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ON THE HYBRID STABILITY OF THE COLLOCATED VIRTUAL HOLONOMIC CONSTRAINTS BASED WALKING DESIGN*

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Abstract

This paper presents proof of the hybrid stability of the pre-designed walking like trajectory and its feedback tracking controller for the so-called four-link. The four-link is a planar mechanical chain having four degrees of freedom and three actuators placed between its links. In such a way it resembles a pair of legs with knees. The proof of hybrid stability is based on computing the appropriate Poincaré map linear approximation and showing numerically that its eigenvalues are inside the unit disk in the complex plane. Unlike the frequent approach in robotic walking showing the stable path following, nature of our designed trajectory enables to prove its tracking including the time dependence. The tested trajectory and the feedback controller were obtained via combination of the design for the so-called Acrobot and suitable selected collocated holonomic constraints enforced by feedback imposed in knees actuators. This approach was published before but it will be briefly repeated here for the sake of completeness. Finally, the simulations showing the hybrid stability of 150 steps walking of the four-link with lengths and masses configurations corresponding to an existing laboratory model will be presented.

Key words

Underactuated walking, Virtual holonomic constraints, Poincaré section method, collocated constraints.

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1 Introduction

The purpose of the paper is to demonstrate the concept of the collocated virtual holonomic constraint by showing the hybrid stability of the walking-like movement of the mechanical four-link which was designed using that concept. The hybrid stability will be proved using the well-known Poincaré section method. More precisely, the corresponding map will be computed together with the eigenvalues of its approximate linearization, these eigenvalues are then shown to be inside the unit circledisk in the complex plane.

The walking-like trajectory and its corresponding tracking feedback were designed in detail in [Čelikovský and Anderle, 2016a] using the combination of two key ideas.

First, the property that the unactuated variable is also the cyclic one allows to exactly linearize its three dimensional subsystem using state space and feedback transformations. Property of unactuated variable being cyclic is typical for walking-like configurations. It means that kinetic energy of the system does not depend on the cyclic variable. This variable is called also as the absolute orientation angle, while other angles are the so-called shape variables. Alternatively, with a slight abuse of notation, we may also call that property as the kinetic symmetry with respect to unactuated variable. Based on such favorable property, for the so-called Acrobot, the hybrid cyclic walking-like trajectory and its tracking feedback were designed and shown to be exponentially stable in [Čelikovský et al., 2008], [Anderle et al., 2009], [Anderle et al., 2010], [Anderle and Čelikovský, 2009], [Anderle and Čelikovský, 2011], [Anderle and Čelikovský, 2010]. For a more complete picture, see [Olfati-Saber, 2002], [Grizzle et al., 2005] where the cyclic property of unactuated variable was introduced and discussed as well.

The second cornerstone of the approach presented in [Čelikovský and Anderle, 2016a] is to enforce the so-called **collocated virtual holonomic constraints** thereby converting any general planar underactuated mechanical system having n degrees of freedom and $n-1$ actuators and unactuated cyclic variable into the same kind of system but with $n = 2$. Such a system may be tentatively called as the **generalized Acrobot** [Čelikovský et al., 2013].

As a consequence, $n-2$ actuators are used to enforce the collocated virtual holonomic constraints in a exponentially stable way while the remaining one is used to control the residual system - the mentioned generalized Acrobot. During the swing phase it can be shown that such a combination provides exponentially stable tracking of the pre-designed walking-like trajectory.

Yet, it remain to address the hybrid essence of the walking. During the double support phase the so-called impact map occurs and changes impulsively the angular velocities. Pre-designed walking-like trajectory for the generalized Acrobot obtained in [Anderle and Čelikovský, 2010] handles this issue ensuring cyclic character of the multi-step walking. Nevertheless, it remains to ensure that the appropriate virtual holonomic constraints are also hybrid invariant, *i.e.* they are preserved by the impact map. Necessary conditions to do so were obtained in [Čelikovský and Anderle, 2016a]. Based on them, one can pre-design virtual holonomic constraints in such a way that they are hybrid invariant and simultaneously fulfill other constraints requirement. As a matter of fact, these constraints may express dependence of bending knees on the hips angle, thereby defining the so-called walking shape.

Previously described contribution belongs to the area of the underactuated walking design based on the use of the above mentioned **virtual holonomic constraints (VHCs)**. The walking robots design area and even its more narrow field of underactuated walking design has been intensively studied during recent decades. See *e.g.* [Westervelt et al., 2007; Chevallereau et al., 2009; Grizzle et al., 2014] for one possible stream of research and references within there for other approaches. Another line of the research can be found in [Pchelkin et al., 2015], [Spong and Bullo, 2005], [La Herra et al., 2013], to complete the picture at least to some extent, as the real survey of the field is out of this current paper scope and purpose. Though the previous references as well as the present contribution focus are mainly concerned with the analysis and the design of walking concepts assuming a perfect model knowledge, identification and adaptive control are important and widely studied subfield of the underactuated walking, see *e.g.* [Dolinský and Čelikovský, 2012], [Dolinský and Čelikovský, 2017], [Westervelt et al., 2007] and references within there.

The VHCs are constraints imposed on generalized coordinates that are not consequence of some physical limitations but rather are to be artificially enforced using external generalized forces provided by actuators. These forces are not the natural constraining forces, *i.e.* they are not the reaction forces triggered by some physical constraints. While the reaction forces are postulated by basic principle of mechanics not to perform work [Greiner, 2003], the external forces imposing the VHCs may perform some work.

