OVERALL CHARACTERIZATION OF NON-REGULAR RESPONSES OF THERMOMECHANICAL PSEUDOELASTIC OSCILLATORS BY THE METHOD OF WANDERING TRAJECTORIES

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Abstract
The Method of Wandering Trajectories is applied to the study of the asymptotic response of pseudoelastic oscillators under harmonic forcing to build behavior charts both in initial displacement-initial velocity plane as well as in the excitation frequency-amplitude plane. This provides an overall characterization of the non-regular responses, that confirms that the occurence of chaotic responses is a robust outcome for the system.
Key Words
Chaos, Nonlinear Dynamics, Shape Memory Materials, Method Wandering Trajectories.

1 Introduction

Previous studies on the nonlinear dynamics of pseudoelastic oscillators showed the occurrence of chaotic responses in some ranges of the system parameters (Bernardini and Rega 2005, 2007). In order to understand whether nonregular responses only occur in isolated zones or are actually robust outcomes, the analyses need to be carried out through some synthetic measure of nonregularity that allow for systematic investigations in meaningful parameter spaces.

Whereas the numerical characterization of chaos in smooth dynamical systems is often carried out via the computation of Lyapunov exponents, in the present case the computation of such exponents, following, for example, (Müller 1995), does not seem to be a convenient strategy.

The attention has thus been focused on the simpler direct numerical tool represented by the method of wandering trajectories (Awrejcewicz et al. 2004). This method has been successfully applied in the literature to estimate regular and chaotic responses for non-smooth mechanical oscillators with up to two degrees of freedom (Awrejcewicz et al. 2005) and has been validated and calibrated in (Bernardini and Rega 2005).

The purpose of this paper is to present some results on the overall characterization of the chaotic response of pseudoelastic oscillators.

2 Description of the system

The system under consideration is a simple oscillator where the restoring force is provided by a device with pseudoelastic behavior. The model used for the restoring force has been introduced in (Bernardini and Rega 2005) and fits into the family of models proposed in (Bernardini and Pence 2002) based on two constitutive functions: the free energy and the dissipation function.

The evolution of the system is described, besides by the displacement, velocity and temperature (x,v,J), also by the martensitic phase fraction ?∈ [0,1]. The typical pseudoelastic loops occur as a consequence of the Forward (FwT) and Reverse (RvT) Transformation, respectively associated to increasing and decreasing of *?*.

The nondimensional dynamics of the oscillator is modeled by the following system of four ordinary differential equations in the variables **x**:=[$x,v,?, J$] (for details see (Bernardini and Rega 2005))

$$
\begin{aligned}
&\dot{x} = v \\
&\dot{v} = -x + (s\,l)x - z\,v + g\cos a\,t \\
&\dot{x} = H[sv - Jh(J_e - J)] \qquad (1) \\
&\dot{J} = L\frac{\Lambda + J\,l\,J}{J\,l}H[sv - Jh(J_e - J)] + h(J_e - J)
\end{aligned}
$$

where H and *?* are constitutive functions that take different expressions depending on the kind of transformation (Bernardini and Rega 2007) as well as on the value ?$_0$ of the martensite fraction at the end of the last transformation process. The vector field (1) can thus take three different forms depending on the expressions of H and *?*. However, once activated, each kind of behavior is smooth. The time evolution of ?$_0$ is almost everywhere constant as it jumps from a value to another whenever there is a switch between different kinds of behavior.

The parameters l, L, J, h represent respectively: the length of the pseudoelastic plateaus, the latent heat of transformation, the linear temperature dependence of the transformation forces and the coefficient of convective heat exchange with the environenment. Moreover s=sgn(x) whereas z, g, a ?denote respectively the viscous damping and the excitation amplitude and frequency.

It is noted that not every 5-ple of initial conditions (i.c.) represents a physically admissible state of the system. A procedure to determine admissible i.c. is described in (Bernardini and Rega 2007).

3 The method of the wandering trajectories

The Method of Wandering Trajectories (MWT) is a tool for the characterization of the asymptotic behavior of dynamical systems under periodic forcing excitation. The basic idea is very simple: a motion is classified as non-regular if the separation with a neighboring trajectory overcomes a given threshold. The key issue of the method is thus the proper definition of the perturbations and of the threshold.

