NEAR-COLLISION SOLUTIONS ON NON-NEWTONIAN CENTRAL FORCE PROBLEM

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Abstract
Central-force problems appear in various phenomena in science. This paper aims to study the near-collision solutions in Non-Newtonian central force.
Through a judicious change of variables we remove the singularity presented in the cases in which the solutions approach the origin. Through that, a fictitious flow about the singularity is created which allows the understanding of the near-collision dynamics.

Key words
Central-force, collision, singularity.

1 Introduction

Several systems in physics are described by a vector field variant in time and or space. Let us take a particle moving in a force field $\vec{F}$.

Mathematically, $\vec{F}$ is a vector belonging to $\mathbb{R}^n$. In a physical view, $\vec{F}(\vec{X})$ can represent the force exerted on a particle located at $\vec{X} \in \mathbb{R}^n$

The Newton's second law stablishes the link between the physical and the mathematical concepts of force field. It asserts that, at any instant, a particle inserted in such force field moves in a such way that the force vector $\vec{F}$ in a location $\vec{X}$ of the particle is equal to the aceleration vector of the particle times his mass m, i.e. $\vec{F} = m\vec{X}''$.

This system of n secont order differential equations may be writen as a system of $2n$ 1st-order differential equations in $\mathbb{R}^n \times \mathbb{R}^n$ such

$$\vec{X}'' = \vec{V} \tag{1}$$

$$\vec{V}' = \frac{1}{m}\vec{F}(\vec{X}) \tag{2}$$

where $\vec{V}$ is the velocity of the particle.

The solution $\vec{X}(t) \subset \mathbb{R}^n$ of 2nd-order system lies in a configuration space while the solution $(\vec{X}(t), \vec{X}(t)) \subset \mathbb{R}^n \times \mathbb{R}^n$ lies in the phase space of the system.

2 Central Force Systems

A force field $\vec{F}$ is said to be a central type if $\vec{F}(\vec{X})$ is a vector field points away or toward to every $\vec{X}$. Particulaly, if the vector $\vec{F}(\vec{X})$ is a scalar multiple of $\vec{X}$,

$$\vec{F}(\vec{X}) = \lambda(\vec{X})\vec{X} \tag{3}$$

where the coefficient $\lambda(\vec{X})$ depends on $\vec{X}$.

A central force model is often used to describes several phenomena in real world like a motion of an object orbiting around a massive body according to the *Newton's law of gravitation*, or the interaction between charged particles describer by the *Coulomb law*.

Since the force is conservative, $\vec{F}$ can be writen as a gradient of a potential scalar function $U(\vec{X})$, such that

$$\vec{F}(\vec{X}) = -\nabla U(\vec{X}) \tag{4}$$

In the special case known as a Newtonian central force system, the potential is of type

$$U(\vec{X}) = -\frac{1}{|\vec{X}|}$$

and the force varies with the inverse of the square distance.

3 Non-Newtonian central force problem

However, in some systems, such as those classical treated in [Buckingham, 1938], [Lennard-Jones, 1931], [Nicholson, 1989] and in many others related to Schrdinger's equation described in [Ikhdair and Sever, 2007], the scalar function varies in different ways from Newtonian cases.

Our focus is to discuss the classical cases of a Non-Newtonian central force problem. In those cases, the potencial scalar function, which may represents an potential gravitational energy, is given by

$$U(\vec{X}) = \frac{1}{|\vec{X}|^{\nu}} \tag{5}$$

where $\nu > 1$.
In this case, the force

$$\vec{F} = \frac{\nu}{|\vec{X}|^{(\nu+1)}}\hat{X} \tag{6}$$

is not defined in $\vec{X} = \vec{0}$; indeed the force becomes infinite as the moving particle approaches collision with the massive body.
Although the solutions in $\vec{X} = \vec{0}$ may have not much importance in the real systems, it is very interesting to elucidate how solutions behave near to singularity. Our objective here is to remove this singularity by a judicious change of variables and time scalings.
Let us restrict our analysis to particles moving in plane ($\vec{X} = (x, y) \in \mathbb{R}^2$), such that position can be writen in polar coordinates (Fig. 1) as

$$(x, y) = (\cos\theta, \sin\theta)$$

and the velocity has components

$$(x', y') = (r'\cos\theta - r\theta'\sin\theta, r'\sin\theta + r\theta'\cos\theta).$$

