Optimization of the Proportional Navigation Law with Time Delay

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***Abstract*— The controlled relative motion of a spacecraft nearby an orbital station is considered. A rendezvous method used on active spacecraft is an algorithm of proportional navigation which is realized with some constant time delay. The coefficient of the guidance law is considered as a control variable. The problem of choice of the mentioned coefficient which provides minimizing of rendezvous time, is analyzed. It is found that the optimal solution includes regular and singular control values both. The results of computer simulation are given.**

I. Introduction

Proportional navigation, where the intercepting command acceleration is applied normal to the line of sight (LOS) [1], is one of the most widely used rendezvous algorithms for moving objects of different destination, including space vehicle. It was shown in [2] that this method is optimal in the control problem of linear system with quadratic criterion. In the papers [1], [3] – [5] the influence of parameters of proportional navigation law and its various modifications on the engagement time and closeness characteristics of the rendezvous process was investigated. Those investigations were held in assumption that guidance is realized without delay. In [1] it was shown that in the absence of delay to minimize the rendezvous time the coefficient of proportional navigation law should be chosen maximal. At the same time the delay in realistic guidance systems is inevitable and it neglecting may lead to erroneous results. The analysis of the pursuit-evasion problems taking into account time delay was held in [6]–[9], where linearized kinematics were applied. In [6] the optimal evasion problem against the pursuer which used the proportional navigation strategy with time delay was considered. It was shown that for the case of presence of time delay the evader strategy changes essentially in comparison with the strategy without delay.

The influence of the delay, considered as perturbation parameter on the engagement time was investigated in [10]. A sensitivity analysis with respect to the value of the time delay was also held in the mentioned paper.

In the present paper the terminal phase of plane rendezvous of an active space vehicle and a passive orbital station on the Earth orbit is considered. A rendezvous method used on active spacecraft is an algorithm of proportional navigation which is realized with some minor constant time delay. The objective of this work is to optimize the guidance law taking time delay into account. The coefficient of navigation law is considered as a control variable. The problem is to determine the coefficient which provides minimizing of rendezvous time.

II. Problem Statement

The relative motion is described by the system of nonlinear differential equations in polar coordinates with center point in the center of the station, which is also a center of mass for the whole system. Both objects are considered as point masses, and the distance between them is neglected in comparison with the orbital radius. Active vehicle possesses the precise information of relative motion parameters.
The equations of relative motion are:

$$\begin{aligned} &\ddot{\rho} - \rho\dot{\varphi}^2 = a_\rho, \\ &\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi} = a_\varphi. \end{aligned} \tag{1}$$

Here ρ is the range between the spacecraft and the station, φ is the angle between the LOS and some constant direction in the chosen polar coordinate system, a_ρ and a_φ are the components of control acceleration applied to the active vehicle, directed along and normal to the LOS (Fig.1).

Assume the commanded acceleration along the LOS is equal to zero, and the acceleration normal to the LOS is programmed with proportional navigation law

$$a_\varphi = k\dot{\varphi}(t - \tau), \tag{2}$$

where τ is the constant time delay, k is the control variable, a piecewise continuous time function bounded by

$$-\overline{k} \le k(t) \le \overline{k}. \tag{3}$$

The control problem is to minimize the goal function

$$J = T \to \min, \tag{4}$$

which is specified on the trajectories of the system (1), (2) with the corresponding boundary conditions and control limitation (3).

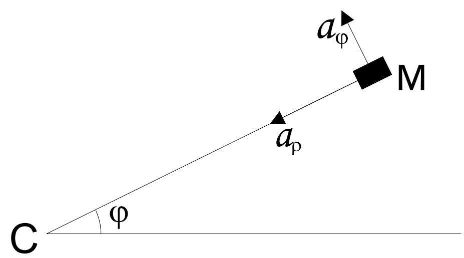

Fig. 1. Relative geometry

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To take into account the time delay we rewrite the acceleration a_φ as follows:

$$a_\varphi = ku(t), \tag{5}$$

where $u(t)$ is described by the equation

$$\tau \dot{u}(t) = \dot{\varphi}(t) - u(t). \tag{6}$$

Time delay was taken into account in a similar way for example in [9].
Substituting (5) and (6) into (1) and rewriting it as a Cauchy-type system, the following system of equations is obtained:

$$\begin{aligned} &\dot{\rho} = v, \\ &\dot{v} = \rho\omega^2, \\ &\dot{\varphi} = \omega, \\ &\dot{\omega} = \frac{-2v\omega + ku}{\rho}, \\ &\dot{u} = \theta(\omega - u), \end{aligned} \tag{7}$$

where $\theta = \dfrac{1}{\tau}$.
The boundary conditions for (7) are:

$$\begin{aligned} t = 0: \quad &\rho(0) = \rho_0,\ v(0) = v_0,\ \varphi(0) = \varphi_0, \\ &\omega(0) = \omega_0,\ u(0) = u_0, \\ t = T: \quad &\rho(T) = \rho_T. \end{aligned} \tag{8}$$

