Variational Formulation for the Optimal Control Problems of Elastic Body Motions

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An approach to modeling and optimization of controlled dynamical systems with distributed elastic and inertial parameters is considered. The general method of integrodifferential relations (MIDR) for solving a wide class of initial-boundary value problems is developed and criteria of solution quality are proposed [1, 2]. A numerical algorithm for discrete approximation of controlled motions has been worked out [3] and applied to design the optimal control law leading an elastic system to the terminal position and minimizing a given objective function [4, 5].

Consider an elastic body occupying some region Ω with an external boundary γ. The body motion is described by the differential equations of linear elasticity:

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}^0, \quad \mathbf{p} = \rho \mathbf{u}^{\boldsymbol{\cdot}}, \tag{1}$$

$$\nabla \cdot \boldsymbol{\sigma} - \mathbf{p}^{\boldsymbol{\cdot}} + \mathbf{f} = 0, \quad \boldsymbol{\varepsilon}^0 = \left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)/2 \tag{2}$$

under the following boundary and initial conditions

$$\begin{aligned} &\alpha_k(x)u_k + \beta_k(x)q_k = v_k, \quad x \in \gamma; \\ &\mathbf{q} = \boldsymbol{\sigma} \cdot \mathbf{n}, \quad k = 1, 2, 3, \end{aligned} \tag{3}$$

$$\mathbf{u}(0, x) = \mathbf{u}^0(x), \quad \mathbf{p}(0, x) = \mathbf{p}^0(x), \quad x \in \Omega. \tag{4}$$

Here $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}^0$ are the stress and strain tensors, **C** is the elastic modulus tensor, **p** and **u** are the momentum density and displacement vectors, **f** and **q** are the vectors of volume and boundary forces, **n** is the unit vector pointing in the direction of the outward normal to the boundary γ; u_k and q_k are the components of the vector functions **u** and **q** of Cartesian coordinates $x = \{x_1, x_2, x_3\}$; α_k and β_k are some fixed functions, defining the type of boundary conditions. The initial vector functions $\mathbf{u}^0$ and $\mathbf{p}^0$ are given and the component v_k of the boundary vector **v** are either given functions or control

$$\mathbf{v} = \mathbf{v}(t, x), \quad x \in \gamma \setminus \gamma_v; \quad \mathbf{v} \in V, \quad x \in \gamma_v. \tag{5}$$

The problem is to find an optimal control $\mathbf{v}(t)$ moving the body from its initial to terminal states in the given time t_f

$$\mathbf{u}(t_f, x) \in U_f, \quad \mathbf{p}(t_f, x) \in P_f, \tag{6}$$

and minimizing an objective function $J[\mathbf{v}]$ in the class V of admissible controls

$$J[\mathbf{v}] \to \min_{\mathbf{v} \in V} . \qquad (7)$$

To solve the initial-boundary value problem (1)–(4), we apply MIDR, in which local relations (1) are replaced by an integral relation, and reduce this problem to a variational one. If a weak solution $\mathbf{p}^*$, $\boldsymbol{\sigma}^*$, and $\mathbf{u}^*$ exists then the following nonnegative quadratic functional Φ under local constraints (2)–(4) reaches on this solution its absolute minimum

$$\Phi(\mathbf{u}^*, \boldsymbol{\sigma}^*, \mathbf{p}^*) = \min_{\mathbf{u},\boldsymbol{\sigma},\mathbf{p}} \Phi(\mathbf{u}, \boldsymbol{\sigma}, \mathbf{p}) = 0,$$

$$\Phi = \int_0^{t_f} \int_\Omega \varphi(t,x) d\Omega\, dt, \quad \boldsymbol{\sigma}^0 \equiv \mathbf{C} : \boldsymbol{\varepsilon}^0, \quad \boldsymbol{\varepsilon} \equiv \mathbf{C}^{-1} : \boldsymbol{\sigma}, \qquad (8)$$

$$\varphi = \frac{1}{2}\left[(\boldsymbol{\sigma} - \boldsymbol{\sigma}^0) : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^0) + (\mathbf{p} - \rho \mathbf{u}^{\boldsymbol{\cdot}}) \cdot (\rho^{-1}\mathbf{p} - \mathbf{u}^{\boldsymbol{\cdot}}) \right].$$

To find an approximate solution of the optimization problem (2)–(8) we use a finite dimensional representation of the unknown functions $\mathbf{p}$, $\boldsymbol{\sigma}$, and $\mathbf{u}$

$$\tilde{\mathbf{p}} = \sum_{k=1}^{N_p} \mathbf{p}^{(k)} \psi_k(t,x), \quad \tilde{\boldsymbol{\sigma}} = \sum_{k=1}^{N_\sigma} \boldsymbol{\sigma}^{(k)} \psi_k(t,x), \quad \tilde{\mathbf{u}} = \sum_{k=1}^{N_u} \mathbf{u}^{(k)} \psi_k(t,x) \qquad (9)$$

If the control $\mathbf{v}$ is restricted to a finite class V of functions in the form

$$\mathbf{v} = \sum_{k=1}^{N_v} \mathbf{v}^{(k)} \overline{\psi}_k, \quad \overline{\psi}_k = \sum_{i=1}^{n_k} a_i^{(k)} \psi_{m_k(i)} , \qquad (10)$$

and the basis functions ψ_k are chosen so that the approximations (9) can exactly satisfy boundary and initial conditions (3), (4) as well as the equation of motion (2) the resulting finite-dimensional unconstrained minimization problem (8) yields an approximate solution $\tilde{\mathbf{p}}^*(t,x,\mathbf{v})$, $\tilde{\boldsymbol{\sigma}}^*(t,x,\mathbf{v})$, $\tilde{\mathbf{u}}^*(t,x,\mathbf{v})$ for an arbitrary control $\mathbf{v} \in V$. The optimal control $\mathbf{v}^*(t,x)$ is determined from minimum condition (7). In the paper we consider a quadratic functional $J[\mathbf{v}]$ (total mechanical energy of the body at the terminal time t_f)

$$J = W(t^f), \quad W(t) = \frac{1}{2}\int_\Omega A(\mathbf{u}) d\Omega, \quad A = \frac{1}{2}\left(\boldsymbol{\sigma}^0 : \boldsymbol{\varepsilon}^0 + \rho(\mathbf{u}^{\boldsymbol{\cdot}} \cdot \mathbf{u}^{\boldsymbol{\cdot}}) \right). \qquad (11)$$

The corresponding optimization problem is reduced to a system of linear equations with respect to the parameters of the control $\mathbf{v}^{(k)}$.

The value of the functional Φ in (8) can be considered as an integral quality criterion for the optimal solution whereas the integrand φ can be used as a local error characteristic.

As an example, the 3D problem of optimal motions of a rectilinear elastic prism with a quadratic cross section is considered for the terminal total mechanical energy to be minimized. The numerical results and their error estimates are presented and discussed.

References

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