ASYMPTOTIC SOLUTION AND STABILITY OF AUTOPARAMETRICAL SYSTEMS

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Abstract
The analysis of vibrations of the weak nonlinear smooth auto-parametrical non autonomous two degree of freedom systems was made. The pendulum of changing length is an example of such a system. A multiple scales method of investigation of small vibrations is applied to the analysis of resonance. The obtained results were confirmed numerically. Analytical calculation was made with the use of *Mathematica*.

Key words
nonlinear dynamics, parametrical resonance, asymptotic method

1. Introduction

Dynamical systems including mathematical or physical pendulum play significant role in technology. In such systems one can observe an auto-parametric resonance phenomena, because of the coupling occurring in the equation of motion. The phenomenon of energy transfer from one of the mode of vibration to the other was widely discussed in [Karamyskin].

Dynamical analysis of nonlinear vibrations of spring pendulum (Figure 1) was presented in the paper.

The main goal of the analysis is recognition of primary and parametric resonances using the multiple scales method. The program in the computer algebra system *Mathematica* was elaborated. It enables automatizing many transformations in the perturbation method use for this purpose [Starosta, Awrejcewicz].

The phenomenon of energy transferring between the modes of vibrations occurs in the vicinity of the parametric resonance. The multiple scale method makes possible to recognize parameters of the system that are dangerous due to the resonance and allows to make a time history for the assumed generalized co-ordinates.

There are many technical applications of the systems containing various types of pendulums. Discussion on such models with a view to damping vibrations may be found in [Sado; Shivamoggi; Genin and Ryabow].

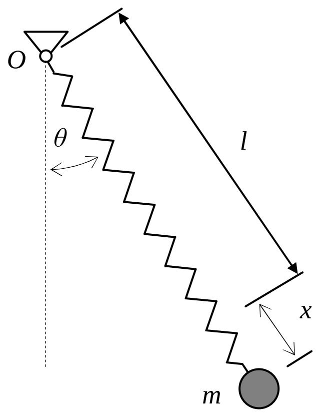

Figure 1. Spring pendulum.

All the calculations presented in the paper, both analytical and numerical transformations, were made with the help of computer algebra system Mathematica.

2. Spring pendulum

The kinetic and potential energy of the examined system has the form:

$$T = \frac{1}{2} m\left[\dot{x}^2 + (l+x)^2 \dot{\theta}^2\right]$$
$$V = mg(l+x)(1-\cos\theta) + \frac{1}{2} kx^2 \tag{1}$$

where m – mass of the pendulum, l – the length, k – stiffness of the spring, g Earth’s acceleration and x

and θ – generalized co-ordinates admitted according to Fig.1.

Applying Lagrangian equation and taking into account external excitation as a force acting on the mass along the pendulum the following equation of motion are obtained:

$$m\ddot{x} + kx - \dot{\theta}^2 m(l+x) - gm\frac{\theta^2}{2} = f\cos\Omega t$$
$$\ddot{\theta} + \frac{g}{l+x}\theta + \frac{2}{l+x}\dot{x}\dot{\theta} = 0 \tag{2}$$

Expansion of trigonometric functions

$$\sin\theta \cong \theta - \frac{1}{3!}\theta^3 \text{ and } \cos\theta \cong 1 - \frac{1}{2}\theta^2,$$

are admitted, assuming that vibrations are small.

Let us introduce dimensionless generalized co-ordinates (with tilde) $\frac{x}{l} = \varepsilon\tilde{x}$, $\theta = \varepsilon\tilde{\theta}$. Let us assume the forcing amplitude in the form $f = \varepsilon^2\tilde{f}$ Equations (2) can be rewritten in the form:

$$\ddot{\tilde{x}} + \omega_1^2\tilde{x} = \varepsilon\left(\dot{\tilde{\theta}}^2 - G_1\tilde{\theta}^2\right) + \varepsilon F\cos\Omega t$$
$$\ddot{\tilde{\theta}} + \omega_2^2\tilde{\theta} = -\varepsilon\left(2\dot{\tilde{x}}\dot{\tilde{\theta}} + \tilde{x}\ddot{\tilde{\theta}}\right) \tag{3}$$

where

$F \equiv \frac{\tilde{f}}{lm}$, $\omega_1^2 \equiv \frac{k}{m}$, $\omega_2^2 \equiv \frac{g}{l}$, $G_1 \equiv \frac{g}{2l}$, $\varepsilon << 1$, is a so-called small parameter.

In order to simplify the notation, the sign ~ (tilde) will be omitted below.