The analysis of controlled mechanical systems subject to forced kinematic and dynamic relations goes back to early 1920's [Bèghin, 1921]. These relations were called as the "servo-constraint" later on [Appel, 1953]. The recent and detailed survey can be found in [Kozlov, 2015]. The term "VHCs" emerges and is widely used during recent two decades only as an alternative terminology for some types of those forced relations.

The use of the VHCs concept in the underactuated walking is quite broad and diverse subarea as itself. Interesting is the use of the VHCs to study the passive walking down the moderate slope in [Freidovich et al., 2009]. Passive walking down the moderate slope is broadly studied and referred since the seminal paper [McGeer, 1990]. Further use of the VHCs can be found in [Shiriaev et al., 2014], [Shiriaev et al., 2006], [Shiriaev et al., 2005], or alternatively in [Westervelt et al., 2007; Chevallereau et al., 2009; Grizzle et al., 2014], [Chevallereau et al., 2003], to mention just a few.

As already noted, the main specific feature of the approach to be tested in the current paper in comparison with the above quoted results is that it uses the collocated VHCs that include the directly actuated coordinates only. Moreover, the number of constraints is usually lower than number of the available input torques. The restricted system has therefore at least one free input to be used for its further control. This restricted system is usually the above mentioned **generalized Acrobot** - the system with 2 degrees of freedom, one actuator and unactuated cyclic variable. Such a system can be transformed into almost linear chain of integrators that allow to generate directly a hybrid periodic trajectory and then to stabilize it using some robust techniques [Čelikovský et al., 2008], [Anderle et al., 2009], [Anderle et al., 2010], [Anderle and Čelikovský, 2009], [Anderle and Čelikovský, 2011], [Anderle and Čelikovský, 2010].

Note also, that while the above mentioned results [Shiriaev et al., 2014], [Shiriaev et al., 2006], [Shiriaev et al., 2005], [Westervelt et al., 2007; Chevallereau et al., 2009; Grizzle et al., 2014] are mostly related with the **path following**, or also **orbital stability**, the approach presented in this paper leads to the design of the periodic time trajectory and its tracking, as *e.g.* in [Song and Zefran, 2006a; Song and Zefran, 2006b; Majumdar et al., 2013].

Concept of the general VHCs was further and more abstractly investigated in [Mohammadi et al., 2013], [Maggiori and Consolini, 2013], while the collocated VHCs were introduced in [Čelikovský, 2015] and further used in [Čelikovský and Anderle, 2016b], [Čelikovský and Anderle, 2016a].

Though the extensive simulations show exponentially stable walking up to 150 steps, the more rigorous stability test of the walking designed in [Čelikovský and Anderle, 2016a] was not performed there. This is actually the main purpose of the present paper. To handle the hybrid character of the stability, the well-known method of Poincaré sections is used. Such a stability test is performed here for the case of the so-called four link, *i.e.* the case $n = 4$. Still, essence of the result is numerical, it computes the appropriate Poincaré map, then numerically finds its eigenvalues and show its exponential stability. Nevertheless, corresponding test can be repeated for wide range of system configurations and pre-designed trajectories and thereby to obtain for each particular case its own rigorous proof of the stability.

Note also, that, strictly speaking, the map computed here is not the classical Poincaré map corresponding to orbital stability of the corresponding limit cycle. Here, the exponentially stable tracking with respect to time dependence is achieved as well. This means that eight dimensional matrix is evaluated, rather than the seven dimensional for the classical approach. This is bit redundant, since orbital stability is considered in robotic applications quite sufficient. In robotic terminology, this corresponds to the above mentioned **path following** rather than time dependent trajectory tracking. Nevertheless, the terminology here is still acceptable as one could add artificial extra state variable being time. In such a way, our eight dimensional matrix describes properties of the classical Poincaré map for that extended nine dimensional system. Such a slight redundancy is due the essence of the method used to design the walking-like trajectory and its tracking feedback.

The rest paper is organized as follows. The next section briefly presents the model of the four link as the demonstrative example including its continuous-time and discrete-time dynamics. The Section 3 describes some preliminary definitions and results from [Čelikovský and Anderle, 2016b], [Čelikovský and Anderle, 2016a] necessary for the further stability analysis. Numerical stability analysis of the four link walking control via the VHCs and remaining restricted system control is done together with simulations of the robot walking in Section 4. Final section draws briefly some conclusions and discusses some future research outlooks.

2 The Model of the Four-link Mechanical Chain

The four link depicted in Figure 1 is in our particular case a representative of n-link mechanical chains with $n - 1$ actuators attached by one of its ends to a pivot point where is no actuator. The 4-link has four degrees of freedom and three actuators placed among its rigid

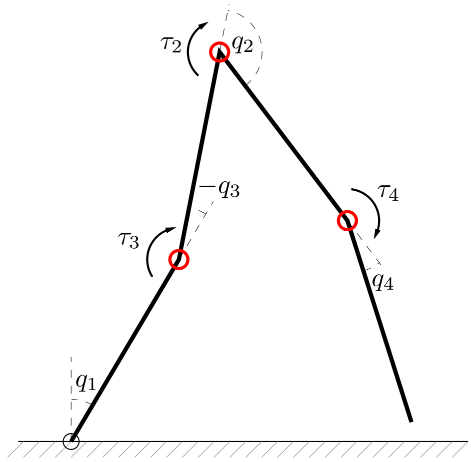

Figure 1. The four-link.

links, or in another words, it has two legs with knees and three actuators. Two actuators are placed in knees and one actuator is placed between its legs in the hip. Therefore the four link belongs to the class of the underactuated walking robots with degree of underactuation equal to one.

