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DYNAMIC STABILITY OF PLATE INTERACTING WITH VISCOUS FLUID

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Abstract
We consider the problem of stability of the motion of an deformable plate which is a part of the border dividing the areas filled with viscous incompressible fluid. Two models of a deformable solid are considered. The first (nonlinear) model of an elastic body takes into account the longitudinal and transverse deformation of the plate, but does not take into account its aging. The second (linear) model of a viscoelastic body takes into account the aging of the plate, but does not take into account longitudinal deformation. The stability research method is based on creation of a functional of Lyapunov type for the partial differential equations which satisfy aerohydrodynamic functions and deformations of plate. Sufficient stability conditions of the motion of an elastic plate imposing restrictions for parameters of mechanical system are received.

Key words
Fluid-structure interaction; Stability; Elastic plate; Viscous fluid.

1 Introduction

At the design and exploitation of structures, devices, mechanisms for various applications, interacting with a fluid, an important problem is to ensure the reliability of their functionality and longer life. Similar problems are common to many branches of engineering. In particular, such problems arise in missilery, aircraft construction, instrumentation, and so on. The essential value in the calculation of structures that interact with the fluid has a stability study of the deformable elements, as the impact of the fluid may lead to its loss.

Thus, at designing of the structures and devices interacting with the fluid, it is necessary to solve problems related to the investigation of stability required for their functioning and operational reliability.

High number of theoretical and experimental studies deal with the stability of elastic bodies interacting with gas and fluid. Among them should be noted the studies [Ageev, Kuznetsova, Kulikov, Mogilevich and Popov, 2014; Kheiri and Paidoussis, 2015; Kontzialis, Moditis and Paidoussis, 2017; Moditis, Paidoussis and Ratigan, 2016; Mogilevich, Popov, Popova and Christoforova, 2016; Mogilevich, Popov, Rabinsky and Kuznetsova, 2016; Naumova, Ivanov, Voloshinova and Ershov, 2015; Sokolov and Razov, 2014; Zvyagin and Gur'ev, 2017] and many others. Among the works of the authors of this article about fluid-structure interaction, note the monographs and articles [Ankilov and Velmisov, 2013, 2015, 2016; Velmisov, Ankilov and Semenova, 2016].

The definition of stability of an elastic body used in this work corresponds to the Lyapunov concept of stability of dynamical systems. The problem can be formulated as follows: for any values of the parameters characterizing the system fluid-solid to the small deformations of bodies at the initial time $t = 0$ (i.e., a small initial deviations from the equilibrium position) will correspond to small deformations and at any time $t > 0$.

2 Mathematical Model

We investigate stability of the motion (by Lyapunov) of an elastic plate which is a part ($x = a, y_0 < y < y_*$) of the border L_0 dividing two areas S_1 and S_2, filled with viscous incompressible fluid. Areas S_1, S_2 have borders L_1, L_2 and L_0 of any form (for example, Fig.1).

We introduce the notations: $u(y, t)$ and $w(y, t)$, $y \in (y_0, y_*)$ are deformations of an elastic plate in the direction of axes of *Oy* and *Ox* respectively;

$$v_1(x,y,t) = \begin{cases} v_{11}(x,y,t), \ (x,y) \in S_1, \\ v_{12}(x,y,t), \ (x,y) \in S_2, \end{cases}$$

$$v_2(x,y,t) = \begin{cases} v_{21}(x,y,t), \ (x,y) \in S_1, \\ v_{22}(x,y,t), \ (x,y) \in S_2 \end{cases}$$

are fluid velocity vector projections;

$$P(x,y,t) = \begin{cases} P_1(x,y,t), \ (x,y) \in S_1, \\ P_2(x,y,t), \ (x,y) \in S_2 \end{cases}$$

is pressure in fluid.

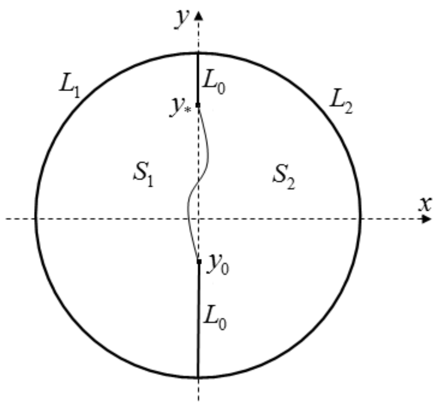


Figure 1. Example of the areas S_1, S_2

Function $w(y,t) \in C^{4,2}\{[y_0,y_*] \times R^+\}$, i.e. it belongs to four times continuously differentiable functions with respect to the variable y on the interval (y_0,y_*) and twice continuously differentiable with respect to the variable t at $t \geq 0$ and takes real values.

Function $u(y,t) \in C^{2,2}\{[y_0,y_*] \times R^+\}$, i.e. it belongs to twice continuously differentiable functions with respect to the variable y on the interval (y_0,y_*) and twice continuously differentiable with respect to the variable t at $t \geq 0$ and takes real values.

Functions $v_{1i}(x,y,t)$, $v_{2i}(x,y,t)$, $P_i(x,y,t) \in C^{2,1}\{S_i \times R^+\}$, i.e. it belongs to twice continuously differentiable functions with respect to the variables x, y in the area S_i and continuously differentiable with respect to the variable t at $t \geq 0$ and takes real values.