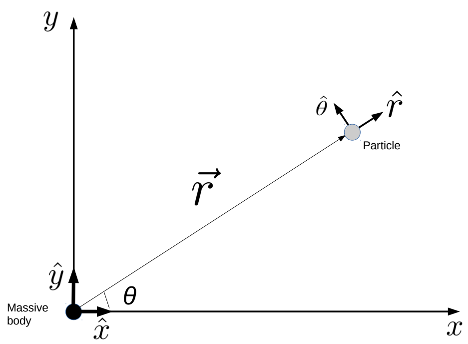

Figure 1. Reference system for Central force problem

The solutions belong to configuration space $\mathbb{C} = \mathbb{R}^2 - 0$. Therefore, the phase space is denoted by $\mathbb{P} = (\mathbb{R}^2 - 0) \times \mathbb{R}^2$.
This phase space can be seen as a colection of all tangent vectors at each point $\vec{X} \in \mathbb{C}$. For a given $\vec{X} \in \mathbb{R}^2 - 0$, let $\mathbb{T}_X = \{(\vec{X}, \vec{Y}) | \vec{V} \in \mathbb{R}^2\}$. $\mathbb{T}_X$ is the tangent plane on the configuration space at $\vec{X}$.

Hence, $\mathbb{P} = \bigcup_{\vec{X}\in\mathbb{C}} \mathbb{T}_X$ is the tangent space of the phase space.
We may introduce new variables (v_r, v_θ)in the tangent space as

$$\vec{V} = v_r \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} + v_\theta \begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix} \tag{7}$$

where $v_r = r'$ and $v_\theta = r\theta'$.
Differentiating once more, we have

$$\vec{V}' = \frac{\nu}{|\vec{X}|^{(\nu+1)}} \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$$

$$= \left(v_r' - \frac{v_\theta^2}{2}\right) \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} + \left(\frac{v_r v_\theta}{r} + v_\theta'\right) \begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix} \tag{8}$$

Rewrinting the system in terms of the new coordinates

$$\begin{cases} r' = v_r \\ \theta' = \frac{v_\theta}{r} \\ v_r' = \frac{\nu}{r^{(\nu+1)}} + \frac{v_\theta^2}{r} \\ v_\theta' = -\frac{v_r v_\theta}{r} \end{cases} \tag{9}$$

Both the mechanical energy and angular momentum are integrals of motion, given respectively by

$$E = \frac{1}{2}(v_r^2 + v_\theta^2) + \frac{1}{r^\nu} \tag{10}$$

$$\vec{h} = r^2\theta'(\hat{x} \times \hat{y}) \tag{11}$$

4 Removing the singularity

In a similar way to what was done in [McGehee, 1974] we "blow up" the singularity via the following change of variables.

$$u_r = r^{\nu/2} v_r \tag{12}$$

$$u_\theta = r^{\nu/2} v_\theta \tag{13}$$

so that the system of equations becomes

$$\begin{cases} r' = r^{\nu/2}u_r \\ \theta' = r^{-\left(\frac{\nu}{2}+1\right)}u_\theta \\ u_r' = r^{-\left(\frac{\nu}{2}+1\right)}\left[\frac{\nu u_r^2}{2} + u_\theta^2 + \nu\right] \\ u_\theta' = r^{-\left(\frac{\nu}{2}+1\right)}u_r u_\theta \left[\frac{\nu}{2} - 1\right] \end{cases} \tag{14}$$

If we multiply this equations by $r^{\left(\frac{\nu}{2}+1\right)}$ we will remove the singularity that exists when $r \to 0$. In that case, the solution curve of the systems remain the same but are parameterized differently.
Strictly, we introduce a new time variable τ such such that

$$\frac{dt}{d\tau} = r^{\left(\frac{\nu}{2}+1\right)} \tag{15}$$

In this new timescale the system becomes

$$\dot{r} = r^{\nu+1}u_r \tag{16}$$

$$\dot{\theta} = u_\theta \tag{17}$$

$$\dot{u_r} = \left[\frac{\nu u_r^2}{2} + u_\theta^2 + \nu\right] \tag{18}$$

$$\dot{u_\theta} = u_r u_\theta \left[\frac{\nu}{2} - 1\right] \tag{19}$$

where the dot indicates differentiation with respect to τ.
As can be seen, when r is small, the variation of τ with respect to t is close to zero. Therefore, the scaled time τ moves slowly then t near to the origin. In summary, the change of variables removes the singularity and allows to maintain the continuity of the solution.

