ALGORITHMIC AND SOFTWARE IMPLEMENTATION OF ANISOTROPY-BASED ANALYSIS AND CONTROLLER DESIGN PROBLEMS

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Abstract: This paper makes brief mention of numerical algorithms and software tools for anisotropy-based performance analysis and synthesis of control systems using anisotropic norm of closed-loop system as a cost function.

Keywords: Stochastic robust control, Riccati equations, numerical algorithms.

1. INTRODUCTION

This paper is devoted to numerical algorithms of anisotropy-based analysis and controller design problems for finite-dimensional linear discrete time-invariant control systems. The detailed formulation and grounds for applied stochastic approach to $\mathcal{H}_\infty$-optimization have been presented by (Semyonov *et al.* 1994), (Vladimirov *et al.* 1995a), (Diamond *et al.* 2001).

The subject of the numerical methods of anisotropic analysis includes algorithms for computing mean anisotropy and anisotropic norm.

For an open-loop automatic control system and given external disturbance mean anisotropy level a, the stochastic $\mathcal{H}_\infty$-optimization problem consists in finding internally stabilizing controller that minimizes the a-anisotropic norm of the closed-loop system transfer matrix.

For finite-dimensional systems, the optimal controller design problem using the anisotropic norm as performance criterion reduces to solving algebraic equation system consisting of three cross-coupled Riccati equations, Lyapunov equation and special-type nonlinear matrix equation (Vladimirov *et al.* 1996). This nonlinear equation system smoothly depends on a scalar parameter, and when it is equal to zero (that corresponds to the zero level of mean anisotropy $a = 0$), reduces to two independent Riccati equations determining $\mathcal{H}_2$-optimal controller.

For an arbitrary $a > 0$, the respective a-anisotropic controller can be obtained from $\mathcal{H}_2$-optimal one by smooth deformation or homotopy described by differential equations.

2. BASIC CONCEPTS OF ANISOTROPIC ANALYSIS

2.1 *Mean anisotropy of Gaussian signal*

Let $V = (v_k)_{-\infty<k<+\infty}$ be the discrete-time m-dimensional Gaussian white noise with zero

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[1] Supported by Russian Foundation for Basic Research, grant 05-08-18131-a
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mean and identity covariance matrix:

$$\mathbf{E}\,(v_k) = 0, \mathbf{E}\,\left(v_k v_j^\top\right) = \delta_{kj} I_m, -\infty < k < +\infty, \quad (1)$$

where δ_{kj} is the Kronecker delta. Consider m-dimensional stationary Gaussian sequence

$$W = (w_k)_{-\infty<k<+\infty} = G * V,$$

generated from white noise V by a stable generating filter with impulse transient response $g_k \in \mathbb{R}^{m\times m}$, $k \geqslant 0$:

$$w_j = \sum_{k=0}^{+\infty} g_k\, v_{j-k}\,, \quad -\infty < j < +\infty. \quad (2)$$

Such the filter is identified with its transfer matrix $G(z) = \sum_{k=0}^{+\infty} g_k\, z^k$, $z v_j = v_{j-1}$, belonging to the Hardy space $\mathcal{H}_2^{m\times m}$.

Define the mean anisotropy of the sequence W as (Vladimirov *et al.* 1995a)

$$\overline{\mathbf{A}}(W) \doteq -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det\left(\frac{m}{\|G\|_2^2}\,\widehat{G}(\omega)\left(\widehat{G}(\omega)\right)^*\right) d\omega, \quad (3)$$

where $\widehat{G}(\omega) = \lim_{r\to 1-0} G(r\,\mathrm{e}^{j\omega})$. Expression (3) also is the formula for computing the mean anisotropy of Gaussian stationary random sequence in frequency domain. The sequence W is completely determined by its generating filter G, therefore, along with the notation $\overline{\mathbf{A}}(W)$, the equivalent notation $\overline{\mathbf{A}}(G)$ can be used.