In the model of the viscous incompressible medium we write the equations describing the motion of the fluid in the fields S_1, S_2:

$$\begin{gathered} \rho(v_{1t} + v_1 v_{1x} + v_2 v_{1y}) = \mu(v_{1xx} + v_{1yy}) - \\ -P_x, \ \ (x,y) \in S_1 \bigcup S_2; \end{gathered} \tag{1}$$

$$\begin{gathered} \rho(v_{2t} + v_1 v_{2x} + v_2 v_{2y}) = \mu(v_{2xx} + v_{2yy}) - \\ -P_y, \ \ (x,y) \in S_1 \bigcup S_2; \end{gathered} \tag{2}$$

$$v_{1x} + v_{2y} = 0, \quad (x,y) \in S_1 \bigcup S_2\,. \tag{3}$$

We introduce the conditions for the sticking of a viscous fluid:

$$v_1(L_k) = v_2(L_k) = 0, \quad k = 1,2; \tag{4}$$

$$v_1\left(L_0 \backslash (y_0,y_*)\right) = v_2\left(L_0 \backslash (y_0,y_*)\right) = 0; \tag{5}$$

$$\begin{gathered} v_1(a,y,t) = \dot{w}(y,t), \ v_2(a,y,t) = 0, \\ y \in (y_0,y_*). \end{gathered} \tag{6}$$

Consider two models of the deformable solid – the two-stage nonlinear model of a elastic body (7), taking into account the longitudinal and transverse deformation of the plate, but not taking into account its aging, and the single-stage linear model of a viscoelastic body (8), taking into account the aging of the plates, but not taking into account longitudinal deformation:

1) first model

$$\begin{cases} -EF\left(u'(y,t) + \frac{1}{2}w'^2(y,t)\right)' + M\ddot{u}(y,t) = 0, \\ -EF\left[w'(y,t)\left(u'(y,t) + \frac{1}{2}w'^2(y,t)\right)\right]' + \\ +Dw''''(y,t) + M\ddot{w}(y,t) + N(t)w''(y,t) + \\ +\beta_2\dot{w}''''(y,t) + \beta_1\dot{w}(y,t) + \beta_0 w(y,t) = \\ = P_1(a,y,t) - P_2(a,y,t), \ \ y \in (y_0,y_*); \end{cases} \tag{7}$$

2) second model

$$\begin{gathered} D\left[w''(y,t) - \int_0^t \frac{\partial Q_1(y,\tau,t)}{\partial \tau} w''(y,\tau)d\tau\right]'' + \\ +M\ddot{w}(y,t) + N(t)w''(y,t) + \beta_2\dot{w}''''(y,t) + \beta_1\dot{w}(y,t) + \\ +\beta_0\left[w(y,t) - \int_0^t \frac{\partial Q_2(y,\tau,t)}{\partial \tau} w(y,\tau)d\tau\right] = \\ = P_1(a,y,t) - P_2(a,y,t), \ \ y \in (y_0,y_*). \end{gathered} \tag{8}$$

The indices *x*, *y*, *t* below denote partial derivatives with respect to *x*, *y*, *t*; the bar and the point denote the partial derivatives with respect to *y* and *t*, respectively; ρ, μ are density and dynamic coefficient of viscosity of the fluid; $D = Eh^3/(12(1-\nu^2))$ is flexural stiffness of the element; *h* is the thickness of the plate; $M = h\rho_p$ is the linear mass of the plate; $F = h/(1-\nu^2)$; *E*, ρ_p are elasticity modulus and the linear density of the plate; ν is the Poisson coefficient; $N(t)$ is compressing $(N>0)$ or tensile $(N<0)$ forces of the plate; β_2, β_1 are coefficients of internal and external damping; β_0 is stiffness coefficient of the base (bed); $Q_1(y,\tau,t), Q_2(y,\tau,t)$ – measures of relaxation of the plate and base materials.

Compressive (tensile) force $N(t)$ element may depend on time. For example, if a non-stationary heat exposure to the plate the $N(t)$ is as follows:

$$N\left(t\right)=N_0+N_T(t),$$

$$N_T(t)=-\frac{T_0(t)}{1-\nu}, T_0(t)=E\alpha_T\int\limits_{-h/2}^{h/2}T\left(z,t\right)dz,$$

where α_T is the temperature coefficient of the linear expansion, $T(z,t)$ is the law of temperature change over the thickness of the plate, N_0 is the constant component of the force generated when fixing the plate.

The equations (1)-(3) describe the motion of the fluid in areas S_1, S_2, the equation (7) describes the dynamic of a plate; conditions (4)-(6) are conditions of sticking of the viscous fluid.

For a two-stage model of an elastic body the boundary conditions at the ends of the plate at $y=y_0$ and $y=y_*$ can take the form:

1) rigid clamping (Fig. 2a):

$$w(y,t)=w'(y,t)=u(y,t)=0; \tag{9}$$

2) hinge securely fastened (Fig. 2b):

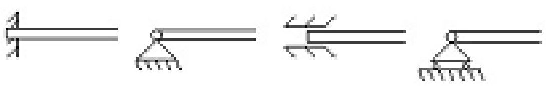
$$w(y,t)=w''(y,t)=u(y,t)=0; \tag{10}$$

3) rigid mobile jamming (Fig. 2c):

$$w(y,t)=w'(y,t)=u'(y,t)=0; \tag{11}$$

4) hinged movable anchorage (Fig. 2d):

$$\begin{aligned} w(y,t)=w''(y,t)=\\ =u'(y,t)+\frac{1}{2}w'^2(y,t)=0. \end{aligned} \tag{12}$$



a) b) c) d)

Figure 2. The method of fixing.

For single-stage model of an elastic body the boundary conditions at the ends of the plate at $y=y_0$ and $y=y_*$ can take the form:

1) rigid fastened:

$$w(y,t)=w'(y,t)=0; \tag{13}$$

2) hinge fastened:

$$w(y,t)=w''(y,t)=u(y,t)=0. \tag{14}$$

We will notice that for the description of the movement of the fluid the nonlinear equations of Navier-Stokes are used, and boundary conditions (6), as well as the right part of the equation (7), are written down in the assumption that deformations of the plate are small.