The movement or walking of the four link consists of continuous-time and discrete-time dynamics. The continuous part of the four link movement occurs during a walking, *i.e.* when one leg, usually called as a *swing leg* is in the air. Whereas, the discrete part occurs when the swing leg hits the ground, *i.e.* when both legs are in contact with the ground. The collision between the swing leg and the ground is in the literature usually called as the impact. Only for completeness, the second leg, which is in contact with the ground during the four link movement, is usually called as a *stance leg*. The impact event is accompanied with change of legs, *i.e.* the swing leg becomes the stance leg and vice versa.

The derivation of continuous-time and discrete-time dynamics below is related to the four-link, however, procedures can be simply extended to a general n-link mechanical chains. In the sequel, the shorter notation ”4-link” will be used as well.

2.1 Continuous-time Dynamics of the 4-link

The continuous part, *i.e.* when the swing leg of the four-link is in the air, is modeled by the usual Lagrangian approach. Consider the so-called Lagrangian for the 4-link being a smooth function of some suitable chosen generalized coordinates and velocities:

$$\mathcal{L}(q, \dot{q}) = K(q, \dot{q}) - V(q) = \frac{1}{2}\dot{q}^T D(q)\dot{q} - V(q). \quad (1)$$

Here $q = (q_1, \ldots, q_4)^\top$ denotes the 4-dimensional vector of (usually angular) generalized coordinates, $\dot{q} = (\dot{q}_1, \ldots, \dot{q}_4)^\top$ is that of the respective generalized velocities, $D(q) = D(q)^\top > 0$ is the inertia matrix, K

is the kinetic and V the potential energy of the system. Dynamic equations are then obtained as follows

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right]^{\top} - \left[\frac{\partial \mathcal{L}}{\partial q}\right]^{\top} = u,\ u \in R^4, \tag{2}$$

where u is the input vector. Any system that can be written in the form (2) with some $\mathcal{L}$ satisfying the above properties will be shortly called as the **Lagrangian system**, cf. [Mohammadi et al., 2013] for the more refined terminology. Evaluating (2) gives the well-known form of the mechanical system dynamics

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = u, \tag{3}$$

with the so-called Coriolis and centrifugal terms $C(q,\dot{q})$ and gravity terms $G(q) = -\nabla V(q)$.

The configuration of the 4-link is described by the generalized coordinates q and it is bounded by one-sided constraint represents the limitation that, in general, two rigid bodies do not penetrate each other. In this particular case it means that the 4-link swing leg cannot move under the ground, *i.e.* the height of the swing leg's end-point has to be $h_{endpoin}(q) > 0$ during the swing leg movement.

2.2 Discrete-time Dynamics of the 4-link

When the swing leg of the 4-link hits the ground at the end of the step, *i.e.* $h_{endpoint}(q) = 0$, the impact occurs. The result of this event is the instantaneous jump in angular velocities $\dot{q}$ whereas the angular positions q remain unchanged. The impact is modeled as a contact between two rigid bodies, see e.g. [Brach, 1989], [Keller, 1986]. Crucial in the impact mapping is an extension of the inertia matrix $D_e(q_e)$ by adding the Cartesian coordinates of the stance leg end point. The impact mapping is based on the following equation

$$D_e\left[\dot{q}_e^+ - \dot{q}_e^-\right] = F_{ext} \tag{4}$$

accompanied by the following equation

$$E_2(q_e^-)\dot{q}_e^+ = 0 \tag{5}$$

corresponding to the condition of no rebound and no slipping of the swing leg. $D_e(q_e)$ is the extended inertia matrix, $\dot{q_e}^+$, $\dot{q_e}^-$ are extended angular velocities of the 4-link just after, just before the impact, respectively, F_{ext} corresponds to the contact impulse over the impact duration and it is given by forces acting in the end of the swing leg and $E_2(q_e) = \frac{\partial \Upsilon(q_e)}{\partial q_e}$, where Υ represents the end point coordinates of the swing leg. For more details see [Chevallereau et al., 2009]. The result of equations (4), (5) is the impact matrix $\Phi(q^-)$ and the relabeling map $\mathcal{R}(q^-)$ with the following effect:

$$q^+ = \mathcal{R}(q^-), \tag{6}$$

$$\dot{q}^+ = \Phi(q^-)\dot{q}^-, \tag{7}$$

where $q^-, \dot{q}^-$ and $q^+, \dot{q}^+$ are angular positions and velocities just before the impact and just after the impact and relabeling, respectively.

3 Walking Design Using the Collocated VHCs

In this section, for the sake of more fluent reading, some well-known facts from literature and some related results formulations from [Čelikovský, 2015], [Čelikovský and Anderle, 2016b], [Čelikovský and Anderle, 2016a] are repeated. More specifically, the application of the VHCs to the underactuated walking robot and the basic idea of the impact invariant VHCs will be repeated here.

3.1 The Collocated VHCs in Lagrangian Systems

In [Čelikovský, 2015], [Čelikovský and Anderle, 2016b] it was shown that for a given underactuated mechanical system the so-called restricted underactuated mechanical system to be further controlled can be obtained using the VHCs.