The solution of (4) is sought in the following form

$$x(t;\varepsilon) \approx x_0(T_0,T_1) + \varepsilon\, x_1(T_0,T_1) + O(\varepsilon^2)$$
$$\theta(t;\varepsilon) \approx \theta_0(T_0,T_1) + \varepsilon\, \theta_1(T_0,T_1) + O(\varepsilon^2) \tag{4}$$

where $T_0 = t$, $T_1 = \varepsilon t$ is a scale of time of slow changing processes.

The original set of equations will be transformed to the set of partial differential equations because of differentiating of the compound functions according to:

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1}$$
$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon\frac{\partial^2}{\partial T_0\,\partial T_1} \tag{5}$$

Substitution of (4) and (5) to (3) leads to the equations:

(order ε^0)

$$\frac{\partial^2 x_0}{\partial T_0^2} + \omega_1^2 x_0 = 0$$
$$\frac{\partial^2\theta_0}{\partial T_0^2} + \omega_2^2\theta_0 = 0 \tag{6}$$

(order ε^1)

$$\frac{\partial^2 x_1}{\partial T_0^2} + \omega_1^2 x_1 + 2\frac{\partial^2 x_0}{\partial T_0\,\partial T_1} + G_1\theta_0^2 - \left(\frac{\partial\theta_0}{\partial T_0}\right)^2 +$$
$$+ F\cos(\Omega T_0) = 0$$
$$\frac{\partial^2\theta_1}{\partial T_0^2} + \omega_2^2\theta_1 + 2\frac{\partial^2\theta_0}{\partial T_0\,\partial T_1} - x_0\frac{\partial^2\theta_0}{\partial T_0^2} + \tag{7}$$
$$+ 2\frac{\partial x_0}{\partial T_0}\frac{\partial\theta_0}{\partial T_0} = 0$$

The solution of (6) may be found as follows

$$x_0 = A(T_1)e^{i\omega_1 T_0} + \overline{A}(T_1)e^{-i\omega_1 T_0}$$
$$\theta_0 = B(T_1)e^{i\omega_2 T_0} + \overline{B}(T_1)e^{-i\omega_2 T_0}. \tag{8}$$

Substitution of (8) to (7) gives the first order equation in the form

$$\frac{\partial^2 x_1}{\partial T_0^2} + \omega_1^2 x_1 = -\frac{1}{2}Fe^{iT_0\Omega} - e^{2iT_0\omega_2}\left(G_1 + \omega_2^2\right)B^2 -$$
$$2G_1 B\overline{B} + 2\omega_2^2 B\overline{B} - 2ie^{iT_0\omega_1}\frac{dA}{dT_1} + c.c.$$
$$\frac{\partial^2\theta_1}{\partial T_0^2} + \omega_2^2\theta_1 = -e^{iT_0(\omega_1+\omega_2)}\left(-2\omega_1\omega_2 - \omega_2^2\right)AB + \tag{9}$$
$$- e^{iT_0(\omega_1-\omega_2)}\left(2\omega_1\omega_2 - \omega_2^2\right)A\overline{B} + 2ie^{i\omega_2 T_0}\omega_2\frac{dB}{dT_1} + c.c.$$

where *c.c.* represents the complex conjugates

In the above equations the arguments of A and B are omitted in order to shorten the notation.

Removal of secular terms in equations (9), requires

$$\frac{dA(T_1)}{dT_1} = 0 \quad \text{and} \quad \frac{dB(T_1)}{dT_1} = 0. \tag{10}$$

The solution of (9) is then

$$
\begin{aligned}
x_1 &= F\,\frac{e^{iT_0\Omega}}{2\left(\Omega^2-\omega_1^2\right)} - \frac{\left(G_1+\omega_2^2\right)B^2}{-\omega_1^2+4\omega_2^2}\,e^{2\,i\,T_0\omega_1} \\
&+ \frac{\left(-G_1+\omega_2^2\right)B\overline{B}}{\omega_1^2} + c.c. \\
\theta_1 &= -\frac{\omega_2\left(2\omega_1+\omega_2\right)}{\omega_1\left(\omega_1+2\omega_2\right)}ABe^{\,i(\omega_1+\omega_2)T_0} + \\
&\frac{\omega_2\left(2\omega_1-\omega_2\right)}{\omega_1\left(\omega_1-2\omega_2\right)}A\overline{B}e^{\,i(\omega_1-\omega_2)T_0} + c.c.
\end{aligned}
\tag{11}
$$