Given the initial conditions:

$$\begin{aligned} w(y,0)=f_1(y), \quad \dot{w}(y,0)=f_2(y),\\ u(y,0)=f_3(y), \quad \dot{u}(y,0)=f_4(y), \end{aligned} \tag{15}$$

which must agree with the boundary conditions (9)-(12). According to the definitions of the functions $w(y,t)$,$u(y,t)$: $f_1(y)$, $f_2(y)\in C^4[y_0,y_*]$, $f_3(y)$, $f_4(y)\in\in C^2[y_0,y_*]$. The norms in the spaces $C^4[y_0,y_*]$ and $C^2[y_0,y_*]$ are defined by the equalities

$$\|f_i(y)\|=\sup_{0\le m\le 4}\max_{y\in[y_0,y_*]}\left|\frac{d^m f_i(y)}{dy^m}\right|,\ i=1,2,$$

$$\|f_i(y)\|=\sup_{0\le m\le 2}\max_{y\in[y_0,y_*]}\left|\frac{d^m f_i(y)}{dy^m}\right|,\ i=3,4.$$

Given also initial conditions:

$$v_1(x,y,0)=f_5(x,y),\ v_2(x,y,0)=f_6(x,y), \tag{16}$$

which must agree with the boundary conditions (4), (5) and (6). According to the definition of the functions $v_1(x,y,t)$, $v_2(x,y,t)$: $f_5(x,y)$, $f_6(x,y)\in\in C^2\{S_1\bigcup S_2\}$. The norm in the space $C^2\left\{G\right\}$ is defined by the equality

$$\|f_i\|=\sup_{0\le n+m\le 2}\ \max_{(x,y)\in S_1\bigcup S_2}\left|\frac{\partial^{m+n}f_i(x,y)}{\partial x^n\partial y^m}\right|,$$

$i=5,6.$

3 Stability Investigation

3.1 Stability Investigation for the Two-stage Model of an Elastic Body

Definition 3.1. *The solution of the problem* (1)–(7)*,* (9)–(12) *for five unknown functions* $u(y,t) \in C^{2,2}\{[y_0,y_*]\times R^+\}$, $w(y,t) \in C^{4,2}\{[y_0,y_*]\times R^+\}$, $v_1(x,y,t)$, $v_2(x,y,t)$, $P(x,y,t) \in C^{2,2}\{S_1 \bigcup S_2 \times R^+\}$ *is called stable with respect to perturbations of the initial data* (15), (16)*, if for any arbitrarily small positive number* $\delta > 0$ *there exists a number* $\varepsilon = \varepsilon(\delta) > 0$*, such that for any functions* $f_1(y)$, $f_2(y) \in C^4[y_0,y_*]$, $f_3(y)$, $f_4(y) \in C^2[y_0,y_*]$ *and* $f_5(x,y)$, $f_6(x,y) \in C^2\{S_1 \bigcup S_2\}$*, satisfying the boundary conditions and the conditions of the smallness by the norm* $\|f_1(y)\| < \varepsilon$, $\|f_2(y)\| < \varepsilon$,$\|f_3(y)\| < \varepsilon$, $\|f_4(y)\| < \varepsilon$, $\|f_5(x,y)\| < \varepsilon$, $\|f_6(x,y)\| < \varepsilon$*, the inequalities* $|w(y,t)| < \delta$, $|u(y,t)| < \delta$, $y \in [y_0,y_*]$ *and* $|v_1(x,y,t)| < \delta$, $|v_2(x,y,t)| < \delta$, $|P(x,y,t)| < \delta$, $(x,y) \in S_1 \bigcup S_2$ *will hold for any time* $t > 0$.

The similar definitions of stability with respect to perturbations of the initial data can be given separately for the functions themselves $u(y,t)$, $w(y,t)$, $v_1(x,y,t)$, $v_2(x,y,t)$, $P(x,y,t)$ and its partial derivatives.

We introduce the followig notations: λ_1, η_1 are the smallest eigenvalues of the boundary value problems for the equations $\psi'''' = -\lambda\psi''$, $\psi'''' = \eta\psi$, $y \in (y_0,y_*)$ with boundary conditions corresponding (9)-(12).

Theorem 3.1. *Let the conditions*

$$\beta_2\eta_1 + \beta_1 \geq 0, \quad \dot{N}(t) > 0, \tag{17}$$

$$N(t) < \lambda_1 D. \tag{18}$$

Then the solution $w(y,t)$,$v_1(x,y,t)$, $v_2(x,y,t)$ *of the problem* (1)–(7)*,* (9)–(12) *and the derivatives* $\dot{u}(y,t)$, $\dot{w}(y,t)$ *are stable with respect to perturbations of the initial data* $v_1(x,y,0)$, $v_2(x,y,0)$, $\dot{u}(y,0)$, $u'(y,0)$, $\dot{w}(y,0)$, $w'(y,0)$, $w''(y,0)$.

Proof. We will write down the equations (1), (2) as

$$\rho(v_{1t} + v_2 v_{1y} - v_2 v_{2x}) = -\left(P + \tfrac{1}{2}\rho V^2\right)_x + \\ +\mu\Delta v_1, \tag{19}$$

$$\rho(v_{2t} + v_1 v_{2x} - v_1 v_{1y}) = -\left(P + \tfrac{1}{2}\rho V^2\right)_y + \\ +\mu\Delta v_2, \tag{20}$$

where $V^2 = v_1^2 + v_2^2$ is fluid velocity square, Δ is laplacian.

Multiplying the equation (19) by $v_1(x,y,t)$, the equation (20) on $v_2(x,y,t)$, and summing the obtained expressions, accounting for the equation of continuity (3), we will obtain

$$\left(\frac{1}{2}\rho V^2\right)_t = -\left[v_1\left(P + \frac{1}{2}\rho V^2\right)\right]_x - \\ -\left[v_2\left(P + \frac{1}{2}\rho V^2\right)\right]_y + \mu\left[(v_1 v_{1x} + v_2 v_{2x})_x + \right. \\ \left. +(v_1 v_{1y} + v_2 v_{2y})_y - v_{1x}^2 - v_{1y}^2 - v_{2x}^2 - v_{2y}^2\right]. \tag{21}$$