Virtual holonomic constraints for the system (2-3) are given by l equalities, $1 \le l \le n$,

$$\varphi_i(q) = 0, \quad i = 1, \ldots, l, \tag{8}$$

where $\varphi_1, \ldots, \varphi_l$ are smooth functions of the generalized coordinates having $\forall q \in R^n$ linearly independent differentials $\mathrm{d}\varphi_i(q), \quad i = 1, \ldots, l$. The VHCs are called as collocated if and only if they are regular [Maggiori and Consolini, 2013] and involve the actuated coordinates $q_{k+1}, \ldots, q_n$. The VHCs are called as flat if after suitable renumbering of the generalized coordinates they take the form

$$q_{n-l+1} \equiv 0, \ldots, q_n \equiv 0, \quad l \le n-k. \tag{9}$$

Consider the given flat collocated VHCs. To study how to impose them introduce the following notation

$$u^R = \begin{bmatrix} 0_k \\ u_{k+1} \\ \vdots \\ u_{n-l} \end{bmatrix},\ u^C = \begin{bmatrix} u_{n-l+1} \\ \vdots \\ u_n \end{bmatrix}, \tag{10}$$

where u^R corresponds to the actuators of the so-called restricted subsystem whereas u^C corresponds to the actuators of the so-called constrained system. In such a way, (3) becomes as follows

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \begin{bmatrix} u^R \\ u^C \end{bmatrix}, \quad (11)$$

$$q = \begin{bmatrix} q^R \\ q^C \end{bmatrix},\ q^R = \begin{bmatrix} q_1 \\ \vdots \\ q_{n-l} \end{bmatrix},\ q^C = \begin{bmatrix} q_{n-l+1} \\ \vdots \\ q_n \end{bmatrix}, \quad (12)$$

$$\dot{q} = \begin{bmatrix} \dot{q}^R \\ \dot{q}^C \end{bmatrix},\ \dot{q}^R = \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_{n-l} \end{bmatrix},\ \dot{q}^C = \begin{bmatrix} \dot{q}_{n-l+1} \\ \vdots \\ \dot{q}_n \end{bmatrix}, \quad (13)$$

where q^R, $\dot{q}^R$ correspond to the coordinates and the velocities of the restricted system whereas q^C, $\dot{q}^C$ correspond to the coordinates and the velocities of the constrained system. Finally, let

$$D(q) = \begin{bmatrix} D_R(q) & D_{RC}(q) \\ D_{RC}^\top(q) & D_C(q) \end{bmatrix}, \quad (14)$$

$$C(q) = \begin{bmatrix} C^R(q,\dot{q}) \\ C^C(q,\dot{q}) \end{bmatrix}, \qquad G = \begin{bmatrix} G^R(q) \\ G^C(q) \end{bmatrix}, \quad (15)$$

where D_R, D_C are square $(n-l)\times(n-l)$ and $l \times l$ matrices, correspondingly, while D_{RC} is $(n-l)\times l$ matrix. Finally, $C^R \in R^{(n-l)\times n}, G^R \in R^{n-l}, C^C \in R^{l\times n}, G^C \in R^l$.

The Lagrangian system in the form (11) with (10) can be transformed using an invertible feedback transformation into a partial exact linearized form which leads to the following subsystem

$$D^R(\begin{bmatrix} q^R \\ 0 \end{bmatrix})\ddot{q}^R + C^R(\begin{bmatrix} q^R \\ 0 \end{bmatrix}, \begin{bmatrix} \dot{q}^R \\ 0 \end{bmatrix})\dot{q} + G^R(\begin{bmatrix} q^R \\ 0 \end{bmatrix}) = (\begin{bmatrix} q^R \\ 0 \end{bmatrix})u^R \quad (16)$$

which can be regarded as a restriction of (3) to flat holonomic VHCs (9). This restriction is obviously Lagrangian system with the same degree of underactuation k as original system (3), for details see [Čelikovský and Anderle, 2016b]. The restriction of original system (3) into (16) can be enforced by the following feedback controller

$$u^C = D_{RC}^\top D_R^{-1} u^R(t) - [D_C - D_{RC}^\top D_R^{-1} D_{RC}] \times [\mathrm{diag}\{K_{P1},\ldots,K_{Pl}\} q^C + \mathrm{diag}\{K_{V1},\ldots,K_{Vl}\}\dot{q}^C] + C^C(q,\dot{q})\dot{q} + G^C(q) - D_{RC}^\top D_R^{-1}[C^R(q,\dot{q})\dot{q} + G^R(q)], \quad (17)$$

where, $K_{P1},\ldots,K_{Pl}, K_{V1},\ldots,K_{Vl}$ are some positive reals.

Roughly speaking, restricted system (16) represents in our particular case the Acrobot model to be controlled by an controller u^R according to a required movement of the restricted system. Whereas controller u^C must fulfil equation (17) in order that the original system (3), in our particular case the four-link, is restricted into the Acrobot model represented by (16) according to the flat holonomic VHCs (9).

This results were demonstrated on example of the four-link model, *i.e.* the walking robot with four degree of freedom with three actuators and one degree of underactuation. In this particular example, two VHCs were defined to control legs bending and straightening. The remaining restricted system in the example is composed from underactuated two legged robot equivalent to the Acrobot, i.e. two degree of freedom with one degree of underactuation as only the angle between the stance leg and the ground is not actuated. By virtue of the VHCs approach, the restricted system, *i.e.* the two-legged robot must be controlled according to desired motion whereas the legs bending and straightening is controlled via VHCs.