2.1.1. *State-space formulas for mean anisotropy.*

Let the generating filter $G \in \mathcal{H}_2^{m\times m}$ has n-dimensional internal state $X = (x_k)_{-\infty<k<+\infty}$ relating the input V and output W by equations

$$\left.\begin{array}{ll} x_{k+1} &= Ax_k + Bv_k \\ w_k &= Cx_k + Dv_k \end{array}\right\}, \quad -\infty < k < +\infty,$$

where A, B, C, D are the matrices of appropriate dimensions, the matrix A is assumed to be asymptotically stable, and the matrix D is nonsingular. Consider the Riccati equation in the matrix $R \in \mathbb{R}^{n\times n}$

$$R = ARA^\top + BB^\top - LTL^\top, \quad (4)$$

$$T = CRC^\top + DD^\top, \quad (5)$$

$$L = \left(ARC^\top + BD^\top\right) T^{-1} \quad (6)$$

associated with the filter G. Equation (4)–(6) has a unique stabilizing solution $R = R^\top > 0$ such that the matrix $A - LC$ is asymptotically stable (Shiryaev and Liptser 1977).

Theorem 1. (Diamond *et al.* 2001) Mean anisotropy (3) of the sequence $W = G * V$ generated by the filter G with the asymptotically stable matrix $A \in \mathbb{R}^{n\times n}$ and the nonsingular matrix $D \in \mathbb{R}^{m\times m}$ is given by formula

$$\overline{\mathbf{A}}(G) = -\frac{1}{2}\ln\det\left(\frac{mT}{\mathrm{tr}\,(CPC^\top + DD^\top)}\right), \quad (7)$$

where the matrix T is determined via the stabilizing solution R of Riccati equation (4)–(6), and the matrix P is the controllability grammian of the filter G satisfying the Lyapunov equation

$$P = APA^\top + BB^\top. \quad (8)$$

2.2 *Computing mean anisotropy of random sequence: function* `meananis`

The numerical procedure for computing the mean anisotropy of Gaussian random sequence is implemented in MATLAB environment as function `meananis` with the following syntax:

```
[a] = meananis(A,B,C,D)
```

that returns the numerical value of variable `a` equal to the mean anisotropy of random sequence produced by the generating filter with the state-space realization matrices `A`, `B`, `C`, `D`, which are the input variables of the function.

The functions `dare` and `dlyap` of MATLAB environment are used for solving algebraic Riccati equation (4)–(6) and Lyapunov equation (8), respectively.

2.3 *Anisotropic norm of linear system*

Let F be the linear discrete time-invariant system with input W and output $Z = F * W$. Assume that its transfer matrix belongs to the Hardy space $\mathcal{H}_\infty^{p\times m}$.

For given $a \geqslant 0$ the a-anisotropic norm of the system F is defined as

$$|\!|\!| F |\!|\!|_a = \sup_{G\in\mathbb{G}_a=\left\{G\in\mathcal{H}_2^{m\times m}:\ \overline{\mathbf{A}}(G)\leqslant a\right\}} \left\{\frac{\|FG\|_2}{\|G\|_2}\right\}. \quad (9)$$

Since, as it has been shown in paper (Diamond *et al.* 2001),

$$\frac{1}{\sqrt{m}}\|F\|_2 = |\!|\!| F |\!|\!|_0 \leqslant \lim_{a\to\infty} |\!|\!| F |\!|\!|_a = \|F\|_\infty,$$

below we will consider the systems F satisfying the inequality $\frac{1}{\sqrt{m}}\|F\|_2 < \|F\|_\infty$.

2.3.1. *State-space formulas for anisotropic norm.*

Let the system $F \in \mathcal{H}_\infty^{p\times m}$ has the following representation in state space

$$F \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \quad (10)$$

There A, B, C, D are the matrices of appropriate dimentions with the asymptotically stable matrix A.

Consider the following algebraic Riccati equation in the matrix $R \in \mathbb{R}^{n\times n}$

$$R = A^\top RA + qC^\top C + L^\top \Sigma^{-1} L, \quad (11)$$
$$\Sigma = (I_m - qD^\top D - B^\top RB)^{-1}, \quad (12)$$
$$L = \Sigma(B^\top RA + qD^\top C). \quad (13)$$

Denote that for $\forall q \in [0, \|F\|_\infty^{-2})$ Riccati equation (11)–(13) has a unique stabilizing solution $R = R^\top \geqslant 0$ such that the matrix $A + BL$ is asymptotically stable and matrix $\Sigma > 0$.