Considering the boundary conditions (9)-(12), we will obtain equalities

$$-\int_{y_0}^{y_*} \dot{w}\left[w'\left(u' + \frac{1}{2}w'^2\right)\right]' dy - \int_{y_0}^{y_*} \dot{u}\left(u' + \frac{1}{2}w'^2\right)' dy = \\ = \int_{y_0}^{y_*} \dot{w}' w'\left(u' + \frac{1}{2}w'^2\right) dy + \int_{y_0}^{y_*} \dot{u}'\left(u' + \frac{1}{2}w'^2\right) dy = \\ = \frac{1}{2}\left(\int_{y_0}^{y_*}\left(u' + \frac{1}{2}w'^2\right)^2 dy\right)_t, \tag{22}$$

$$\int_{y_0}^{y_*} \dot{w}\ddot{w} dy = \frac{1}{2}\left(\int_{y_0}^{y_*} \dot{w}^2 dy\right)_t,$$

$$\int_{y_0}^{y_*} \dot{u}\ddot{u} dy = \frac{1}{2}\left(\int_{y_0}^{y_*} \dot{u}^2 dy\right)_t,$$

$$\int_{y_0}^{y_*} \dot{w} w'''' dy = \int_{y_0}^{y_*} \dot{w}'' w'' dy = \frac{1}{2}\left(\int_{y_0}^{y_*} w''^2 dy\right)_t,$$

$$N(t)\int_{y_0}^{y_*} \dot{w} w'' dy = -N(t)\int_{y_0}^{y_*} \dot{w}' w' dy = \\ = -\frac{1}{2}\left(N(t)\int_{y_0}^{y_*} w'^2 dy\right)_t + \frac{1}{2}\dot{N}(t)\int_{y_0}^{y_*} w'^2 dy,$$

$$\int_{y_0}^{y_*} \dot{w}\dot{w}'''' dy = \int_{y_0}^{y_*} \dot{w}''^2 dy.$$

Multiplying the first equation of the system (7) by $\dot{u}(y,t)$, the second equation of system (7) by $\dot{w}(y,t)$, and summing the received expressions and integrating from y_0 to y_*, accounting for the equalities (22), we will obtain

$$\left(\frac{1}{2}\int_{y_0}^{y_*}\left(EF\left(u'(y,t) + \frac{1}{2}w'^2(y,t)\right)^2 + \right.\right.$$

$$+M(\dot{u}^2(y,t)+\dot{w}^2(y,t))+Dw''^2(y,t)-$$

$$\left.-N(t)w'^2(y,t)+\beta_0 w^2(y,t)\Big)dy\right)_t =$$

$$=-\int_{y_0}^{y_*}\Big(\beta_2\dot{w}''^2(y,t)+\beta_1\dot{w}^2(y,t)+\frac{1}{2}\dot{N}(t)w'^2(y,t)-$$

$$-\left(P_1(a,y,t)-P_2(a,y,t)\right)\dot{w}(y,t)\Big)dy. \qquad (23)$$

Now we consider the following functional

$$J(t)=\frac{1}{2}\iint_S \rho V^2(x,y,t)dS+$$

$$+\frac{1}{2}\int_{y_0}^{y_*}\left(EF\left(u'(y,t)+\frac{1}{2}w'^2(y,t)\right)^2+\right.$$

$$+M\left(\dot{u}^2(y,t)+\dot{w}^2(y,t)\right)+Dw''^2(y,t)- \qquad (24)$$

$$\left.-N(t)w'^2(y,t)+\beta_0 w^2(y,t)\right)dy,$$

where $S=S_1\bigcup S_2$.

For the time-derivative $\frac{\partial J}{\partial t}$ of this functional, using expressions (21), (23) and applying the Green's formula, we find

$$\frac{\partial J}{\partial t}=\oint_{L_1\cup L_0}\left[-v_{11}\left(P_1+\frac{1}{2}\rho V_1^2\right)+\mu(v_{11}v_{11x}+\right.$$

$$\left.+v_{21}v_{21x})\right]dy+\left[v_{21}\left(P_1+\frac{1}{2}\rho V_1^2\right)-\mu(v_{11}v_{11y}+\right.$$

$$\left.+v_{21}v_{21y})\right]dx+\oint_{L_2\cup L_0}\left[-v_{12}\left(P_2+\frac{1}{2}\rho V_2^2\right)+\right.$$

$$\left.+\mu(v_{12}v_{12x}+v_{22}v_{22x})\right]dy+\left[v_{22}\left(P_2+\frac{1}{2}\rho V_2^2\right)-\right.$$

$$\left.-\mu(v_{12}v_{12y}+v_{22}v_{22y})\right]dx-\mu\iint_S\left(v_{1x}^2+v_{1y}^2+\right.$$

$$\left.+v_{2x}^2+v_{2y}^2\right)dS-\int_{y_0}^{y_*}\left(\beta_2\dot{w}''^2(y,t)+\right.$$

$$+\beta_1\dot{w}^2(y,t)+\frac{1}{2}\dot{N}(t)w'^2(y,t)-$$

$$\left.-\left(P_1(a,y,t)-P_2(a,y,t)\right)\dot{w}(y,t)\right)dy. \qquad (25)$$

Considering the boundary conditions (4)-(6) and the equations (7), we have

$$\frac{\partial J}{\partial t}=\int_{y_0}^{y_*}\left[-v_{11}\left(P_1+\frac{1}{2}\rho v_{11}^2\right)\right]dy+$$

$$+\int_{y_*}^{y_0}\left[-v_{12}\left(P_2+\frac{1}{2}\rho v_{12}^2\right)\right]dy-\mu\iint_S\left(v_{1x}^2+\right.$$

$$\left.+v_{1y}^2+v_{2x}^2+v_{2y}^2\right)dS-\int_{y_0}^{y_*}\left(\beta_2\dot{w}''^2(y,t)+\right. \qquad (26)$$

$$+\beta_1\dot{w}^2(y,t)+\frac{1}{2}\dot{N}(t)w'^2(y,t)-(P_1(a,y,t)-$$

$$-P_2(a,y,t))\,\dot{w}(y,t))\,dy=-\int_{y_0}^{y_*}\Big(\beta_2\dot{w}''^2(y,t)+$$

$$+\beta_1\dot{w}^2(y,t)+\frac{1}{2}\dot{N}(t)w'^2(y,t)\Big)dy-$$

$$-\mu\iint_S\left(v_{1x}^2+v_{1y}^2+v_{2x}^2+v_{2y}^2\right)dS.$$

Consider the boundary value problems for the equations $\psi''''=-\lambda\psi''$, $\psi''''=\eta\psi$, $y\in(y_0,y_*)$ with boundary conditions (9)–(12) for the function $w(y,t)$. These problems are self-adjoint and completely defined. For the function $w(y,t)$, using Rayleigh's inequality [Kollatz, 1968], we obtain the estimates

$$\int_{y_0}^{y_*}w(y,t)w''''(y,t)dy\geq-\lambda_1\int_{y_0}^{y_*}w(y,t)w''(y,t)dy,$$

$$\int_{y_0}^{y_*}w(y,t)w''''(y,t)dy\geq\eta_1\int_{y_0}^{y_*}w(y,t)w(y,t)dy.$$

Integrating by parts and taking into account boundary conditions (9)-(12), we will obtain inequalities

$$\int_{y_0}^{y_*}w''^2(y,t)dy\geq\lambda_1\int_{y_0}^{y_*}w'^2(y,t)dy,$$
$$\int_{y_0}^{y_*}w''^2(y,t)dy\geq\eta_1\int_{y_0}^{y_*}w^2(y,t)dy. \qquad (27)$$