5 Collision Solutions Analysis

In a precise manner, the singularity was replaced by a new set (well defined) given by $r = 0$ and θ, with u_r and u_θ being arbitrary. Naturally, the set $r = 0$ is an invariant set for the flow, provided that $\dot{r} = 0$ when $r = 0$.
Since, in terms of new variables, the total energy relation is

$$\frac{1}{2}(v_r^2 + v_\theta^2) + 1 = hr^\nu \tag{20}$$

in $r = 0$ we have a subset Λ, called *collision surface* and defined by

$$u_r^2 + u_\theta^2 = -2 \tag{21}$$

with θ arbitrary, in which the solution behave informs how the movement occurs in singularity's neighborhood.
This collision surface can be seen as a two-dimentional torus (Fig. 2), formed by a circle in θ direction and another circle in the $u_r u_\theta$ plane.

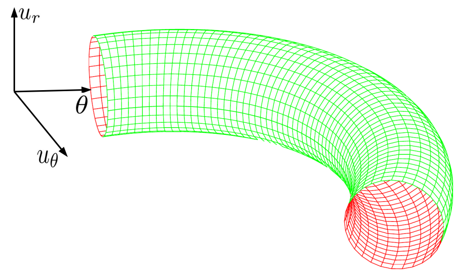

Figure 2. Collision surface Λ

In this subspace, the system reduces to

$$\dot{\theta} = u_\theta \tag{22}$$

$$\dot{u_r} = u_\theta^2 \left[1 - \frac{\nu}{2}\right] \tag{23}$$

$$\dot{u_\theta} = u_r u_\theta \left[\frac{\nu}{2} - 1\right] \tag{24}$$

This system can be analyzed as follow:

– For $\nu < 2$ (provided $u_\theta \neq 0$) the coordinate u_r increases along any solution in Λ, as shown in Figure 3 for $\nu = 1.99$ case.

– For $\nu = 2$, there is a bifurcation, in which u_r has no slope.

– For $\nu > 2$, the coordinate u_r decreases along any solution in Λ (since $u_\theta \neq 0$), as shown in Figure 4 for $\nu = 2.01$ case.

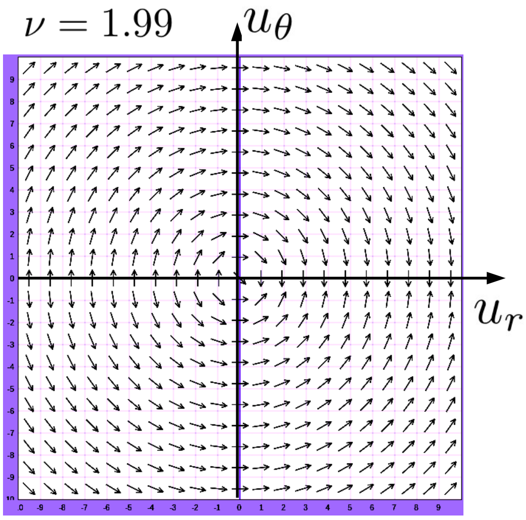

Figure 3. Phase plane for $\nu = 1.99$

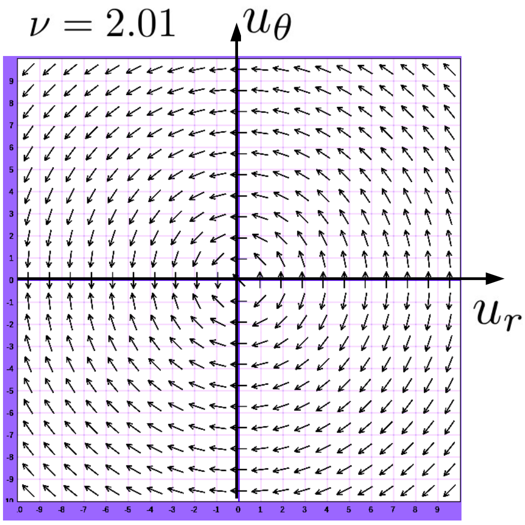

Figure 4. Phase plane for $\nu = 2.01$

6 Final Remarks

This paper deals with an analytical way to remove the singularity existing in near collsion solutions on Non-Newtonian potentials. Through a suitable change of variables, the system is parameterized and the previous divergent solution was replaced by a "*smooth solution*" when the system goes close to the singularity.

This type of artifice is extremely useful, especially in computational treatment of scatteting problems, in which the analysis of dynamics close to singularity would produce an overflow, allowing analysis of configuration and phase spaces, even when the particle approaches the origin.

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