Synchronization Stability in Oscillator Networks: Solution for Asymmetrical Configurations

Igor Belykh*, Vladimir Belykh **, Martin Hasler***

* Department of Mathematics and Statistics, Georgia State University,
30 Pryor Street, Atlanta, GA 30303, USA
e-mail: ibelykh@mathstat.gsu.edu

**Department of Mathematics, Volga State Academy,
5, Nesterov st., Nizhny Novgorod, 603 600 Russia
e-mail: belykh@unn.ac.ru

*** School of Computer and Communication Sciences,
Ecole Polytechnique Fédérale de Lausanne (EPFL),
Station 14, 1015 Lausanne, Switzerland
e-mail: martin.hasler@epfl.ch

Abstract

We extend the connection graph stability method, originally developed for symmetrically coupled networks, to the general asymmetrical case. The principal new component of the method is the transformation of the directed connection graph into an undirected graph. The extension of the method to asymmetrical coupling consists in symmetrizing the graph and associating a weight to each path. This weight involves the node unbalance of the two nodes. The synchronization condition for this symmetrized-and-weighted network then also guarantees synchronization in the original non-symmetrical network.

1 Introduction

The increasing interest in synchronization of limit-cycle and chaotic dynamical systems [1, 2, 3] has led many researchers to consider the phenomenon of synchronization in large complex networks of coupled oscillators (see, e.g. [4, 5] for a sampling of this large field). The strongest form of synchrony in chaotic systems is complete synchronization when all oscillators of the network acquire identical chaotic behavior. The central question about synchronization of periodic chaotic oscillators coupled in a network is: When is such synchronous behavior stable, especially in regard to coupling strengths and coupling configurations of the network?

Complete synchronization in undirected and directed networks of linearly coupled limit-cycle and chaotic oscillators has received much attention (see, e.g., [6, 7, 8, 9, 10, 11, 12, 13, 14]). These studies show that both local and global stabilities of complete synchronization depend on the eigenvalues of the Laplacian connectivity matrix. Complete synchronization was also studied by means of the adaptive control methods (see, e.g., [15] and the references therein).

In a recent paper [16], we proved that the synchronization condition can also be derived from *graph theoretical quantities* using the connection graph method. The main step of the method is to establish a bound on the total length of all paths passing through an edge on the connection graph. This approach was originally developed for undirected graphs and applied to global synchronization in complex networks [17, 18]. More recently, we showed that the method can be directly applied to asymmetrically coupled networks with node balance [19]. Node balance means that the sum of the coupling coefficients of all edges directed to a node equals the sum of the coupling coefficients of the all edges directed outward from the node. We proved that for node balanced networks it is sufficient to symmetrize all connections by replacing a unidirectional coupling with a bidirectional coupling of half the coupling strength. The bound for global synchronization in this undirected network then holds also for the original directed network.

In this paper we review and extend our approach to networks with arbitrary asymmetrical connections [20]. The connection graph of such a network is directed and the coupling coefficient from node i to node j is in general different from the coupling coefficient for the reverse direction. The new ingredient of the method is the transformation of the directed connection graph into an undirected weighted graph. This is done by symmetrizing the graph and associating a weight to each edge of the undirected graph and to each path between any two nodes. This weight involves the "node unbalance" of the two nodes. This quantity is defined to be the difference between the sum of connection coefficients of the outgoing edges and the sum of the connection coefficients of the incoming edges to the node. As in the case of node-balanced networks, the synchronization criterion derived for this symmetrical network then guarantees synchronization in the asymmetrical directed network.