Similarly, considering the boundary value problems for the equation $\psi''''=-\lambda\psi''$, $y\in(y_0,y_*)$ with boundary conditions (9)-(12) for the function $\dot{w}(y,t)$ we obtain the estimate

$$\int_{y_0}^{y_*} \dot{w''}^2(y,t)dy \geq \eta_1 \int_{y_0}^{y_*} \dot{w}^2(y,t)dy. \qquad (28)$$

Using the inequality (28) and (26), we obtain

$$\frac{\partial J}{\partial t} \leq -\mu \iint_S \left(v_{1x}^2 + v_{1y}^2 + v_{2x}^2 + v_{2y}^2\right) dS- \qquad (29)$$

$$-\int_{y_0}^{y_*} \left((\beta_2\eta_1 + \beta_1)\dot{w}^2(y,t) - \frac{1}{2}\dot{N}(t)w'^2(y,t) \right) dy.$$

Under conditions (15) from (29) it follows, that $\frac{\partial J}{\partial t} \leq 0$. Integrating from 0 to *t*, we obtain the inequality

$$J(t) \leq J(0). \qquad (30)$$

We make the evaluations of the functional with the boundary conditions (9)-(12). Using the inequalities (27), we obtain the upper bound on $J(0)$:

$$J(0) = \frac{1}{2} \iint_S \rho \left(v_{10}^2 + v_{20}^2\right) dS+$$

$$+\frac{1}{2} \int_{y_0}^{y_*} \left(M(\dot{u}_0^2 + \dot{w}_0^2) + EF\left(u_0' + \frac{1}{2}{w_0'}^2\right)^2 + \right. \qquad (31)$$

$$\left. + \left(D + \frac{|N(0)|}{\lambda_1} + \frac{\beta_0}{\eta_1}\right){w_0''}^2\right) dy.$$

Here are introduced the notations $v_{10} = v_1(x,y,0)$, $v_{20} = v_2(x,y,0)$, $\dot{u}_0 = \dot{u}(x,0)$, $u_0' = u'(x,0)$, $\dot{w}_0 = \dot{w}(x,0)$, $w_0' = w'(x,0)$, $w_0'' = w''(x,0)$.

Using the first inequality (27), we obtain the lower bound for $J(t)$:

$$J(t) \geq \frac{1}{2} \iint_S \rho V^2 dS + \frac{1}{2} \int_{y_0}^{y_*} \left(M(\dot{u}^2(y,t)+\right.$$

$$\left. +\dot{w}^2(y,t)) + (\lambda_1 D - N(t))w'^2(y,t)\right) dy. \qquad (32)$$

Using the Cauchy-Bunyakovsky inequality at the boundary conditions (9)-(12), we obtain the estimate

$$w^2(y,t) \leq (y_* - y_0) \int_{y_0}^{y_*} w'^2(y,t)dy. \qquad (33)$$

Under conditions (18) the inequality (32) takes the form

$$J(t) \geq \frac{1}{2} \iint_S \rho \left(v_1^2 + v_2^2\right) dS + \frac{1}{2} \int_{y_0}^{y_*} M(\dot{u}^2(y,t)+$$

$$+\dot{w}^2(y,t))dy + \frac{\lambda_1 D - N(t)}{2(y_* - y_0)} w^2(y,t). \qquad (34)$$

Thus, taking into account (30), (31), (34), we obtain the inequality

$$\iint_S \rho \left(v_1^2 + v_2^2\right) dS + \int_{y_0}^{y_*} M(\dot{u}^2(y,t) + \dot{w}^2(y,t))dy+$$

$$+\frac{\lambda_1 D - N(t)}{y_* - y_0} w^2(y,t) \leq \iint_S \rho \left(v_{10}^2 + v_{20}^2\right) dS+$$

$$+\int_{y_0}^{y_*} \left(M(\dot{u}_0^2 + \dot{w}_0^2) + EF\left(u_0' + \frac{1}{2}{w_0'}^2\right)^2 + \right.$$

$$\left. + \left(D + \frac{|N(0)|}{\lambda_1} + \frac{\beta_0}{\mu_1}\right){w_0''}^2\right) dy. \qquad (35)$$

From inequalities $\|f_i(y)\| < \varepsilon$, $i = \overline{1,4}$, $\|f_i(x,y)\| < \varepsilon$, $i = 5,6$ it follows that $|v_{10}| < \varepsilon$, $|v_{20}| < \varepsilon$, $|\dot{u}_0| < \varepsilon$, $|u_0'| < \varepsilon$, $|\dot{w}_0| < \varepsilon$, $|w_0'| < \varepsilon$, $|w_0''| < \varepsilon$. And as $u(y,t) \in C^{2,2}\{[y_0,y_*] \times R^+\}$, $w(y,t) \in C^{4,2}\{[y_0,y_*] \times R^+\}$, $v_1(x,y,t), v_2(x,y,t) \in C^{2,2}\{S_1 \bigcup S_2 \times R^+\}$, then (35) implies, that for any arbitrarily small positive number $\delta > 0$ there exists a number $\varepsilon = \varepsilon(\delta) > 0$, such that the inequalities $|w(y,t)| < \delta$, $|\dot{w}(y,t)| < \delta$, $|\dot{u}(y,t)| < \delta$, $y \in [y_0,y_*]$ and $|v_1(x,y,t)| < \delta$, $|v_2(x,y,t)| < \delta$, $(x,y) \in S_1 \bigcup S_2$ will hold for any time $t > 0$. Hence, according to Definition 3.1, the theorem is proved.