3.2 Impact Invariance of the Collocated VHCs

The impact invariant VHCs are crucial in application of a robot walking in order that the walking can be considered as a periodic orbit, *i.e.* after the impact at the end of the step the walking-like mechanism has the same configuration and velocities as at the beginning of the step. In [Čelikovský and Anderle, 2016a], special conditions for VHCs to be impact invariant for a general n-link system were presented. These conditions are crucial for the cyclic hybrid walking trajectory design provided the cyclic walking trajectory design for the restricted system has already been developed.

During the impact the angular configuration of the walking robot does not change, however, the angular velocities change impulsively, i.e. the angular velocity “just before” the impact $\dot{q}^-$ changes instantaneously into the angular velocity “just after” the impact $\dot{q}^+$. This change can be mathematically expressed as multiplication by the so-called **impact matrix** $\Phi(q^-)$ (7). Entries of the impact matrix depend nonlinearly on the mechanical system configuration q^- “just before” the impact moment. Moreover, due to the switching of legs at the end of the step, the relabeling of angles is necessary, which results into a simple map of q^- into configuration variables “just after” the impact q^+, this mapping is denoted as $\mathcal{R}(q^-)$ (6). The crucial equation to be fulfilled for the cyclic walking trajectory design of the restricted system as well as for impact invariant VHCs design is as follows

$$\dot{q}(0) = \dot{q}(T^+) = \Phi(q^-)\dot{q}(T^-). \quad (18)$$

The design of impact invariant VHCs in [Čelikovský and Anderle, 2016a] was done for general n-link walking robot such that the restricted system consists of two legs. Therefore, $l = n - 2$ impact invariant VHCs in the following special form were defined and studied

$$\varphi_i := q_{i+2} - \phi_{i+2}(q_2), \;\; i = 1, \ldots, n-2, \qquad (19)$$

where $\phi_3, \ldots, \phi_n$ are suitable sufficiently smooth functions to be defined. Their design was aimed at their initial and final derivatives $\frac{\partial \phi_{3,\ldots,n}}{\partial q_2}(q_2^+)$, $\frac{\partial \phi_{3,\ldots,n}}{\partial q_2}(q_2^-)$ to be in a relation given by impact matrix $\Phi(q^-)$ (7). To briefly present the idea of impact invariant VHCs design, introduce the following block notation for the impact matrix $\Phi(q^-)$

$$\Phi(q^-) = \begin{bmatrix} \Phi^{11}(q^-) & \Phi^{12}(q^-) \\ \Phi^{21}(q^-) & \Phi^{22}(q^-) \end{bmatrix}, \qquad (20)$$

where

$$\Phi^{11}(q^-) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}, \;\; \Phi^{12}(q^-) = \begin{bmatrix} \phi_{13} & \cdots & \phi_{1n} \\ \phi_{23} & \cdots & \phi_{2n} \end{bmatrix},$$

$$\Phi^{21}(q^-) = \begin{bmatrix} \phi_{31} & \phi_{32} \\ \phi_{41} & \phi_{42} \\ \vdots & \vdots \\ \phi_{n1} & \phi_{n2} \end{bmatrix}, \; \Phi^{22} = \begin{bmatrix} \phi_{33}(q^-) & \ldots & \phi_{3n}(q^-) \\ \vdots & \vdots & \vdots \\ \phi_{n3}(q^-) & \ldots & \phi_{nn}(q^-) \end{bmatrix}.$$

In [Čelikovský and Anderle, 2016a] it was shown that for impact invariant VHCs (19) the crucial property (7) holds if and only if the following equations hold

$$\begin{bmatrix} \frac{\partial \phi_3}{\partial q_2}(q_2^+) \\ \vdots \\ \frac{\partial \phi_n}{\partial q_2}(q_2^+) \end{bmatrix} = \begin{bmatrix} \frac{\phi_{31}(q^-)}{\phi_{21}(q^-)} \\ \vdots \\ \frac{\phi_{n1}(q^-)}{\phi_{21}(q^-)} \end{bmatrix},$$

$$\begin{bmatrix} \frac{\partial \phi_3}{\partial q_2}(q_2^-) \\ \vdots \\ \frac{\partial \phi_n}{\partial q_2}(q_2^-) \end{bmatrix} = \left[\begin{bmatrix} 0 & \frac{\phi_{31}(q^-)}{\phi_{21}(q^-)} \\ \vdots & \vdots \\ 0 & \frac{\phi_{n1}(q^-)}{\phi_{21}(q^-)} \end{bmatrix} \Phi^{12} - \Phi^{22} \right]^{-1}$$

$$\times \begin{bmatrix} \phi_{32}(q^-) - \frac{\phi_{31}(q^-)}{\phi_{21}(q^-)}\phi_{22}(q^-) \\ \vdots \\ \phi_{n2}(q^-) - \frac{\phi_{n1}(q^-)}{\phi_{21}(q^-)}\phi_{22}(q^-) \end{bmatrix}, \qquad (21)$$

provided $\phi_{21}(q^-) \neq 0$ and the above inverse exists at q^-. Roughly speaking, derivatives of VHCs according (21) are in a relation given by the impact matrix. As a result of this, corresponding angular velocities at the end of the step after relabeling are equal to angular velocities at the beginning of the step. And by virtue of this, their movement correspond to the periodic orbit.