2 Problem statement

We consider a network of n interacting nonlinear l-dimensional dynamical systems (oscillators). We assume that the individual oscillators are all identical, even though our results can be generalized to slightly non-identical systems. The composed dynamical system is described by the $n \times l$ ordinary differential equations

$$\dot{x}_i = F(x_i) + \sum_{k=1}^{n} d_{ik}(t) P x_k, \quad i = 1, ..., n, \qquad (1)$$

where $x_i = (x_i^1, ..., x_i^l)$ is the l-vector containing the coordinates of the i-th oscillator, the function $F : R^l \to R^l$ is nonlinear and capable of exhibiting periodic or chaotic solutions, and P is a projection operator that selects the components of x_i that are involved in the interaction between the individual oscillators. Without loss of generality, we consider a vector version of the coupling with the diagonal matrix $P = \mathrm{diag}(p_1, p_2, ..., p_l)$, where $p_\nu = 1$, $\nu = 1, 2, ..., s$ and $p_\nu = 0$ for $\nu = s + 1, ..., l$.

The connection matrix D with entries d_{ik} is an $n \times n$ matrix with zero row-sums and nonnegative off-diagonal elements such that

$$\sum_{k=1}^{n} d_{ik} = 0 \text{ and } d_{ii} = - \sum_{k=1; k \neq i}^{n} d_{ik}, \quad i = 1, ..., n.$$

This ensures that the coupling is of diffusive nature (on an arbitrary coupling graph) and any solution $x(t)$ for a single oscillator is also a solution of the coupled system (1). The connection matrix D is assumed to be *asymmetrical* without any further constraints. This is in contrast to our previous papers, where we required the symmetry of the connection matrix [16] or the zero column-sums property of an asymmetrical connection matrix [19]. The coupling matrix D is associated with the edge-weighted directed *connection graph* $\mathbf{D}$, where to each individual system corresponds a node and for each pair of nodes i, j with $i \neq j$ and such that $d_{ij} > 0$, there is an edge directed from from j to i. The weight assigned to this edge is d_{ij}. The connection graph is assumed to be connected.

We admit an arbitrary time dependence in the coupling matrix even if t is not explicitly stated everywhere. All constraints and criteria for the coupling matrix are understood to hold for all times t.

The completely synchronous state of the system (1) is defined by the linear invariant manifold $D = \{x_1 = x_2 = ... = x_n\}$, often called the synchronization manifold. Typically, in networks of continuous time oscillators, the synchronization manifold becomes stable when the coupling strength between the oscillators exceeds a critical value. Our main objective is to obtain conditions of global asymptotic stability of synchronization in the system (1). We want to determine threshold values for the coupling strength required for synchronization, and to reveal their dependence on the network topology.

3 Asymmetrically coupled networks

In contrast to symmetrically coupled networks, where any connection graph configuration allows synchronization of all the nodes, synchrony in asymmetrically coupled networks is only possible if there is at least one node which directly or indirectly influences all the others. In terms of the connection graph, this amounts to the existence of a uniformly directed tree involving all the vertices. A star-coupled network where secondary nodes drive the hub is a counter example, where such a tree does not exist and synchronization is impossible.

Before we proceed with the study of asymmetrical networks, we should impose the following constraint on the dynamics of the coupled system (1).

3.1 Main hypothesis

Assumption 1. There exist a parameter $a > 0$ and a matrix

$H = \mathrm{diag}(h_1, ..., h_s, \tilde{H})$, where $h_i = 1$ for $i = 1, ..., s$ and $\tilde{H}$ is positive definite such that the quadratic form defined by H is a Lyapunov function for all the auxiliary linear systems (varying $x \in B_1$)

$$\dot{\xi} = \frac{\partial F}{\partial x}(x)\xi - aP\xi \tag{2}$$

simultaneously. Equivalently, all matrices

$$H\left(\frac{\partial F}{\partial x}(x) - aP\right) + \left(\frac{\partial F}{\partial x}(x) - aP\right)^T H \tag{3}$$

must be negative definite.

This constraint basically requires that the individual dynamical systems can be stabilized by adding a diagonal term for each state component that is involved in the interaction. In other words, we assume that there exists a critical value a^*, sufficient to make the equilibrium state O of the auxiliary system (2) globally stable.