On the basis of the inequality (35) it is possible to receive an assessment of the amplitude of the greatest possible fluctuations of an elastic plate at any time point:

$$|w(y,t)| \leq \sqrt{\frac{y_* - y_0}{\lambda_1 D - N(t)} J(0)}.$$

3.2 Stability Investigation for the Single-stage Model of an Viscoelastic Body

Definition 3.2. *The solution of the problem* (1)–(6), (8), (13)-(14) *for four unknown functions* $w(y,t) \in C^{4,2}\{[y_0,y_*] \times R^+\}$, $v_1(x,y,t)$, $v_2(x,y,t)$, $P(x,y,t) \in C^{2,2}\{S_1 \bigcup S_2 \times R^+\}$ *is called stable with respect to perturbations of the initial* *data* (15), (16), *if for any arbitrarily small positive number* $\delta > 0$ *there exists a number* $\varepsilon = \varepsilon(\delta) > 0$, *such that for any functions* $f_1(y),\ f_2(y) \in C^4[y_0, y_*]$ *and* $f_5(x,y),\ f_6(x,y) \in C^2\{S_1 \bigcup S_2\}$, *satisfying the boundary conditions and the conditions of the smallness by the norm* $\|f_1(y)\| < \varepsilon,\ \|f_2(y)\| < \varepsilon, \|f_5(x,y)\| < \varepsilon,\ \|f_6(x,y)\| < \varepsilon$, *the inequalities* $|w(y,t)| < \delta,\ y \in [y_0, y_*]$ *and* $|v_1(x,y,t)| < \delta$, $|v_2(x,y,t)| < \delta$, $|P(x,y,t)| < \delta$, $(x,y) \in S_1 \bigcup S_2$ *will hold for any time* $t > 0$.

Theorem 3.2. *Let for all* y, t, τ *the conditions*

$$\beta_2\eta_1 + \beta_1 \geq 0, \quad \dot{N}(t) > 0, \tag{36}$$

$$N(t) < \lambda_1 D \inf_y \left(1 + Q_1(y,0,t)\right), \tag{37}$$

$$\begin{gathered} Q_i(y,t,t)=0, \frac{\partial Q_i}{\partial t}(y,0,t) \leq 0, \frac{\partial Q_i}{\partial \tau}(y,\tau,t) \geq 0, \\ \frac{\partial^2 Q_i}{\partial \tau \partial t}(y,\tau,t) \leq 0, 1+Q_i(y,0,t)>0, (i=1,2). \end{gathered} \tag{38}$$

Then the solution $w(y,t),\ v_1(x,y,t),\ v_2(x,y,t)$ *of the problem* (1)–(6), (8), (13)-(14) *and the derivative* $\dot{w}(y,t)$ *are stable with respect to perturbations of the initial data* $v_1(x,y,0),\ v_2(x,y,0),\ \dot{w}(y,0), w'(y,0), w''(y,0)$.

Proof. We note that the inequalities (27), (28), (33) are valid under the boundary conditions (13) - (14).

Considering the conditions (13) - (14), (38) we will obtain equalities

$$2\int_{y_0}^{y_*} \dot{w} \left[w''(y,t) - \int_0^t \frac{\partial Q_1(y,\tau,t)}{\partial \tau} w''(y,\tau) d\tau \right]'' dy =$$

$$= 2\int_{y_0}^{y_*} \dot{w}'' \left[w''(y,t) - \int_0^t \frac{\partial Q_1(y,\tau,t)}{\partial \tau} w''(y,\tau) d\tau \right] dy =$$

$$= \Bigg(\int_{y_0}^{y_*} \Big(w''^2(y,t)\left(1 + Q_1(y,0,t)\right) +$$

$$+ \int_0^t \frac{\partial Q_1}{\partial \tau}(y,\tau,t)\left(w''(y,t) - w''(y,\tau)\right)^2 d\tau \Big) dy \Bigg)_t +$$

$$+ \int_{y_0}^{y_*} \frac{\partial Q_1}{\partial \tau}(y,t,t) w''^2(y,t) dy -$$

$$- \int_{y_0}^{y_*} \left(\int_0^t \frac{\partial^2 Q_1}{\partial \tau \partial t}(y,\tau,t)\left(w''(y,t) - w''(y,\tau)\right)^2 d\tau \right) dy +$$

$$+ \int_{y_0}^{y_*} w''^2(y,t) \left(\frac{\partial Q_1}{\partial t}(y,t,t) - \frac{\partial Q_1}{\partial t}(y,0,t) \right) dy,$$

$$2\int_{y_0}^{y_*} \dot{w} \left[w(y,t) - \int_0^t \frac{\partial Q_2(y,\tau,t)}{\partial \tau} w(y,\tau) d\tau \right] dy =$$

$$= \Bigg(\int_{y_0}^{y_*} \Bigg(\int_0^t \frac{\partial Q_2}{\partial \tau}(y,\tau,t)\left(w(y,t) - w(y,\tau)\right)^2 d\tau +$$

$$+ w^2(y,t)\left(1 + Q_2(y,0,t)\right) \Bigg) dy \Bigg)_t + \tag{39}$$

$$+ \int_{y_0}^{y_*} \frac{\partial Q_2}{\partial \tau}(y,t,t) w^2(y,t) dy -$$

$$- \int_{y_0}^{y_*} \left(\int_0^t \frac{\partial^2 Q_2}{\partial \tau \partial t}(y,\tau,t)\left(w(y,t) - w(y,\tau)\right)^2 d\tau \right) dy +$$

$$+ \int_{y_0}^{y_*} w^2(y,t) \left(\frac{\partial Q_2}{\partial t}(y,t,t) - \frac{\partial Q_2}{\partial t}(y,0,t) \right) dy.$$