Let the fiduciary trajectory to be characterized be denoted as $\mathbf{u}(t)$ with initial condition $\mathbf{u}(0)=\mathbf{u}_0$. For any other trajectory $\tilde{\mathbf{u}}(t)$ such that $\tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0$ initial and current separations are defined as follows

$$\mathbf{h}_0 := \left|\tilde{\mathbf{u}}_0 - \mathbf{u}_0\right| \qquad \mathbf{h}(t) := \left|\tilde{\mathbf{u}}(t) - \mathbf{u}(t)\right| \quad (2)$$

The MWT proceeds as follows:

- integrate $\mathbf{u}(t)$ over T:=[0,T] and compute, over the subinterval T_1:=[t_1,T] where transients expired, for each component, the vibration amplitude of the fiduciary trajectory, defined as follows
$$A_i = \frac{1}{2}\left|\max_{t\in T_1} u_i(t) - \min_{t\in T_1} u_i(t)\right| \qquad (3)$$
this yields a vector **A** measuring the scale of the motion;
- another trajectory $\tilde{\mathbf{u}}(t)$ is defined by the initial condition $\tilde{\mathbf{u}}_0$ derived from $\mathbf{u}_0$ by perturbing each component proportionally to the corresponding vibration amplitude $\mathbf{h}_0 := e\mathbf{A}$;
- define the normalized separations relative to the vibration amplitudes
$$a_i := \frac{|h_i(t)|}{A_i} \qquad (4)$$

Provided they correspond to an admissible state of the system, the initial values $\alpha_i(0)$ are thus equal to the chosen parameter ε.

If the motion is regular the normalized separations either take values of the same order of magnitude as ε or decay to zero. By contrast, non-regular motions may lead, after the transients, to normalized separations much higher than e; trajectories initiated from two nearby points on a chaotic attractor separate away from each other until the separation levels off at the size of the attractor.

The main assumption of the MWT is that, with some preliminary knowledge of the system dynamics, it is possible to determine a threshold level $\bar{a}$ for the normalized separation that characterizes the occurrence of non-regular motions. In particular, a trajectory $\mathbf{u}(t)$ is characterized as non-regular if, at some $\bar{t} \in T_1$, the normalized separation with respect to a trajectory with initial separation $\mathbf{h}_0$=ε$\mathbf{A}$, exceeds the chosen threshold

$$a_i(\bar{t}) > \bar{a}\ . \qquad (5)$$

This test certainly detects the sensitivity to initial conditions of the trajectory. However this is only a necessary but not sufficient condition for the motion to be chaotic. In facts, the sensitivity to initial conditions alone only indicates that the perturbation may have taken the trajectory outside the basin of attraction of the attractor. A chaotic motion, besides being sensitive to initial conditions, is also wandering in the sense that it attempts to fill a bounded region K of the phase space (for any $\mathbf{q} \in K$ there is a time t such that $\mathbf{u}(t)=\mathbf{q}$). Therefore the MWT tends to overestimate the number of non-regular trajectories, which turns out to be in favour of safety from an engineering viewpoint.

4 Comparison with bifurcation diagrams

The constitutive model for the restoring force covers a great variety of situations. The following set of parameters, as in (Bernardini and Rega 2005), is considered as reference and in the following is referred to as RMP

?=8.125,	J=3.1742,	L=0.124,
q_1=0.98,	q_2=1.2,	q_3=1.0246,
h=0.08,	a=0.03	?=0.03

These parameters correspond to a typical pseudoelastic cycle in a mildly convective environment. For the physical meaning of the parameters see (Bernardini and Rega 2005).

To obtain an overall picture of the system behavior, a constant excitation amplitude (?=1) bifurcation diagram with the frequency a as control parameter has been computed.