Assumption 1 is closely related to the requirement that the network (1) composed of *two* unidirectionally coupled systems globally synchronizes when the coupling c_{12} exceeds the critical value a. Many networks of linearly coupled limit-cycle and chaotic oscillator exhibit global synchronization such that Assumption 1 is satisfied. It was proved for coupled Lorenz systems [16], Chua circuits, Hindmarsh-Rose neuron models [18], etc.

For example, for two unidirectionally x-coupled Lorenz oscillators:

$$\begin{cases} \dot{x}_1 = \sigma(y_1 - x_1) + c_{12}(x_2 - x_1) \\ \dot{y}_1 = rx_1 - y_1 - x_1z_1 \\ \dot{z}_1 = -bz_1 + x_1y_1 \\ \dot{x}_2 = \sigma(y_2 - x_2) \\ \dot{y}_2 = rx_2 - y_2 - x_2z_2 \\ \dot{z}_2 = -bz_2 + x_2y_2 \end{cases} \tag{4}$$

Assumption 1 is true, and the bound for the synchronization coupling threshold is calculated as follows [16] $a = c_{12}^* = \frac{b(b+1)(r+\sigma)^2}{16(b-1)} - \sigma$.

3.2 Main theorem for asymmetrical coupling

Theorem 1 [20].
Under Assumption 1, complete synchronization is globally stable in the network (1) with an arbitrary coupling graph $\mathbf{D}$ *if for all* k

$$d_k + D_k > \frac{a}{n}b_k, \text{ where } b_k = \sum_{j>i;\ k\in P_{ij}}^{n} L(P_{ij}) \tag{5}$$

is the sum of the "lengths" $L(P_{ij})$ *of all chosen paths* P_{ij} *which pass through a given edge* k *that belongs to the symmetrized undirected graph. This weighted path length* $L(P_{ij})$ *is defined as follows*

$$L(P_{ij}) = \begin{cases} |P_{ij}|, & \text{if } D_i^c + D_j^c < 0; \textit{ and there is a link } k \textit{ between } i \textit{ and } j \\ |P_{ij}|\,\chi(1 + \frac{D_i^c + D_j^c}{2a}) = |P_{ij}|\,\chi(1 + \frac{D_{ij}}{a}), & \textit{otherwise,} \end{cases}$$

where the function χ *is the identity for positive and* 0 *for negative arguments, and* $|P_{ij}|$ *is the number of edges in each* P_{ij}.

The mean coupling coefficient $d_k = \frac{d_{ij}+d_{ji}}{2}$ *defines an edge* k *on the undirected symmetrized graph. An extra coupling strength* $D_k = |\frac{D_i^c + D_j^c}{2n}|$ *is added to the edges of the symmetrized connection graph for which the mean node unbalance* $D_i^c + D_j^c$ *is negative.*

In the case where the directed connection graph is not a uniformly directed tree involving all nodes and complete synchronization of all the nodes is impossible, the condition for synchronization is simply impossible to satisfy.

Theorem 1 directly leads to the following method to establish our stability condition for complete synchronization.

Step 1. Determine the "node unbalance" for each node $D_i^c = \sum_{j=1}^{n} d_{ji}$. In terms of graphs, the column sum $D_i^c = \sum_{k=1}^{n} d_{ki} = \sum_{k\neq i} d_{ki} + d_{ii} = \sum_{k\neq i} d_{ki} - \sum_{k\neq i} d_{ik}$ amounts to the difference between the sum of the coupling coefficients of all edges directed outward from node i and the sum of the coupling coefficients of all the edges directed to node i.

Step 2. Symmetrize the connection graph by replacing the edge directed from node i to node j by an undirected edge with half the coupling coefficient $d_{ij}/2$. In the case where there is an edge directed from node i to node j and another edge in the reverse direction, the pair of directed edges is replaced by an undirected edge with mean coupling coefficient $d_k = \frac{d_{ij}+d_{ji}}{2}$.