Theorem 2. (Diamond *et al.* 2001) Let asymptotically stable system (10) and the input mean anisotropy level $a > 0$ be given. Then there exists a unique pair (q, R) of the parameter $q \in (0, \|F\|_\infty^{-2})$ and stabilizing solution R of Riccati equation (11)–(13) such that

$$-\frac{1}{2}\ln\det\left(\frac{m\,\Sigma}{\mathrm{tr}\,(LPL^\top + \Sigma)}\right) = a, \quad (14)$$

where P is the controllability grammian of the filter

$$G \sim \left[\begin{array}{c|c} A+BL & B\Sigma^{1/2} \\ \hline L & \Sigma^{1/2}\end{array}\right], \quad (15)$$

determined by the Lyapunov equation

$$P = (A + BL)P(A + BL)^\top + B\Sigma B^\top. \quad (16)$$

At that, filter (15) is the worst-case generating filter, and a-anisotropic norm (9) of the system F is given by

$$|\!|\!| F |\!|\!|_a = \left(\frac{1}{q}\left(1 - \frac{m}{\mathrm{tr}\,(LPL^\top + \Sigma)}\right)\right)^{1/2}. \quad (17)$$

2.3.2. *Computing anisotropic norm by homotopy method with Newton's iterations*

Lemma 1. For given asymptotically stable system (10), the stabilizing solution of Riccati equation (11)–(13) and accompanying matrices (12), (13), (16) are analytic in $q \in \left[0, \|F\|_\infty^{-2}\right)$ and satisfy the differential equations

$$\begin{aligned}\dot R &= (A + BL)^\top \dot R\,(A + BL) \\ &\quad + (C + DL)^\top (C + DL), \quad (18)\\ \dot P &= (A + BL)\dot P(A + BL)^\top \\ &\quad + B\dot L P(A + BL)^\top \\ &\quad + (A + BL)P\dot L^\top B^\top + B\dot\Sigma B^\top, \\ \dot L &= \Sigma(B^\top \dot R\,(A + BL) + D^\top (C + DL)) \quad (19)\\ \dot\Sigma &= \Sigma(D^\top D + B^\top \dot R B)\Sigma. \quad (20)\end{aligned}$$

Define the functions

$$\mathcal{A}\colon\ [0, \|F\|_\infty^{-2}) \to \mathbb{R}_+,$$
$$\mathcal{N}\colon\ [0, \|F\|_\infty^{-2}) \to [m^{-1/2}\|F\|_2, \|F\|_\infty)$$

by the left and right parts of (14) and (17), respectively:

$$\mathcal{A}(q) = -\frac{1}{2}\ln\det\left(\frac{m\,\Sigma}{\mathrm{tr}\,(LPL^\top + \Sigma)}\right),$$
$$\mathcal{N}(q) = \left(\frac{1}{q}\left(1 - \frac{m}{\mathrm{tr}\,(LPL^\top + \Sigma)}\right)\right)^{1/2}.$$

Restating Theorem 2, we have the following expression for the a-anisotropic norm of the system F

$$|\!|\!| F |\!|\!|_a = \mathcal{N}\left(\mathcal{A}^{-1}(a)\right). \quad (21)$$

Since the function $\mathcal{A}$ is strictly increasing and convex, the a-anisotropic norm can be computed by formula (21) via Newton's iterations whose convergence is provided by strict monotonicity and convexity of the function $\mathcal{A}$.

2.4 *Computing anisotropic norm of linear discrete time-invariant system: function* `aninorm`

The computational algorithm for the anisotropic norm using Newton's iterations is implemented in function `aninorm`:

```
[anorm] = aninorm(A,B,C,D,a).
```

This function returns the numeric value of variable `anorm` equal to the anisotropic norm of the system with the state-space realization matrices `A`, `B`, `C`, `D` given the mean anisotropy level `a` of the input Gaussian random sequence.