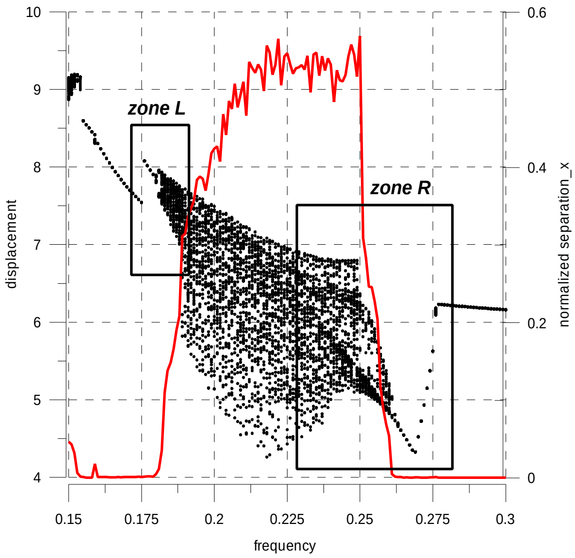

Fig. 1. Bifurcation diagram and normalized separations.

The diagram is obtained by decreasing frequency with variable initial conditions taken from the adjacent computation point, and the region $a \in (0.15,0.3)$ is reported in Figure 1.

In the same frequency interval a systematic application of the MWT has been done for comparison. In particular, for each frequency, the response has been computed for T=200 periods while checking the normalized separations with respect to trajectories perturbed by e=0.01 on T_1=100 periods.

On the same figure a curve depicting the results of the MWT is superposed. For each frequency the curve (to be read with respect to the right vertical axis) shows the maximum value over T_1 of the normalized separation of the displacement. It turns out that, whenever the trajectories are periodic, the separation remains practically 0. On the contrary, when the separation overcomes values of about 0.1, a slightly chaotic behavior is already observed. Values of the separation above 0.3 are definitely associated with consolidated chaos.

5 Overall characterization of the non-regular solutions and effect of the hysteresis

The robustness of the chaotic response within the overall behavior of the system can now be investigated by computing behavior charts in which some control parameters are varied and the MWT is systematically applied to distinguish between regular and nonregular responses. A natural choice for the control parameters is the pair excitation frequency-amplitude at fixed initial conditions and material parameters.

In particular the analysis has been carried out for the above mentioned set of material parameters RMP as well for another set, called MP1, obtained from RMP by decreasing q_2 from 1.2 to 1.02. The parameters MP1 correspond to a pseudoelastic loop with lower hysteresis with respect to RMP. The comparison between the two provide information about the effect of the hysteresis on the chaotic response.

According to the previous analyses the threshold level for the normalized separations has been chosen as $\bar{a}$ =0.3. Integration of the trajectories has been carried out for 200 excitation periods, while restricting the interval T_1 to the last 100 periods. Due to the complexity of the trajectories occurring in some parameter regions, the application of the method requires a rather fine numerical integration. After calibration of various explicit and implicit integration algorithms, a reasonable compromise between accuracy and computational time has been reached by using a standard fourth-order Runge-Kutta algorithm with 2000 steps per period.

Preliminarily, an investigation has been carried out in the initial conditions domain. More specifically, the MWT has been first applied to build a section of a kind of basin of attraction of chaotic responses in the plane of initial displacement x_0 and velocity v_0. Initial conditions $x_0 \in [-1, 1]$ and $v_0 \in [-1, 1]$ have been considered together with $x = x_0 = 0$ and $J = J_0 = 1$. These values can be shown to be all admissible and correspond to the device in elastic, purely austenitic, phase.

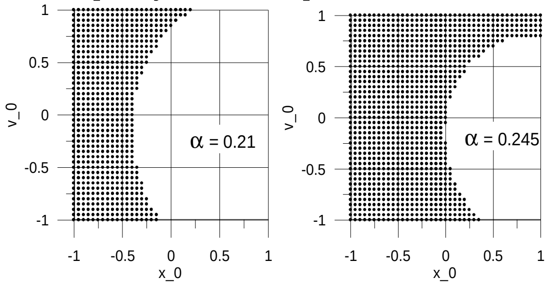

Fig. 2. Regions of nonregular response in initial conditions plane (white: regular, black dot: non-regular).