Step 3. Choose a path P_{ij} between each pair of nodes. Usually, the shortest path is chosen. Sometimes, however, a different choice of paths can lead to lower bounds [18].

Step 4. For each path P_{ij} determine the mean node unbalance of the endnodes i and j.

Identify paths of length 1, i.e edges of the symmetrized graph, with negative mean node unbalance $D_i^c + D_j^c$. For these edges, calculate and add extra strength $D_k = |\frac{D_i^c + D_j^c}{2n}|$ to the symmetrized coupling d_k.

For all other paths P_{ij}, namely, paths of length 1 with nonnegative mean node unbalance and any paths composed of at least two edges, calculate the quantities $D_{ij} = \frac{D_i^c + D_j^c}{2}$ and $1 + \frac{D_{ij}}{a}$. Associate weight $1 + \frac{D_{ij}}{a}$ to the path length of P_{ij} if $1 + \frac{D_{ij}}{a} > 0$, and zero weight, otherwise.

Step 5. For each edge k of the symmetrized-and-weighted connection graph determine the inequality

$$d_k + D_k > \frac{a}{n} b_k, \text{ where } b_k = \sum_{j>i;\ k \in P_{ij}}^{n} L(P_{ij}).$$

Step 6. Combine the inequalities either to describe the set of common values for all connection coefficients that guarantee global complete synchronization or to describe in general the set of connection coefficients vectors that guarantee synchronization if we allow for coefficients that vary from link to link. Finally, the bound for global synchronization in the symmetrized-and-weighted network holds also for the original asymmetrical network.

Let us show how to apply the general method to two examples of asymmetrical networks.

4 Examples

4.1 Two unidirectionally coupled oscillators

Consider the simplest directed network with $n = 2$ and coupling strength d (Fig. 1a).

(a) (b)

Figure 1: Simplest directed network and its symmetrized analog. The directed link is replaced by the undirected edge with half the coupling strength. Here, the mean node unbalance, $\frac{D_1^c + D_2^c}{2} = 0$, so that the symmetrize-and-weight operation amounts to the symmetrization. The path length P_{12} remains unweighted.

Step 1. Determine the node unbalance for node 1 and 2 : $D_1^c = -d$ and $D_2^c = d$.
Step 2. Symmetrize the graph as shown in Fig. 2b: $d_1 = \frac{d+0}{2} = \frac{d}{2}$.
Step 3. Choose a path between each pair of nodes. Here, the graph has only one branch.

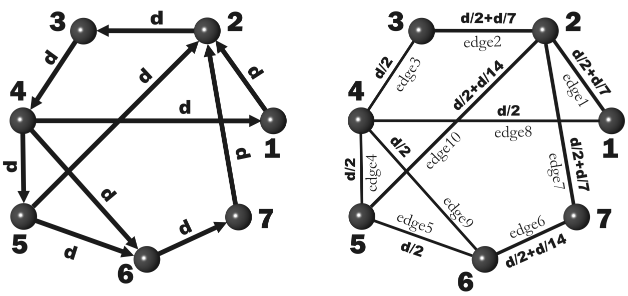
Steps 4. For each path determine the mean node unbalance of the endnodes. Here, this quantity is equal to 0 : $D_1^c + D_2^c = d - d = 0$. Therefore, $D_k \equiv D_1 = \frac{0}{2 \cdot 2} = 0$ and $D_{ij} \equiv D_{12} = 0$.

Steps 5-6. For the edge 1 determine the inequality: $\frac{d}{2} + 0 > \frac{a}{2}|P_{12}|$. The path length $|P_{12}| = 1$ such that the final inequality becomes $d > a$.

Recall that by Assumption 1, a is an upper bound for synchronization in this network such that our method gives the correct synchronization bound.

4.2 Irregular network

Consider the asymmetrical seven-node network of Fig. 2.