Functions `dare` and `dlyap` of MATLAB environment are used for solving Riccati equation (11)–(13) and Lyapunov equation (16), respectively. For solving equations (18)–(20) with respect to the derivatives $\dot R$, $\dot P$, $\dot L$, and $\dot\Sigma$, function `dlyap` is called.

Computing $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norm of the system is implemented by functions `h2norm` and `hinfnorm`, respectively:

```
[norm] = h2norm(A,B,C,D),
[norm] = hinfnorm(A,B,C,D).
```

The function `norminf` calls function `varsvd1` with syntax

```
[Z] = varsvd1(X,Y)
```

that returns r-dimensional column vector `Z` of the first-order directional derivatives

$$Z = \left[\begin{array}{c}\dfrac{\partial\sigma_1(X + aY)}{\partial a}\\ \vdots \\ \dfrac{\partial\sigma_r(X + aY)}{\partial a}\end{array}\right]_{a=0}$$

of the ordered set of singular values $\sigma_1 > \ldots > \sigma_r$ of the matrix X with respect to the matrix Y, where r is the minimum dimension of X, as well as function `varsvd2`:

```
[W] = varsvd2(X,Y,Z).
```

The latter returns in variable `W` r-dimensional column vector of the second-order directional derivatives

$$W = \begin{bmatrix} \dfrac{\partial^2 \sigma_1(X + aY + bZ)}{\partial a \partial b} \\ \vdots \\ \dfrac{\partial^2 \sigma_r(X + aY + bZ)}{\partial a \partial b} \end{bmatrix}_{a,b=0}$$

of the ordered set of singular values $\sigma_1 > \ldots > \sigma_r$ of the matrix X with respect to the matrices Y and Z, where r is the minimum dimension of the matrix X. The functions `varsvd1` and `varsvd2` use the standard MATLAB function `svd` for computing singular value decomposition of the matrix X.

3. ALGORITHMS FOR ANISOTROPY-BASED CONTROLLER DESIGN

3.1 *Optimal control minimizing anisotropic norm*

Consider linear discrete time-invariant system F with two inputs: m_1-dimensional disturbance W and m_2-dimentional control U; and two outputs: p_1-dimensional controlled signal Z and p_2-dimensional observation Y. The system F has the block structure

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}. \tag{22}$$

With the feedback $U = K * Y$, the closed-loop system transfer matrix from W to Z is given by

$$\mathcal{L}(F, K) = F_{11} + F_{12} K \left(I_{p_2} - F_{22} K \right)^{-1} F_{21}. \tag{23}$$

Let the disturbance $W = G * V$ be the stationary Gaussian sequence produced from m_1-dimensional Gaussian white noise V by unknown generating filter G in the family

$$\mathbb{G}_a = \left\{ G \in \mathcal{H}_2{}^{m_1 \times m_1} : \overline{\mathbf{A}}(G) \leqslant a \right\} \tag{24}$$

The controller K is called to be admissible if it is strictly causal and internally stabilizes closed-loop system (23).

Denoting $\mathcal{K}$ the set of admissible controllers for given system (22), let us formulate the stochastic $\mathcal{H}_\infty$-optimization problem (Vladimirov *et al.* 1995b), (Vladimirov *et al.* 1996): given the mean anisotropy level $a \geqslant 0$, find a controller $K \in \mathcal{K}$ minimizing the a-anisotropic norm of closed-loop system (23):

$$|\!|\!| \mathcal{L}(F, K) |\!|\!|_a = \sup_{\forall G \in \mathbb{G}_a} \frac{\| \mathcal{L}(F, K) G \|_2}{\| G \|_2} \searrow \inf_{\forall K \in \mathcal{K}}. \tag{25}$$

3.1.1. *State-space equations for optimal controller.* Let the system F has the following state-space realization

$$F \sim \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right] \tag{26}$$

where A, C_i, B_j, D_{ij} are the matrices of appropriate dimensions.