The above solution becomes singular when the primary or internal resonances occurs i.e. when $\omega_1 = \Omega$ and/or $\omega_1 = 2\omega_2$. In order to treat this resonances case, we can introduce the new parameter σ_1 and σ_2 according to:

$$
\Omega = \omega_1 + \varepsilon\,\sigma_1 \quad 2\omega_2 = \omega_1 + \varepsilon\sigma_2 \tag{12}
$$

The equations (9) can be written now as follows:

$$
\begin{aligned}
&\frac{\partial^2 x_1}{\partial t_0^2} + \omega_1^2 x_1 =^2 -2G_1B\overline{B} + 2\omega_2^2 B\overline{B} \\
&- e^{iT_0\omega_1}\left(\frac{1}{2}e^{iT_1\sigma_1}F + e^{iT_1\sigma_2}G_1B^2 + e^{iT_1\sigma_2}\omega_2^2B^2 + 2i\omega_1\frac{dA}{dT_1}\right) + c.c. \\
&\frac{\partial^2\theta_1}{\partial t_0^2} + \omega_2\theta_1 = e^{3iT_0\omega_2}\left(e^{-iT_1\sigma_2}AB\left(2\omega_1\omega_2+\omega_2^2\right)\right) + \\
&e^{iT_0\omega_2}\left(e^{-iT_1\sigma_2}2\omega_1\omega_2A\overline{B} - \omega_2^2A\overline{B} + 2i\omega_2\frac{dB}{dT_1}\right) + c.c.
\end{aligned}
\tag{13}
$$

Removal of secular terms from (13) requires:

$$
\begin{aligned}
&\frac{1}{2}e^{iT_1\sigma_1}F + e^{iT_1\sigma_2}G_1B^2 + e^{iT_1\sigma_2}\omega_2^2B^2 + 2i\omega_1\frac{dA}{dT_1} = 0 \\
&e^{-iT_1\sigma_2}2\omega_1\omega_2A\overline{B} - e^{-iT_1\sigma_2}\omega_2^2A\overline{B} + 2i\omega_2\frac{dB}{dT_1} = 0
\end{aligned}
\tag{14}
$$

In order to present the solution of (14) in more familiar form:

$$
\begin{aligned}
x &= a\cos(\omega_1 t + \alpha) \\
\theta &= b\cos(\omega_2 t + \beta)
\end{aligned},
\tag{15}
$$

the following substitution can be made [Awrejcewicz and Krysko]:

$$
A \to \frac{a}{2}e^{i\alpha}, \qquad B \to \frac{b}{2}e^{i\beta} \tag{16}
$$

The introduced amplitudes a and b and phases α and β are functions of T_1.

With the use of the above substitution (16), the equations (14) lead to expressions of derivatives of the sought functions looked for:

$$
\begin{aligned}
\frac{da}{dT_1} &= -\frac{2F\sin\xi + \left(G_1+\omega_2^2\right)b^2\sin\eta}{4\omega_1} \\
\frac{d\alpha}{dT_1} &= -\frac{2F\cos\xi + \left(G_1+\omega_2^2\right)b^2\cos\eta}{4\omega_1 a} \\
\frac{db}{dT_1} &= \frac{1}{4}\left(2\omega_1-\omega_2\right)ab\sin\eta \\
\frac{d\beta}{dT_1} &= \frac{1}{4}\left(2\omega_1-\omega_2\right)a\cos\eta
\end{aligned}
\tag{17}
$$

where

$$
\xi = T_1\,\sigma_1 - \alpha(T_1) \text{ and } \eta = T_1\,\sigma_2 - \alpha(T_1) + 2\beta(T_1). \tag{18}
$$

Solving the above equations lets to obtain the information about modulations of amplitudes $a(T_1)$ and $b(T_1)$, and phases $\alpha(T_1)$ and $\beta(T_1)$ respectively.

Equations (17) are non-autonomous system because right-hand side depends on the independent variable T_1. They can be transformed into an autonomous system of equations by expressing $\frac{d\alpha}{dT_1}$ and $\frac{d\beta}{dT_1}$ in terms of $\frac{d\xi}{dT_1}$ and $\frac{d\eta}{dT_1}$.

$$
\frac{d\alpha}{dT_1} = \sigma_1 - \frac{d\xi}{dT_1} \text{ and } \frac{d\beta}{dT_1} = \frac{1}{2}\left(\frac{d\eta}{dT_1} - \sigma_2 + \sigma_1 - \frac{d\xi}{dT_1}\right) \tag{19}
$$