Multiplying the equation (8) by $\dot{w}(y,t)$ and integrating from y_0 to y_*, accounting for the equalities (39), we will obtain

$$\Bigg(\frac{1}{2} \int_{y_0}^{y_*} \Big(M\dot{w}^2(y,t) + D\left(1 + Q_1(y,0,t)\right) w''^2(y,t) +$$

$$+ D \int_0^t \frac{\partial Q_1}{\partial \tau}(y,\tau,t)\left(w''(y,t) - w''(y,\tau)\right)^2 d\tau +$$

$$+ \beta_0 \left(1 + Q_2(y,0,t)\right) w^2(y,t) - N(t) w'^2(y,t) +$$

$$+ \beta_0 \int_0^t \frac{\partial Q_2}{\partial \tau}(y,\tau,t)\left(w(y,t) - w(y,\tau)\right)^2 d\tau \Big) dy \Bigg)_t =$$

$$= - \int_{y_0}^{y_*} \Big(\frac{1}{2}\dot{N}(t) w'^2(y,t) - D\frac{\partial Q_1}{\partial \tau}(y,t,t) w''^2(y,t) +$$

$$+ D \int_0^t \frac{\partial^2 Q_1}{\partial \tau \partial t}(y,\tau,t)\left(w''(y,t) - w''(y,\tau)\right)^2 d\tau -$$

$$- D w''^2(y,t) \left(\frac{\partial Q_1}{\partial t}(y,t,t) - \frac{\partial Q_1}{\partial t}(y,0,t) \right) +$$

$$+ \beta_2 \dot{w}''^2(y,t) + \beta_1 \dot{w}^2(y,t) - \beta_0 \frac{\partial Q_2}{\partial \tau}(y,t,t) w^2(y,t) +$$

$$+\beta_0 \int_0^t \frac{\partial^2 Q_2}{\partial\tau\partial t}(y,\tau,t)\left(w(y,t)-w(y,\tau)\right)^2 d\tau-$$

$$-\beta_0 w^2(y,t)\left(\frac{\partial Q_2}{\partial t}(y,t,t)-\frac{\partial Q_2}{\partial t}(y,0,t)\right)-$$

$$-\left(P_1(a,y,t)-P_2(a,y,t)\right)\dot{w}(y,t)\Big)dy. \quad (40)$$

Now we consider the following functional

$$\Phi(t)=\frac{1}{2}\iint_S \rho V^2(x,y,t)dS+$$

$$+\frac{1}{2}\int_{y_0}^{y_*}\Big(M\dot{w}^2(y,t)+D\left(1+Q_1(y,0,t)\right)w''^2(y,t)+$$

$$+D\int_0^t \frac{\partial Q_1}{\partial\tau}(y,\tau,t)\left(w''(y,t)-w''(y,\tau)\right)^2 d\tau+$$

$$+\beta_0\int_0^t \frac{\partial Q_2}{\partial\tau}(y,\tau,t)\left(w(y,t)-w(y,\tau)\right)^2 d\tau- \quad (41)$$

$$-N(t)w'^2(y,t)+\beta_0\left(1+Q_2(y,0,t)\right)w^2(y,t)\Big)dy.$$

For the time-derivative $\frac{\partial\Phi}{\partial t}$ of this functional, using expressions (21), (40) and inequality (28), similarly (29), we obtain

$$\frac{\partial\Phi}{\partial t}\le-\int_{y_0}^{y_*}\Big((\beta_2\eta_1+\beta_1)\dot{w}^2(y,t)+\frac{1}{2}\dot{N}(t)w'^2(y,t)+$$

$$+D\frac{\partial Q_1}{\partial\tau}(y,t,t)w''^2(y,t)+$$

$$+D\int_0^t \frac{\partial^2 Q_1}{\partial\tau\partial t}(y,\tau,t)\left(w''(y,t)-w''(y,\tau)\right)^2 d\tau-$$

$$-Dw''^2(y,t)\left(\frac{\partial Q_1}{\partial t}(y,t,t)-\frac{\partial Q_1}{\partial t}(y,0,t)\right)-$$

$$-\beta_0\frac{\partial Q_2}{\partial\tau}(y,t,t)w^2(y,t)+$$

$$+\beta_0\int_0^t \frac{\partial^2 Q_2}{\partial\tau\partial t}(y,\tau,t)\left(w(y,t)-w(y,\tau)\right)^2 d\tau-$$

$$-\beta_0 w^2(y,t)\left(\frac{\partial Q_2}{\partial t}(y,t,t)-\frac{\partial Q_2}{\partial t}(y,0,t)\right)\Big)dy-$$

$$-\mu\iint_S\left(v_{1x}^2+v_{1y}^2+v_{2x}^2+v_{2y}^2\right)dS. \quad (42)$$

Under conditions (36), (38) from (42) it follows, that $\frac{\partial\Phi}{\partial t}\le 0$. Integrating from 0 to *t*, we obtain the inequality

$$\Phi(t)\le\Phi(0). \quad (43)$$

We make the evaluations of the functional with the boundary conditions (13)–(14). Using the inequalities (30), we obtain the upper bound on $\Phi(0)$:

$$\Phi(0)=\frac{1}{2}\iint_S \rho\left(v_{10}^2+v_{20}^2\right)dS+$$
$$+\frac{1}{2}\int_{y_0}^{y_*}\left(M\dot{w}_0^2+\left(D+\frac{|N(0)|}{\lambda_1}+\frac{\beta_0}{\eta_1}\right){w_0''}^2\right)dy. \quad (44)$$

Using the first inequality (27) and inequalities (38), we obtain the lower bound for $\Phi(t)$:

$$\Phi(t)\ge\frac{1}{2}\iint_S \rho V^2 dS+\frac{1}{2}\int_{y_0}^{y_*}\Big(M\dot{w}^2(y,t)+ \quad (45)$$

$$+\left(\lambda_1 D\inf_y\left(1+Q_1(y,0,t)\right)-N(t)\right)w'^2(y,t)\Big)dy.$$

Using the Cauchy-Bunyakovsky inequality (33) under condition (37) the inequality (45) takes the form

$$\Phi(t)\ge\frac{1}{2}\iint_S \rho\left(v_1^2+v_2^2\right)dS+\frac{1}{2}\int_{y_0}^{y_*}M\dot{w}^2(y,t)dy+$$

$$+\frac{\lambda_1 D\inf_y\left(1+Q_1(y,0,t)\right)-N(t)}{2(y_*-y_0)}w^2(y,t). \quad (46)$$