4 Underactuated Walking Hybrid Stability

The stability analysis of the underactuated robot walking controlled via the VHCs is done here using a method of Poincaré sections. The Poincaré method is a numerical method based on a movement evaluation of a particular robot and, therefore, it is not possible to apply the Poincaré method on a general n-link walking robot in order to test the stability in general. By virtue of this, the representative example from [Čelikovský and Anderle, 2016b], [Čelikovský and Anderle, 2016a] being the so-called four-link will be used here for stability analysis of its walking controlled using the combination of restricted system control and VHCs via the Poincaré method. The four-link was described at the end of the Section 2 and it is depicted in the Fig. 1.

4.1 Method of Poincaré Sections

The Poincaré sections method for stability determination of a biped walking is frequently used in robotics, e.g. in [Wang and Chevallereau, 2011], [Westervelt et al., 2007], Chapter 4. This method will be used here to analyze hybrid stability of the periodic walking-like trajectories described in the previous section.

Before going into the details of the method application, let us discuss the specifics of the designed hybrid periodic trajectory to be analyzed. The walking-like movement designed by the method of the collocated VHCs is the result of the controller that uses target system and a specially designed open loop control component. This open loop control component applied to the target system provides hybrid cyclic walking-like target trajectory. Then, the same open loop control component is feeded into the robot model to be controlled and it is supplemented with the carefully designed feedback component ensuring that the robot model tracks that target system. Such a design during the continuous time swing phase is based on imposing suitable selected collocated VHCs and then controlling the residual restricted system using the method described in the next subsection. In such a way a stable periodic hybrid trajectory may be imposed in the walking robot model.

Unfortunately, the exponential convergence can be proved only during the continuous time phase, called also as the swing phase. To prove overall stability, including the impulsive effects of the impacts, one need to compute numerically the joint effect of the continuous time and the impact phases on the small deviations from ideal target trajectory. This is done by the evaluating the mentioned Poincare sections map.

Note, that the designed periodic trajectory is a result of the tracking of the target system and therefore the robot model is, in fact, forced both by open loop component and the state of that target system, *i.e.* the robot model with controller is a non autonomous system. Due to such a design, one can expect not only the orbital stability, but actually exponentially stable convergence of the robot model trajectory to the target periodic trajectory, **including** the time dependence. In such a way, instead of the expectable 7 eigenvalues of the linear approximation of the Poincare map inside the unit circle of the complex plane, one will get 8 such eigenvalues.

As a matter of fact, the terminology "Poincare sections map" is still acceptable as the map described bellow is the standard Poincare sections map for the extended autonomous system obtained by augmenting the state by the another component representing the time. The augmented component is then hybrid reset at the end of the each step to zero which is possible due to the periodicity of the time dependency of the controlled robot model right hand side. Then, as it can be seen bellow, the appropriate Poincare section is defined by the simple condition that the augmented state component is equal to $T/2$, where T is the step duration, *i.e.* also the period of the investigated periodic trajectory. Finally, note that along the mentioned augmented 9-dimensional trajectory the hybrid matrix of the approximate linearization of the dependence on initial conditions would posses the additional 9th eigenvalue equal to 1. That is thanks to the mentioned reset of the augmented component. Summarizing, the augmented system will be 9 dimensional and its Poincare sections map linear approximation will have 8 eigenvalues inside the unit disk of the complex plane.

The application of the method of Poincaré sections is as follows. Roughly speaking, a solution $\phi(t,x)$ of a system is sampled according to usually event-based or time-based rule and then the stability of an equilibrium point of the sampled system is evaluated. The event-based or time-based rule is in the literature usually called Poincaré section $\mathcal{S}$, which is determined by crossing a plane being transversal to a trajectory of the system solution $\phi(t,x)$. The correspondence between two subsequent crossing of $\mathcal{S}$ by the trajectory of the system solution $\phi(t,x)$ is called in the literature as the Poincaré return map $\mathcal{P}$, $\mathcal{P}:\mathcal{S}\to\mathcal{S}$. In another words, the Poincaré return map $\mathcal{P}$ is a mapping from an initial point $x\in\mathcal{S}$ to the intersection of the surface $\mathcal{S}$ with the solution $\phi(t,x)$, i.e. $\mathcal{P}(x):=\phi(t,x)$.

In our case, the Poincaré section is defined at the middle of the step time $\frac{T}{2}$, where T is total step time. The Poincaré return map is defined by the Poincaré section $\mathcal{S}$ and it represents the evolution of four-link swing phase from this point until the end of the step through the impact phase including change of legs and 4-link swing phase in the next step until it intersects the Poincaré section $\mathcal{S}$ in the middle of the next step.

A point $x^*\in\mathcal{S}$ is called as a fixed point of the Poincaré map if $\mathcal{P}(x^*)=x^*$. The known cyclic motion of coordinates $q,\dot{q}$ gives a unique fixed point $x^*=(q_{1,2,3,4}(\frac{T}{2}),\dot{q}_{1,2,3,4}(\frac{T}{2})$. By definition, the Poincaré return map

$$x[k+1]=\mathcal{P}(x[k]) \tag{22}$$

is a discrete-time system on the Poincaré section $\mathcal{S}$. Define

$$\delta x^z[k]=x^z[k]-x^* \tag{23}$$

the Poincaré return map linearized about the fixed-point x^*, then it gives rise to a linearized system

$$\delta x^z[k+1]=A^z\delta x^z[k], \tag{24}$$

where the (8×8) square matrix A^z is the Jacobian of the Poincaré map and it is computed as follows

$$A^z=[A_1^z\ A_2^z\ A_3^z\ A_4^z\ A_5^z\ A_6^z\ A_7^z\ A_8^z]_{8\times 8}\,, \tag{25}$$

where

$$A_i^z=\frac{P(x^*+\Delta x_i^z)-P(x^*-\Delta x_i^z)}{2\,\Delta x_i^z},\quad i=1,\dots,8, \tag{26}$$

and $\Delta x_i^z=\Delta q_i$ for $i=1,2,3,4$ and $\Delta x_i^z=\Delta\dot{q}_i$ for $i=5,6,7,8$. The fixed-point x^* of the Poincaré return map is locally exponentially stable if, and only if, the eigenvalues of A^z lie inside the unit circle. For more details see e.g. [Westervelt et al., 2007].