Below we consider equations for solving stated stochastic $\mathcal{H}_\infty$-optimization problem (25). At that, we use standard assumptions in respect of relations between dimensions of the system input and output, the subsystem matrices rank fullness, and on stabilizability and detectability of the system.

Let K be the admissible controller with n-dimensional internal state H relating observation Y and control U by equations

$$\left. \begin{array}{ll} h_{k+1} & = \widehat{A} h_k + \widehat{B} y_k \\ u_k & = \widehat{C} h_k \end{array} \right\}, \quad -\infty < k < +\infty, \tag{27}$$

where $\widehat{A}, \widehat{B}, \widehat{C}$ are the matrices of appropriate dimensions. Then for closed-loop system (23)

$$\mathcal{L}(F, K) \sim \left[\begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & D_{11} \end{array} \right] = \left[\begin{array}{cc|c} A & B_2 \widehat{C} & B_1 \\ \widehat{B} C_2 & \widehat{A} & \widehat{B} D_{21} \\ \hline C_1 & D_{12} \widehat{C} & D_{11} \end{array} \right], \tag{28}$$

where the matrix $\overline{A}$ is asymptotically stable. Associate with system (28) algebraic Riccati equation in the matrix $R \in \mathbb{R}^{2n \times 2n}$

$$R = \overline{A}^\top R \overline{A} + q \overline{C}^\top \overline{C} + L^\top \Sigma^{-1} L, \tag{29}$$

$$\Sigma = (I_{m_1} - q D_{11}^\top D_{11} - \overline{B}^\top R \overline{B})^{-1}, \tag{30}$$

$$L = \left[\, L_1 \;\; L_2 \, \right] = \Sigma (\overline{B}^\top R \overline{A} + q D_{11}^\top \overline{C}), \tag{31}$$

where $0 \leqslant q < \| \mathcal{L}(F, K) \|_\infty^{-2}$ and the matrix L is partitioned into the blocks $L_1, L_2 \in \mathbb{R}^{m_1 \times n}$. Whatever admissible controller (27), for $\forall q \in \left[0, \| \mathcal{L}(F, K) \|_\infty^{-2} \right)$ equation (29)–(31) has a unique stabilizing solution $R = R^\top \geqslant 0$ such that the matrix $\overline{A} + \overline{B} L$ is asymptotically stable and matrix $\Sigma > 0$.

Theorem 3. (Vladimirov *et al.* 1996) Let system (26) satisfy the standard assumptions mentioned before. Then for any admissible controller (27) and any mean anisotropy level $a > 0$ there exists a unique pair of the parameter q

$\in (0, \|\mathcal{L}(F,K)\|_\infty^{-2})$ and stabilizing solution R of Riccati equation (29)–(31) such that

$$-\frac{1}{2}\ln\det\left(\frac{m_1\,\Sigma}{\mathrm{tr}\,(LPL^\top+\Sigma)}\right)=a, \qquad (32)$$

where $P \in \mathbb{R}^{2n\times 2n}$ is the controllability grammian of the generating filter

$$G \sim \left[\begin{array}{c|c}\overline{A}+\overline{B}L & \overline{B}\Sigma^{1/2}\\ \hline L & \Sigma^{1/2}\end{array}\right] = \left[\begin{array}{cc|c} A+B_1L_1 & B_1L_2+B_2\widehat{C} & B_1\Sigma^{1/2}\\ \widehat{B}\,(C_2+D_{21}L_1) & \widehat{A}+\widehat{B}D_{21}L_2 & \widehat{B}D_{21}\Sigma^{1/2}\\ \hline L_1 & L_2 & \Sigma^{1/2}\end{array}\right], \qquad (33)$$

determined by the Lyapunov equation

$$P=(\overline{A}+\overline{B}L)P(\overline{A}+\overline{B}L)^\top+\overline{B}\Sigma\overline{B}^\top. \qquad (34)$$

This filter is the worst-case input generating filter for closed-loop system (28), moreover,

$$\|\mathcal{L}(F,K)\|_a=\left(\frac{1}{q}\left(1-\frac{m_1}{\mathrm{tr}(LPL^\top+\Sigma)}\right)\right)^{1/2}. \qquad (35)$$