Thus, taking into account (43), (44), (46), we obtain the inequality

$$\iint_S \rho\left(v_1^2+v_2^2\right)dS+\int_{y_0}^{y_*}M\dot{w}^2(y,t)dy+$$

$$+\frac{\lambda_1 D\inf_y\left(1+Q_1(y,0,t)\right)-N(t)}{2(y_*-y_0)}w^2(y,t)\le$$

$$\le\iint_S \rho\left(v_{10}^2+v_{20}^2\right)dS+ \quad (47)$$

$$+\int\limits_{y_0}^{y_*}\left(M\dot{w}_0^2+\left(D+\frac{|N(0)|}{\lambda_1}+\frac{\beta_0}{\eta_1}\right){w_0''}^2\right)dy.$$

From inequalities $\|f_i(y)\| < \varepsilon$, $i = 1,2$, $\|f_i(x,y)\| < \varepsilon$, $i = 5,6$ it follows that $|v_{10}| < \varepsilon$, $|v_{20}| < \varepsilon$, $|\dot{w}_0| < \varepsilon$, $|w_0'| < \varepsilon$, $|w_0''| < \varepsilon$. And as $w(y,t) \in C^{4,2}\{[y_0,y_*]\times R^+\}$, $v_1(x,y,t)$, $v_2(x,y,t) \in C^{2,2}\{S_1\bigcup S_2\times R^+\}$, then (47) implies, that for any arbitrarily small positive number $\delta > 0$ there exists a number $\varepsilon = \varepsilon(\delta) > 0$, such that the inequalities $|w(y,t)| < \delta$, $|\dot{w}(y,t)| < \delta$, $y \in [y_0,y_*]$ and $|v_1(x,y,t)| < \delta$, $|v_2(x,y,t)| < \delta$, $(x,y) \in S_1\bigcup S_2$ will hold for any time $t > 0$. Hence, according to Definition 3.2, the theorem is proved.

Remark. For example, the kernels of the form $Q_i(x,s,t) = a_i\left(e^{s-t}-1\right)$ satisfy the conditions (38), where $0 < a_i \le 1$ are some positive parameters.

4 Conclusion

In this work to study the dynamics of a deformable body proposed two models. We constructed the two relevant functionals of Lyapunov to investgate the dynamic stability of an elastic plate contacting a fluctuating viscous incompressible fluid. Accounting for the aging of the plate material in the second model led to more stringent constraints on the parameters of the system (on the flexural stiffness of the plate D and on the compressing forces N).

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References

Ageev, R.V., Kuznetsova, E.L., Kulikov, N.I., Mogilevich, L.I., and Popov, V.S. (2014). Mathematical model of movement of a pulsing layer of viscous liquid in the channel with an elastic wall. *PNRPU Mechanics Bulletin* (Perm National Research Polytechnic University), **2014**(3), pp. 17–35 (In Russian).

Ankilov, A.V. and Velmisov, P.A. (2013). *Mathematical modeling in problems of dynamic stability of deformable elements of constructions at aerohydrodynamic influence*. Ulyanovsk State Technical University. Ulyanovsk (In Russian).

Ankilov, A.V. and Velmisov, P.A. (2015). *Lyapunov functionals in some problems of dynamic stability of aeroelastic constructions*. Ulyanovsk State Technical University. Ulyanovsk (In Russian).

Ankilov, A.V. and Velmisov, P.A. (2016). Stability of solutions to an aerohydroelasticity problem. *Journal of Mathematical Sciences (United States)*, **219**(1), pp. 14–26.

Ankilov, A.V. and Velmisov, P.A. (2016). Stability of solutions of initial-boundary value problem of aerohydroelasticity. *Modern mathematics. Fundamental directions*. **59**, pp. 35–52 (In Russian).

Kheiri, M. and Paidoussis, M.P. (2015). Dynamics and stability of a flexible pinned-free cylinder in axial flow. *Journal of Fluids and Structures*, **55**, pp. 204–217.

Kollatz, L. (1968). *Problems on eigenvalues*. Science. Moscow (In Russian).

Kontzialis, K., Moditis, K., and Paidoussis, M.P. (2017). Transient simulations of the fluid-structure interaction response of a partially confined pipe under axial flows in opposite directions. *Journal of Pressure Vessel Technology, Transactions of the ASME*, **139**(3).

Moditis, K., Paidoussis, M., and Ratigan, J. (2016). Dynamics of a partially confined, discharging, cantilever pipe with reverse external flow. *Journal of Fluids and Structures*, **63**, pp. 120–139.

Mogilevich, L.I., Popov, V.S., Popova, A.A., and Christoforova, A.V. (2016). Mathematical modeling of hydroelastic walls oscillations of the channel on Winkler foundation under vibrations. *Vibroengineering Procedia*, **8**, pp. 294–299.

Mogilevich, L.I., Popov, V.S., Rabinsky, L.N., and Kuznetsova, E.L. (2016). Mathematical model of the plate on elastic foundation interacting with pulsating viscous liquid layer. *Applied Mathematical Sciences*, **10**, pp. 1101–1109.

Naumova, N., Ivanov, D., Voloshinova, T., and Ershov, B. (2015). Mathematical modelling of cylindrical shell vibrations under internal pressure of fluid flow. *International Conference on Mechanics - Seventh Polyakhov's Reading: Saint Petersburg State University*.

Sokolov, V.G. and Razov, I.O. (2014). Parametrical vibrations and dynamic stability of long-distance gas pipelines at above-ground laying. *Bulletin of Civil Engineers*, **2**(43), pp. 65–68 (In Russian).

Velmisov, P.A., Ankilov, A.V., and Semenova, E.P. (2016). Dynamic stability of deformable elements of one class of aeroelastic constructions. *Applications of Mathematics in Engineering and Economics (AMEE'16), AIP Conference Proceedings*, **1789**, edited by V. Pasheva, G. Venkov, N. Popivanov, pp. 1–12.

Zvyagin, A.V. and Gur'ev, K.P. (2017). A fluid-saturated porous medium under the action of a moving concentrated load. *Moscow University Mechanics Bulletin*, **72**(2), pp. 34–39 (In Russian).