The problem (3), (4), (7), (8) is Mayer optimal control problem, to analyze it we'll use Pontryagin maximal principle [2].

III. Optimal Problem Analysis

The Hamiltonian of the problem (3), (4), (7), (8) is

$$H = v\psi_\rho + \rho\omega^2\psi_v + \omega\psi_\varphi + \frac{ku - 2v\omega}{\rho}\psi_\omega + \theta(\omega - u)\psi_u \tag{9}$$

and the equations for adjoint variables are

$$\begin{aligned} &\dot{\psi}_\rho = -\omega^2\psi_v + \frac{ku - 2v\omega}{\rho^2}\psi_\omega, \\ &\dot{\psi}_v = -\psi_\rho + \frac{2\omega}{\rho}\psi_\omega, \\ &\dot{\psi}_\varphi = 0, \\ &\dot{\psi}_\omega = -2\rho\omega\psi_v - \psi_\varphi + \frac{2v}{\rho}\psi_\omega - \theta\psi_u, \\ &\dot{\psi}_u = -\frac{k}{\rho}\psi_\omega + \theta\psi_u. \end{aligned} \tag{10}$$

Using transversality conditions we obtain the adjoint variables values at the final moment:

$$\psi_v(T) = \psi_\varphi(T) = \psi_\omega(T) = \psi_u(T) = 0,\ H(T) = 1. \tag{11}$$

From (10), (11) and the stationary condition for H on the optimal trajectory we get $H(t) = 1$ on $[0;T]$, $\psi_\varphi \equiv 0$ on $[0;T]$.
It follows from the condition for maximum of Hamiltonian (9) that the extremal control satisfies the following criterion:

$$k = \begin{cases} \overline{k}, & H_1 > 0 \\ -\overline{k}, & H_1 < 0, \end{cases} \tag{12}$$

where $H_1 = \dfrac{u}{\rho}\,\psi_\omega$ is a switching function.
Clear up whether the singular control can exist in the considered problem. Suppose switching function H_1 is identically equal to zero on some time interval $[t_1;t_2] \subseteq [0;T]$. From the condition

$$H_1 = \frac{u}{\rho}\,\psi_\omega \equiv 0$$

the equality $\psi_\omega \equiv 0$ follows, otherwise $u \equiv 0$, and the system (7) becomes uncontrolled. As far as $\dot{H}_1 \equiv 0$ on $[t_1;t_2]$, then, differentiating H_1 by time by virtue of the systems (7) and (10), we obtain:

$$\dot{\psi}_\omega = -2\rho\omega\psi_v - \theta\psi_u \equiv 0. \tag{13}$$

Note that $\psi_v \neq 0$, otherwise it follows from (10) that all the adjoint variables identically equal to zero, that contradicts to maximum principle. From (13) we find:

$$\frac{\psi_u}{\psi_v} = -\frac{2\rho\omega}{\theta}. \tag{14}$$

Further, $\ddot{H}_1 \equiv 0$ on $[t_1;t_2]$:

$$\begin{aligned} \ddot{H}_1 = &-\frac{\psi_u\theta^2\omega}{\rho} + \frac{\psi_u\theta uv}{\rho^2} - 2\theta\omega^2\psi_v + 2\theta u\omega\psi_v - \\ &-\frac{2u^2\psi_v}{\rho}k + \frac{4uv\omega\psi_v}{\rho} + 2u\omega\psi_\rho \equiv 0. \end{aligned} \tag{15}$$

Using stationary condition $H = 1$ on $[t_1;t_2]$ we can find:

$$\psi_\rho + \rho\omega\psi_v(2u - \omega) = 1. \tag{16}$$

From (15) using (14), we find the singular control:

$$k_s = \frac{\omega\rho}{u}\Big(\frac{v}{\rho} + \frac{\psi_\rho}{\psi_v} + \theta\Big). \tag{17}$$