(a) (b)

Figure 2: (a) Unidirectionally coupled network with uniform coupling d. (b) Symmetrized analog of (a) with weighted bidirectional connections. Arrows indicate the direction of coupling along an edge; edges without arrows are coupled bidirectionally. The width of the links may be thought of as the coupling strength.

As before, we use the six-step process to derive the synchronization condition of Theorem 1.
Step 1. Calculate the difference between the sum of the coupling coefficients of all edges directed outward from node i and the sum of the coupling coefficients of all the edges directed to node i. Thus, determine the node balance for each node of the graph:

$$D_1^c = d - d = 0 \qquad D_2^c = d - 3d = -2d$$
$$D_3^c = d - d = 0 \qquad D_4^c = 3d - d = 2d$$
$$D_5^c = 2d - d = d \qquad D_6^c = d - 2d = -d$$
$$D_7^c = d - d = 0.$$

Step 2. Symmetrize the graph by replacing each directed edge by an undirected edge with half the coupling strength: $d_k = \frac{d}{2}$, $k = 1, ..., 10$ (see Fig. 2b).
Step 3. Choose a path P_{ij} between any pair of nodes i, j of the symmetrized graph. It turns out that it is often advantageous to choose paths that contain edges with negative mean node unbalance (this quantity will be calculated in Step 4.)

Our choice of paths is

P_{12} : edge1	P_{13} : edges1, 2	P_{14} : edge8
P_{15} : edges1, 10	P_{16} : edges1, 7, 6	P_{17} : edges1, 7
P_{23} : edge2	P_{24} : edges2, 3	P_{25} : edge10
P_{26} : edges7, 6	P_{27} : edge7	P_{34} : edge3
P_{35} : edges2, 10	P_{36} : edges2, 7, 6	P_{37} : edges2, 7
P_{45} : edge4	P_{46} : edge9	P_{47} : edges9, 6
P_{56} : edge5	P_{57} : edges5, 6	P_{67} : edge 6

Step 4. For each path P_{ij} determine the mean node unbalance $\frac{D_i^c+D_j^c}{2}$ for endnodes i and j :

$$\begin{array}{llll} P_{12}: -d & P_{13}: 0 & P_{14}: d & P_{15}: \frac{d}{2} \\ P_{16}: -\frac{d}{2} & P_{17}: 0 & P_{23}: -d & P_{24}: 0 \\ P_{25}: -\frac{d}{2} & P_{26}: -\frac{3d}{2} & P_{27}: -d & P_{34}: d \\ P_{35}: \frac{d}{2} & P_{36}: -\frac{d}{2} & P_{37}: 0 & P_{45}: \frac{3d}{2} \\ P_{46}: \frac{d}{2} & P_{47}: \frac{d}{2} & P_{56}: 0 & P_{57}: \frac{d}{2} \\ P_{67}: -\frac{d}{2}. & & & \end{array}$$

We now categorize the mean node unbalance terms as follows.

If $\frac{D_i^c+D_j^c}{2} < 0$ and there is an edge k of the symmetrized graph linking directly i and j, we set $D_k = \left|\frac{D_i^c+D_j^c}{2\cdot 7}\right|$ and add this additional coupling strength to d_k. This relates to edges 1, 2, 6, 7, 10 (see Fig. 2b):

$$\begin{array}{ll} D_1 = \left|\frac{D_1^c+D_2^c}{2\cdot 7}\right| = \frac{d}{7} & D_2 = \left|\frac{D_2^c+D_3^c}{2\cdot 7}\right| = \frac{d}{7} \\ D_6 = \left|\frac{D_6^c+D_7^c}{2\cdot 7}\right| = \frac{d}{14} & D_7 = \left|\frac{D_2^c+D_7^c}{2\cdot 7}\right| = \frac{d}{7} \\ D_{10} = \left|\frac{D_2^c+D_5^c}{2\cdot 7}\right| = \frac{d}{14}. & \end{array}$$

In all other cases, the terms $\frac{D_i^c+D_j^c}{2}$ are either nonnegative or negative but there is no direct link between i and j, so that all these terms become D_{ij}.