Consider the algebraic Riccati equation in the matrix $S\in\mathbb{R}^{n\times n}$

$$\begin{aligned} S&=(A+B_1L_1)S(A+B_1L_1)^\top\\ &\quad+B_1\Sigma B_1^\top-\Lambda\Theta\Lambda^\top, && (36)\\ \Theta&=(C_2+D_{21}L_1)S(C_2+D_{21}L_1)^\top\\ &\quad+D_{21}\Sigma D_{21}^\top, && (37)\\ \Lambda&=((A+B_1L_1)S(C_2+D_{21}L_1)^\top\\ &\quad+B_1\Sigma D_{21}^\top)\Theta^{-1} && (38)\end{aligned}$$

(here the matrices Σ and L are defined in Theorem 3). Denote that equation (36)–(38) has no more than one stabilizing solution $S=S^\top\geqslant 0$ such that the matrix $A+B_1L_1-\Lambda(C_2+D_{21}L_1)$ is asymptotically stable.

Finally, consider the Riccati equation in the matrix $T\in\mathbb{R}^{2n\times 2n}$

$$\begin{aligned} T&=\underline{A}^\top T\underline{A}+\underline{C}^\top\underline{C}-N^\top\Pi N\,, && (39)\\ \Pi&=\underline{B}^\top T\underline{B}+D_{12}^\top D_{12}\,, && (40)\\ N&=-\Pi^{-1}(\underline{B}^\top T\underline{A}+D_{12}^\top\underline{C}) && (41)\end{aligned}$$

where the matrix $N=\left[\,N_1\ \ N_2\,\right]$ is partitioned into the blocks N_1, $N_2\in\mathbb{R}^{m_2\times n}$, and the matrices $\underline{A}\in\mathbb{R}^{2n\times 2n}$, $\underline{B}\in\mathbb{R}^{2n\times m_2}$, and $\underline{C}\in\mathbb{R}^{p_1\times 2n}$ are given by

$$\left[\begin{array}{c|c}\underline{A} & \underline{B}\\ \hline \underline{C} & *\end{array}\right]=\left[\begin{array}{cc|c} A & B_1M & B_2\\ 0 & A+B_1M+B_2\widehat{C} & 0\\ \hline C_1 & D_{11}M & *\end{array}\right], \qquad (42)$$

and $M=L_1+L_2$ is the sum of the blocks of matrix (31). Equation (39)–(41) has a unique stabilizing solution $T=T^\top\geqslant 0$ such that the matrix $\underline{A}+\underline{B}N$ is asymptotically stable.

Theorem 4. (Vladimirov *et al.* 1996) Let system (26) satisfy the standard assumptions, and let the state-space realization matrices of admissible controller (27) be given by

$$\begin{aligned} \widehat{A}&=B_2\widehat{C}+\left[\,I_n\ \ -\Lambda\,\right]\begin{bmatrix}A & B_1\\ C_2 & D_{21}\end{bmatrix}\begin{bmatrix}I_n\\ M\end{bmatrix}, && (43)\\ \widehat{B}&=\Lambda\,, && (44)\\ \widehat{C}&=N_1+N_2, && (45)\end{aligned}$$

where the matrices N_1, N_2 are determined by the stabilizing solution of Riccati equation (39)–(41). Then such the controller is the solution of problem (25).

Theorems 3 and 4 give the complete equation system for the optimal controller state-space realization matrices in problem (26). This system consists of three algebraic Riccati equations (two $(2n\times 2n)$-dimensional equations (29)–(31), (39)–(41), and $(n\times n)$-dimensional equation (36)–(38)), Lyapunov equation (34), special-type nonlinear matrix algebraic equation (32), as well as relations (43)–(45), (28), and (42).