Now (12) can be rewritten in the following way:

$$k = \begin{cases} \overline{k}, & H_1 > 0 \\ k_s, & H_1 \equiv 0 \\ -\overline{k}, & H_1 < 0, \end{cases} \tag{18}$$

Consequently the optimal control problem (3), (4), (7), (8) is formally reduced to the boundary problem for the system of ordinary differential equations (7), (8), (10), (11) with the rule for choosing of control (18). It is generally known that numerical solution of the problems of optimal control with singular paths leads to much difficulties. These difficulties can be overcome if, for example, the structure of the optimal trajectory is known, or the combination of singular and nonsingular arches. To clear up the optimal control structure, let's consider the auxiliary problem for the reduced dynamic system (7).

IV. Auxiliary Problem

Notice that in (7) the control variable $k(t)$ appears only in one equation. Hence we can pass on to an auxiliary problem for the reduced system releasing the initial conditions for the variable $\omega(t)$ and to strike off the equation for $\dot{\omega}$:

$$\begin{aligned} &\dot{\rho} = v, \\ &\dot{v} = \rho\omega^2, \\ &\dot{\varphi} = \omega, \\ &\dot{u} = \theta(\omega - u). \end{aligned} \tag{19}$$

Consider $\omega(t)$ as a control in this problem, the control limitation is lacking. The similar procedure in general form is described in [11], [12]. The goal function and the boundary conditions for the other variables remain the same:

$$\begin{aligned} t = 0: \quad & \rho(0) = \rho_0,\ v(0) = v_0, \\ & \varphi(0) = \varphi_0,\ u(0) = u_0, \\ t = T: \quad & \rho(T) = \rho_T. \end{aligned} \tag{20}$$

The Hamiltonian of the problem (4), (19), (20) is:

$$H = \rho\psi_v\omega^2 + (\psi_\rho + \theta\psi_u)\omega + v\psi_\rho - \theta u\psi_u. \tag{21}$$

The system for adjoint variables is:

$$\begin{aligned} &\dot{\psi}_\rho = -\omega^2\psi_v, \\ &\dot{\psi}_v = -\psi_\rho, \\ &\dot{\psi}_\varphi = 0, \\ &\dot{\psi}_u = \theta\psi_u. \end{aligned} \tag{22}$$

Using transversality conditions we obtain the adjoint variables values at $t = T$:

$$\psi_v(T) = \psi_\varphi(T) = \psi_u(T) = 0,\ H(T) = 1. \tag{23}$$

From the third and the forth equations of the system (22), using adjoint variables values at the final moment (23), we find: $\psi_\varphi \equiv 0$, $\psi_u = c_1 e^{\theta t} \equiv 0$ on $[0;T]$.
To find the optimal control notice that the function H (21) is quadratic in control $\omega(t)$, and its maximum is reached when $\omega = \dfrac{-\psi_\varphi - \theta\psi_u}{2\rho\psi_v} \equiv 0$ under the condition $\rho\psi_v < 0$.
Therefore, the optimal motion in the problem (4), (19), (20) is constant motion along the LOS.

The auxiliary problem solution corresponds to the singular arch of the optimal trajectory of the problem (3), (4), (7), (8) and is obtained under the assumption that the starting condition $\omega(0)$ is free, and the limitations (3) are lacking. Hence, if the optimal trajectory of the problem (3), (4), (7), (8) includes the singular arch and k_s satisfies the condition (3), then this arch adjoins the extremity of the trajectory.
Substituting the auxiliary problem solution $\omega(t) \equiv 0$ in (17), we find the corresponding singular control value $k_s \equiv 0$.

We cleared up the optimal control structure: the coefficient $k(t)$ of the proportional navigation law should be chosen maximal in absolute value from the allowable class $k = \overline{k}$ till the angular velocity ω decreases to zero, then follows the switching to the singular mode $k = 0$ till the reach of the given terminal range ρ_T. Thus, the optimal control synthesis looks like:

$$k = \begin{cases} \left[\begin{array}{ll} -\overline{k}, & \omega \neq 0 \\ \overline{k}, & \omega \neq 0 \end{array} \right. \\ k_s = 0,\ \omega = 0. \end{cases} \tag{24}$$

It follows from the physical meaning of proportional navigation that inclusion of motion path with $k = -\overline{k}$ in the trajectory is not optimal, it is also confirmed by simulation.