The calculation of the matrix A^z requires sixteen evaluations of the Poincaré return map $\mathcal{P}$, two evaluations for each coordinate. Each evaluation of the Poincaré return map is composed of the integration of the swing phase from $t=\frac{T}{2}$ to the collision with the ground, the calculation of the influence of the impact on angular velocities including their relabeling due to switching the swing and the stance leg and relabeling of angular positions and the integration of the swing phase until $t=\frac{T}{2}$.

4.2 Simulations

The evaluation of the Poincaré stability test for the 4-link together with simulation of the 4-link walking during approximately 150 steps was done by virtue of a restricted system control and application of impact invariant VHCs. The restricted system consists of the underactuated angle in the pivot point and the hip angle, i.e. q_1 and q_2 according to Figure 1, respectively. As a result of this, the restricted system is equivalent to two legged walking robot called as the Acrobot and therefore the developed method to its control can be adapted and used to control the restricted system.

The knee angles of the four-link are controlled via VHCs. The special form of two impact invariant VHCs were defined according equation (19) such that the conditions on initial and final derivatives given by equation (21) were fulfilled. Moreover, VHCs include the movement of the swing and the stance leg during the step such that the swing leg do not hit the ground somewhere else then it is proposed, i.e. the VHCs include bending of the swing leg and straightening of the stance leg during the step as well. According to (19) the VHCs can be defined as follows

$$\bar{q}_3 = q_3 - \phi_3(q_2), \quad \dot{\bar{q}}_3 = \dot{q}_3 - \frac{\partial \phi_3(q_2)}{\partial q_2}\dot{q}_2, \\ \bar{q}_4 = q_4 - \phi_4(q_2), \quad \dot{\bar{q}}_4 = \dot{q}_4 - \frac{\partial \phi_4(q_2)}{\partial q_2}\dot{q}_2. \tag{27}$$

Using the partial feedback linearization method originally developed in [Čelikovský et al., 2008] the restricted system dynamics (16) is transformed into partially linear form and using a feedback regulator the restricted system in the partial linear form is controlled along a cyclic walking-like reference trajectory from [Anderle and Čelikovský, 2010]. For the purpose of uniform labeling of coordinates let us to relabel the coordinates q_1 and q_2 as follows

$$\bar{q}_1 = q_1, \quad \dot{\bar{q}}_1 = \dot{q}_1, \quad \bar{q}_2 = q_2, \quad \dot{\bar{q}}_2 = \dot{q}_2. \tag{28}$$

The partial exact feedback linearization method is based on a system transformation into a new system of coordinates that displays linear dependence between some auxiliary output and new (virtual) input. In the case of the Acrobot there are two independent functions with relative degree 3 which transform the original system into the desired partial linearized form with one dimensional zero dynamics [Olfati-Saber, 2002], [Grizzle et al., 2005], namely

$$\sigma = \frac{\partial \mathcal{L}}{\partial \dot{\bar{q}}_1}, \quad p = \bar{q}_1 + \int_0^{\bar{q}_2} \bar{d}_{11}(s)^{-1}\bar{d}_{12}(s)\mathrm{d}s, \tag{29}$$

which can also be used together with the following transformation defined in [Čelikovský et al., 2008]

$$\xi = \mathcal{T}(\bar{q}, \dot{\bar{q}}): \quad \xi_1 = p, \ \xi_2 = \sigma, \ \xi_3 = \dot{\sigma}, \ \xi_4 = \ddot{\sigma} \tag{30}$$

to transform the nonlinear dynamics of the restricted system (16) into the partial linear form. Therefore, transformation (30) together with the property $\dot{p} = \bar{d}_{11}(\bar{q}_2)^{-1}\sigma$ results in the restricted system dynamics in the following partial exact linearized form

$$\dot{\xi}_1 = \bar{d}_{11}(\bar{q}_2)^{-1}\xi_2, \ \dot{\xi}_2 = \xi_3, \ \dot{\xi}_3 = \xi_4, \\ \dot{\xi}_4 = \alpha(\bar{q}, \dot{\bar{q}})\tau_2 + \beta(\bar{q}, \dot{\bar{q}}) = w \tag{31}$$

with the new coordinates ξ and the new input w being well defined wherever $\alpha(\bar{q}, \dot{\bar{q}})^{-1} \neq 0$. Assume, the following reference system is used to generate a reference trajectory via open loop control w^r

$$\dot{\xi}_1^r = \bar{d}_{11}(\bar{q}_2^r)^{-1}\xi_2^r, \dot{\xi}_2^r = \xi_3^r, \dot{\xi}_3^r = \xi_4^r, \dot{\xi}_4^r = w^r. \tag{32}$$