3.2 *Numerical solution of optimal anisotropic controller design problem: function* `anicont`

The computational homotopy-based algorithm with Newton's iterations for solving the optimal anisotropic controller design problem is implemented in function `anicont`. Syntax for calling this function is as follows

```
[hatA,hatB,hatC,N]
= anicont(A,B1,B2,C1,C2,D11,D12,D21,a).
```

The function `anicont` returns the optimal anisotropic controller matrices $\widehat{A}$, $\widehat{B}$, $\widehat{C}$ in variables `hatA`, `hatB`, and `hatC`, respectively, together with the value of the anisotropic norm of system with the state-space realization matrices, corresponding to the input variables `A`, `B1`, `B2`, `1`, `2`, `D11`, `D12`, and `D21`, closed by the anisotropic controller for given disturbance mean anisotropy level `a`.

The $\mathcal{H}_2$-optimal controller being the initial point of homotopy-based algorithm with $q=0$ is computed by function `h2cont` with call syntax

```
[hatA,hatB,hatC,norm]
= h2cont(A,B1,B2,C1,C2,D11,D12,D21).
```

The function returns the matrices $\widehat{A}$, $\widehat{B}$, $\widehat{C}$ of $\mathcal{H}_2$-optimal controller as well as the value of $\mathcal{H}_2$-norm of the system with the realization matrices `A`, `B1`, `B2`, `1`, `2`, `D11`, `D12`, and `D21` closed by $\mathcal{H}_2$-optimal controller. For solving Riccati equations determining the controller matrices, the function `h2cont` calls the standard function `dare`. The function `h2norm` is called for computing $\mathcal{H}_2$-norm of the closed-loop system.

In computations of matrix-valued mappings in homotopy method, the function `tocol` is used:

```
[Y] = tocol(A,B,C).
```

It returns vectorized matrices `A`, `B`, and `C`. For vectorizing each of the matrices, the function `tocol`, in turn, calls function `mat2col` with syntax

```
[X] = mat2col(Z),
```

returning the column vector with dimension $p\,q$ constituted by sequentially written columns of the $(p \times q)$-dimensional matrix `Z`.

Recovery of the matrix `A`, `B`, and `C` with dimensions $n \times n$, $n \times m$, and $p \times n$, respectively, from their vectorized representation `Y` is implemented in function `tomat`:

```
[A,B,C] = tomat(n,m,p,Y),
```

which, in turn, use the function

```
[X] = col2mat(p,q,Z)
```

that returns $(p{\times}q)$-matrix `X` such that its sequentially written columns form the column vector `Z`.

The matrices of the system closed by the controller are formed by function

```
[Acl,Bcl,Ccl,Dcl]
= loop(A,B1,B2,C1,C2,D11,D12,D21,...
hatA,hatB,hatC).
```

Function `ric1` is used for solving Riccati equation (29)–(31) and computing derivatives of the matrices Σ and L with respect to the parameter q and the controller matrices $\widehat{A}$, $\widehat{B}$ $\widehat{C}$. It has syntax

```
[Sigma,L,dSigma,dL] = ric1(q,A,B,C,D),
```

and calls the standard function `dare` for solving the Riccati equation.

The function `dlyap` of MATLAB environment is called for solving Lyapunov equation (34). $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms are computed by functions `h2norm` and `hinfnorm`, respectively (see subsection 2.4).

Computational algorithm for solving algebraic Riccati equation (36)–(38) and computing the derivative of the matrix Λ with respect to the matrices Σ and L_1 is implemented in function `ric2`:

```
[Lambda,dLambda]
= ric2(A,B1,C2,D21,Sigma,L1),
```

which calls function `estric`:

```
[L,dL] = estric(A,B,C,D)
```

for solving Riccati equation and computing the derivative. The function `estric`, in turn, calls the standard function `dare`. Finally, function `ric3` solves Riccati equation (39)–(41) and computes the derivative of the matrix N with respect to the matrices $\underline{A}$ and $\underline{C}$. It has call syntax

```
[N,dN] = ric3(A,B,C,D)
```

and also use the function `estric`.

4. CONCLUSION

This paper makes a brief mention of algorithms and MATLAB software tools for solving anisotropy-based performance analysis and optimal controller design problems. Some numerical examples with computations made using these software tools can be found, e.g., in papers (Diamond *et al.* 2001), (Kurdyukov *et al.* 2004). The presented software package is freely available for download at `www.apkurdukov.narod.ru`.

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