Substituting (19) into $(17)_2$ and $(17)_4$ yields

$$
\begin{aligned}
&\frac{d\xi}{dT_1} = \left(4\sigma_1\omega_1 a - 2F\cos\xi - \left(G_1+\omega_2^2\right)b^2\cos\eta\right) \\
&/\,4\omega_1 a \\
&\frac{d\eta}{dT_1} = \left(4\sigma_2\omega_1 a + 2\omega_1\left(2\omega_1-\omega_2\right)a^2\cos\eta - \right. \\
&\left.\left(G_1+\omega_2^2\right)b^2\cos\eta - 2F\cos\xi\right)/\,4\omega_1 a
\end{aligned}
\tag{20}
$$

Equations $(17)_1$, $(17)_3$ and (20) form an autonomous system of equations. The fixed points of these equations correspond to $\frac{da}{dT_1}=0$, $\frac{db}{dT_1}=0$, $\frac{d\xi}{dT_1}=0$, $\frac{d\eta}{dT_1}=0$. They can be written in the form:

$$\frac{2F\sin\xi + \left(G_1+\omega_2^2\right)b^2\sin\eta}{4\omega_1} = 0$$

$$\frac{-4\sigma_1\omega_1 a + 2F\cos\xi + \left(G_1+\omega_2^2\right)b^2\cos\eta}{4\omega_1 a} = 0$$

$$\frac{1}{4}(2\omega_1-\omega_2)ab\sin\eta = 0 \qquad (21)$$

$$\big(4\sigma_2\omega_1 a + 2\omega_1(2\omega_1-\omega_2)a^2\cos\eta - \left(G_1+\omega_2^2\right)b^2\cos\eta - 2F\cos\xi\big)/4\omega_1 a = 0$$

3. Stability of the fixed points

To analyze the fixed points let us to write a disturbed equations. Lets

$$a = a_0 + a_1,\ \xi = \xi_0 + \xi_1,\ b = b_0 + b_1,\ \eta = \eta_0 + \eta_1 \quad (22)$$

where a_0, ξ_0, b_0, η_0 are solutions of the equations (21) and a_1, ξ_1, b_1, η_1 are perturbations which are assumed to be small compared to a_0, ξ_0, b_0, η_0. Now (22) are substituted into $(17)_{1,3}$ and (19) and the resulting equations are linearized. Taking into account that steady state values satisfy equations (21), we get

$$\begin{aligned}
\dot a_1 &= C_1 b_0^2\cos(\eta_0)\eta_1 + C_2 2b_0\sin(\eta_1)b_1 + C_2\cos(\xi_0)\xi_1\\
a_0\dot\xi_1 &= -\sigma_1 a_1 + C_1 2b_0\cos(\eta_0)b_1 + C_1 b_0^2\sin(\eta_0)\eta_1 - C_2\sin(\xi_0)\xi_1\\
\dot b_1 &= C_3 b_0\sin\eta_0\, a_1 + C_3 a_0\sin(\eta_0)b_1 + C_3 a_0 b_0\cos(\eta_0)\eta_1\\
a_0\dot\eta_1 &= (\sigma_2 + 4C_3 a_0\cos(\eta_0))a_1 - 2C_1 b_0\cos(\eta_0)b_1 + \left(C_1 b_0^2\sin(\eta_0) - 2C_3 a_0^2\sin(\eta_0)\right)\eta_1 + C_2\sin(\xi_0)\xi_1
\end{aligned} \qquad (23)$$

where

$$C_1 = \frac{G_1+\omega_2^2}{4\omega_1},\ C_2 = \frac{F}{2\omega_1},\ C_3 = \frac{2\omega_1-\omega_2}{4}.$$

If the steady state solution a_0, ξ_0, b_0, η_0 is asymptotically stable then the real parts of the roots of matrix of set of equations (23) should be negative. For stability of the particular fixed point obtained from (21) eigenvalues are given by the equation

$$\lambda^4 + \Gamma_1\lambda^3 + \Gamma_2\lambda^2 + \Gamma_3\lambda + \Gamma_4 = 0 \qquad (24)$$

were $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ are functions of the parameters $a_0, b_0, \xi_0, \eta_0, \omega_1, \omega_2, \sigma_1, \sigma_2, F$. They are given in the appendix. According to the Routh-Hurwitz criterion the necessary and sufficient conditions for all the roots of equation (24) to possess negative real parts is that

$$\Gamma_1 > 0,\ \Gamma_3(\Gamma_1\Gamma_2 - \Gamma_3) - \Gamma_4\Gamma_1^2 > 0,\ \Gamma_3 > 0,\ \Gamma_4 > 0. \quad (25)$$

3 Examples

The set (17) was solved numerically by Runge-Kutta method for various parameters of the system.