Step 5. For each edge of the graph determine the inequality (5).

Edge 1 (link between nodes 1 and 2):

$$d_1+D_1 = \frac{d}{2}+\frac{d}{7} > \frac{a}{7}b_k, \text{ where } b_k = \sum_{j>i;\ k\in P_{ij}}^{n} L(P_{ij}).$$

The chosen paths that pass through the edge 1 are P_{12}, P_{13}, P_{15}, P_{16}, P_{17}. Their weighted lengths $L(P_{ij})$ are calculated in accordance with Theorem 1:

$$\begin{array}{l} L(P_{12}) = |P_{12}| = 1 \text{ since } D_1^c + D_1^c < 0 \\ \qquad \text{and there is an edge between 1 and 2} \\ L(P_{13}) = |P_{13}|\chi(1+\frac{D_{13}}{a}) = 2 \\ L(P_{15}) = |P_{15}|\chi(1+\frac{D_{15}}{a}) = 2(1+\frac{d}{2a}) \\ L(P_{16}) = |P_{16}|\chi(1+\frac{D_{16}}{a}) = 0 \\ L(P_{17}) = |P_{17}|\chi(1+\frac{D_{17}}{a}) = 2. \end{array}$$

Summing up all the lengths, we obtain

$$\frac{d}{2}+\frac{d}{7} > \frac{a}{7}\left[1+2+2\left(1+\frac{d}{2a}\right)+2\right] = \frac{7a+d}{7}.$$

Therefore, the synchronization condition for the edge 1 becomes $d > 2a$.

Exactly as for the edge 1, we can calculate the synchronization bounds for other edges. These bounds can be summarized as follows

$$\begin{array}{lll} \text{edge } 1: d > 2a & \text{edge } 2: d > \frac{18a}{7} & \text{edge } 3: d > \frac{6a}{5} \\ \text{edge } 4: d > \frac{a}{2} & \text{edge } 5: d > \frac{3a}{5} & \text{edge } 6: d > 5a \\ \text{edge } 7: d > \frac{10a}{9} & \text{edge } 8: d > \frac{2a}{5} & \text{edge } 9: d > 3a \\ \text{edge } 10: d > \frac{5a}{2} & & \end{array}.$$

Step 6. Combining the synchronization criteria for all the edges, we take the maximum constraint to achieve global synchronization. This constraint corresponds to the weakest link. Here, the weakest link is the edge 6. This edge is a bottle neck for synchronization of the entire network and requires the maximum coupling strength to synchronize all oscillators of the network. Therefore we conclude that for

$$d > d^* = 5a$$

we can guarantee global synchronization of the network.

5 Conclusions

We have extended the connection graph stability method for synchronization in an arbitrary nonsymmetrical network of coupled identical oscillators. The condition is composed of a set of inequalities which have to be satisfied, one inequality for each edge of the connection graph. Each inequality involves a term that depends only on the individual dynamical systems, namely the coupling strength that guarantees global synchronizing of two systems. The other terms of the inequality depend only on the graph structure and on the coupling coefficients.

In small and also in sufficiently regular networks, the condition can be written down explicitly. In other networks, a combinatorial algorithm of polynomial complexity can establish the inequalities on the coupling coefficients that guarantee global complete synchronization. The main computational task is to determine a path between any two nodes of the graph, typically the shortest path.

We should remark that our generalized method is valid for networks of slightly nonidentical oscillators. In this case, perfect synchronization cannot exist anymore, but approximate synchronization is still possible. We have previously shown that in the case of symmetrically coupled networks, similar global stability conditions of approximate synchronization can be derived within the framework of the connection graph method [16]. This carries over to asymmetrical heterogenous networks.

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