Two sample domains corresponding to the excitation amplitude g=1 and different frequencies a =0.245 and a=0.21 are shown in Figure 2 (with RMP). At both frequencies, nonregular responses occur for various initial conditions. Analogous responses occur at the other frequencies where chaos is found. From consideration of such analyses, the pair (x_0, v_0)=(-1.0,-1.0) has been selected as fixed initial condition, together with x_0=0 and J_0=1, for the subsequent investigations.

The frequency-amplitude behavior chart for the basic set of parameters RMP is shown in Figure 3.

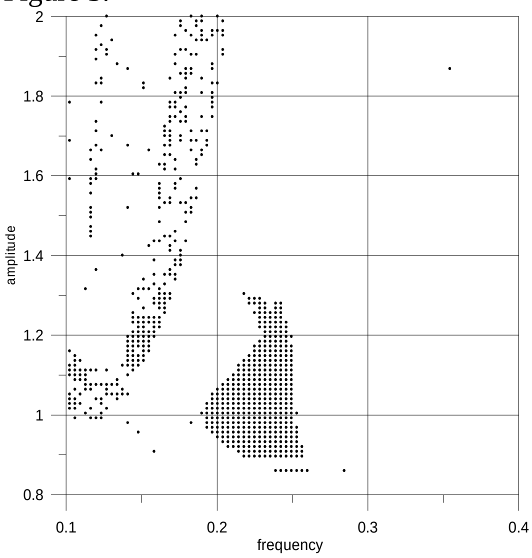

Fig. 3 Behavior chart in excitation frequency-amplitude plane for RMP (white: regular, black dot: non-regular).

For g=1, two clearly separated regions of non-regular motion are found, a compact one on the right, a more scattered one on the left. They are likely to correspond with the two kinds of chaotic motions highlighted in (Bernardini and Rega 2005) by bifurcation diagrams. The presence of scattered points, especially at the higher excitation amplitudes, can be eliminated by a finer numerical integration.

The same chart has then been computed with MP1 material parameters (Figure 4).

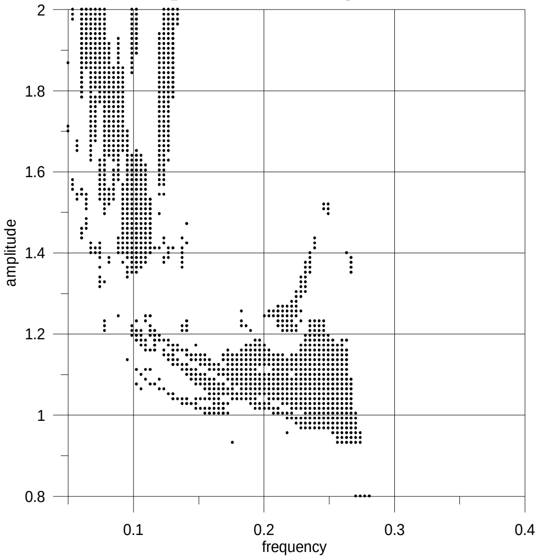

Fig. 4. Behavior chart in excitation frequency-amplitude plane for MP1 (white: regular, black dot: non-regular).

It turns out, as expected, that decreasing hysteresis leads to a significant increase of the size of the regions of irregular motion, with the intermediate region tending to cluster in nearly vertical stripes at lower frequencies.

In-depth understanding of the kind of non-regular motion with respect to the neighbouring regular one would require complementing the chart with a number of bifurcation diagrams with frequency as control parameter (this is left for future investigations). Overall, the charts show that the chaotic motions are robust and persist in significant regions of the excitation amplitude and frequency plane.

6 Conclusions

The Method of Wandering Trajectories has been shown to be effective in detecting the sensitivity to initial conditions of the orbits of a thermomechanically based pseudoelastic oscillator. The occurrence of chaotic reponses has been characterized via excitation frequency-amplitude charts for two sets of material parameters. The results confirm that, although an increase of the hysteresis in the system tends to reduce chaotic motions, even in the reference case the occurrence of chaos is a robust outcome taking place in large regions of the frequency-amplitude plane.

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