To obtain the exponentially stable state feedback, subtract the original system (31) and the reference one (32) and after application of Taylor expansion the following error dynamics system to be stabilized can be obtained

$$\dot{e}_1 = \mu_1(t)e_1 + \mu_2(t)e_2 + \mu_3(t)e_3 + o(e), \\ \dot{e}_2 = e_3, \ \dot{e}_3 = e_4, \ \dot{e}_4 = w - w^r, \tag{33}$$

where $e := \xi - \xi^r$. Definitions of functions $\mu_{1,2,3}(t)$ are given in [Čelikovský et al., 2008], [Čelikovský et al., 2013]. The cyclic walking reference trajectory to be tracked for the restricted system is generated by the reference system (32) in a way developed in [Anderle and Čelikovský, 2010], i.e. the reference input w^r has the form $w^r = a + bt, \ a, b \in R$, see [Anderle and Čelikovský, 2010] for details. The feedback controller w to track the restricted system along the cyclic walking reference trajectory from [Čelikovský et al., 2013] as follows

$$w = K_1 \frac{e_1 - \mu_3 e_2}{\mu_1\mu_3 - \dot{\mu}_3 + \mu_2} + K_2e_2 + K_3e_3 + K_4e_4 \tag{34}$$

was used, for details see [Čelikovský et al., 2013].

The simulations of the above described controller were performed. The reference trajectory tracking during more than 150 steps is demonstrated by the phase plane plots in Figs. 2-5. As usual, the phase plane plots show trajectories by plotting velocities against positions. Red circles depict the initial conditions. One can easily see the convergence to the cyclic reference angular positions and velocities visible as the thick lines. The thin lines represent the transition to these stable limit cycles. The animations on the last figure then nicely demonstrates convergence to the pre-designed walking-like trajectory. Here, dotted line is the target one, the bold line is the actual four-link movement.

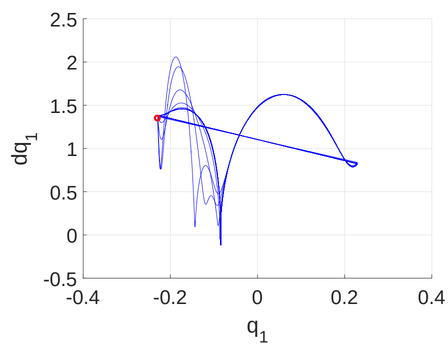

Figure 2. Phase-plane plots for coordinates q_1.

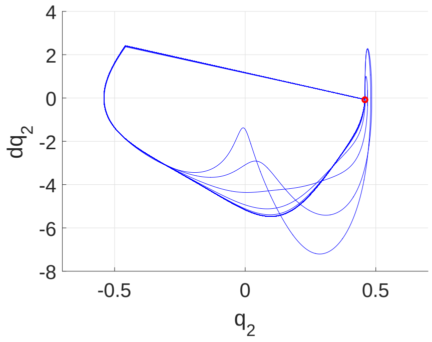

Figure 3. Phase-plane plots for coordinates q_2.

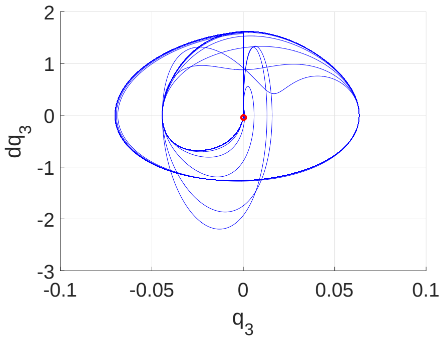

Figure 4. Phase-plane plots for coordinates q_3.

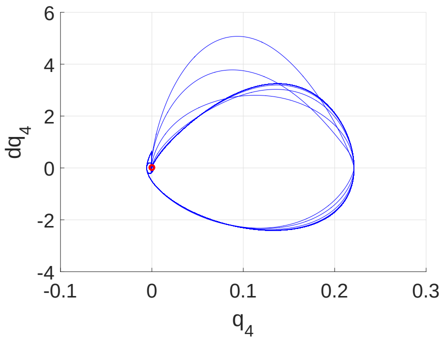

Figure 5. Phase-plane plots for coordinates q_4.

5 Conclusion

The exponential stability of periodic gaits of the four link walking-like mechanical system was demonstrated by numerical computations of the Poincaré map. Both the walking-like gait and its tracking feedback were obtained earlier using the combination of the collocated virtual holonomic constraints concept and the cyclic property of the unactuated variable. The numerical stability test obtained here can be easily adapted to check stability of other hybrid cyclic trajectories as well.

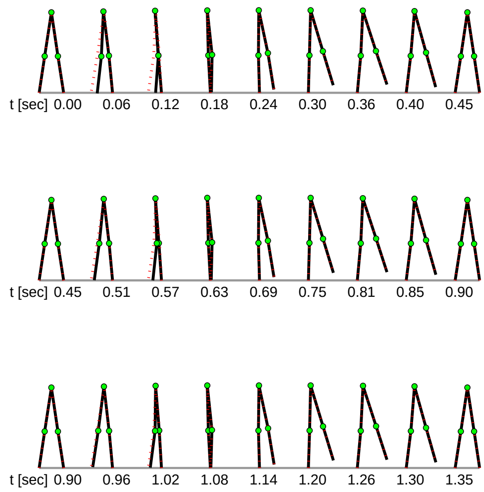

Figure 6. Animation of the 4-link during the initial three steps.

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