlich Preüsischen Akademie der Wissenschaften zu Berlin*, Berlin, Germany, pp. 809–847.

Hagiwara, T., Yuasa, T., and Araki, M. (1993). Stability of the limiting zeros of sampled-data systems with zero- and first-order holds. *Int. J. Contr.* **58**, pp. 1325–1346.

Sobolev, S.L. (1977). On the roots of Euler polynomials. *Soviet Math. Dokl.* **18**, pp. 935–938.

Weller, S.R., Moran, W., Ninness, S., and Pollington, A.D. (2001). Sampling zeros and the Euler-Frobenius polynomials. *IEEE Trans. on Autom. Contr.* **46**(2), pp. 340–343.