V. Computer Simulation

Consider system (7) with various control modes:

$$k = \begin{cases} \overline{k}, & t < t_s \\ 0, & t > t_s, \end{cases} \tag{25}$$

where the switching moment t_s varies from $t_s = 0$ to $t_s = T$. The simulation is realized with the constant time delay $\tau = 0.1$ and the following boundary conditions:

$$\begin{aligned} t = 0: \quad & \rho_0 = 1,\ v_0 = -0.5,\ \varphi_0 = 1, \\ & \omega_0 = 1,\ u_0 = 1, \\ t = T: \quad & \rho_T = 0.4 \end{aligned} \tag{26}$$

and control restriction $\overline{k} = 5$.
The system (7) was numerically integrated with chosen limit values of the variables. The plot of rendezvous time (time of reaching terminal range $\rho_T = 0.4$) vs switching point t_s is shown in Fig.2.
The rightmost point of the plot corresponds to the control mode, on which the proportional navigation law coefficient is chosen constant and maximal $k = \overline{k}$ during the whole approach time. Obviously, this mode is not optimal, because there exists the mode with a switching, on which the required terminal range is reached faster. With the chosen boundary conditions and control limitations minimum of approach time is reached under the control with the switching from $k = \overline{k}$ to $k = 0$ at $t_s = 0.325$.
The plot of $\omega(t)$ vs time with $k = \overline{k}$ on $[0;T]$ is shown in Fig.3. As can be seen from the picture, the angular velocity

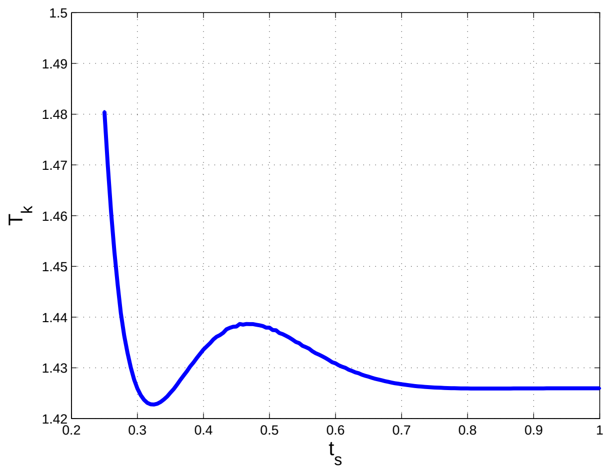

Fig. 2. Rendezvous time vs switching point

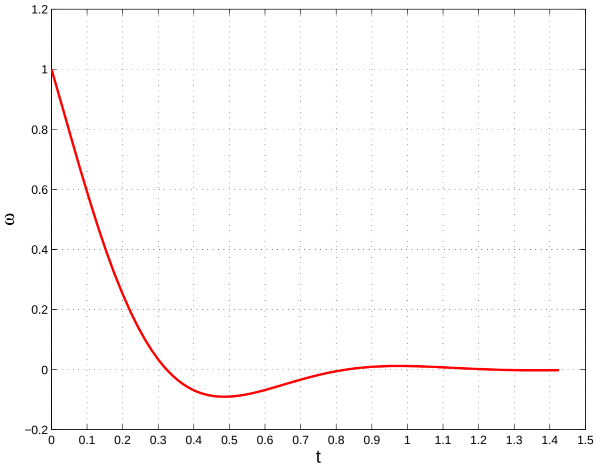

Fig. 3. Relative angular velocity with $k = \bar{k}$

ω decreases to zero at $t = 0.325$, that corresponds to the moment of switching to $k = 0$ under the optimal control.

The system (1) with $a_\varphi = k\dot{\varphi}(t-\tau)$ was also simulated with the help of Matlab function 'dde23', which allows to take delay into account. The qualitative and quantitative closeness of trajectories with small τ was obtained. This fact confirms our right to use the equation (6) for time delay.

VI. Conclusion

The problem of optimization of the proportional navigation law with a constant time delay was considered. It was shown that the time delay essentially effects on the character of optimal guidance. To minimize the rendezvous time of a spacecraft and a passive orbital station the guidance law coefficient should be chosen variable: maximal in absolute value till the moment when the LOS angular velocity reaches zero (regular control) and equal to zero till the end of the process (singular control).

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