3.1 Main and internal resonance

The modulations of amplitudes are shown in the Figs. 2-3. The admitted parameters: $\Omega = 3,\ \omega_1 = 3,\ \omega_2 = 2,\ \varepsilon = 0.1,\ f = 0.1$ and initial conditions $a(0) = 0.1, b(0) = 0.1, \alpha(0) = 0, \beta(0) = 0$.

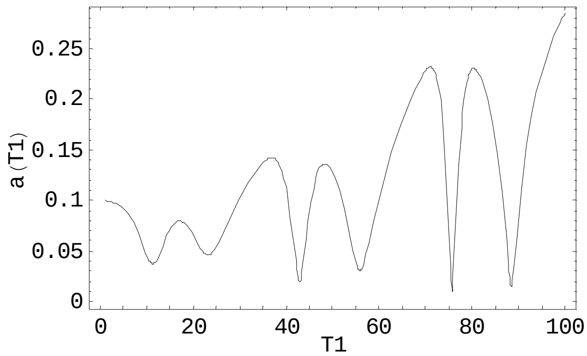

Figure 2. Modulation of amplitude of $x(T_1)$ (combined resonances).

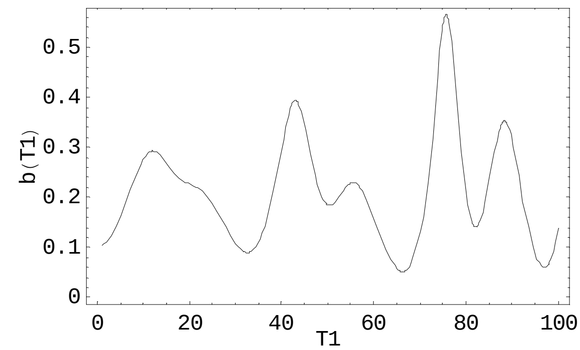

Figure 3. Modulation of amplitude of $\theta(T_1)$ (combined resonances).

Time history of motion in time is presented in the Figures below. In the Fig. 4 the solution is obtained by multiple-scale-method while in Fig. 5 by numerical solution of eq. (3). The results are very similar that confirms correctness of calculation. The numerical solution was also positively verified by analysis of the mechanical energy.

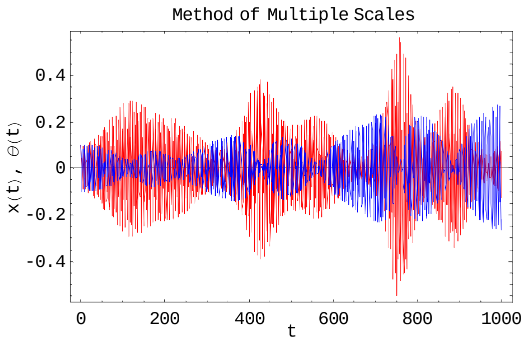

Figure 4. Time history of $x(t)$ —— and $\theta(t)$ —— (combined resonances).

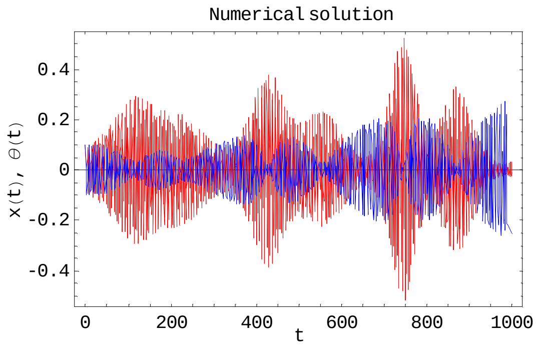

Figure 5. Time history of $x(t)$ —— and $\theta(t)$ —— (combined resonances).

Results presented in Figs. 2-5 suggest that combination of two resonances causes a chaotic behavior. It is confirmed by Poincare map showed in the Fig 6., made for amplitudes *a* and *b*.

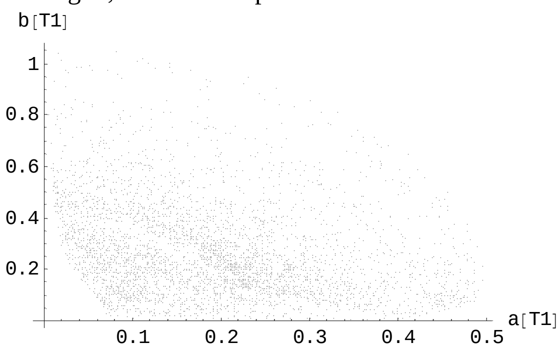

Figure 6. Poincare map for amplitudes.

Let us to drive the largest Lapunov coefficient λ versus frequency of the exciting force (Fig. 7).

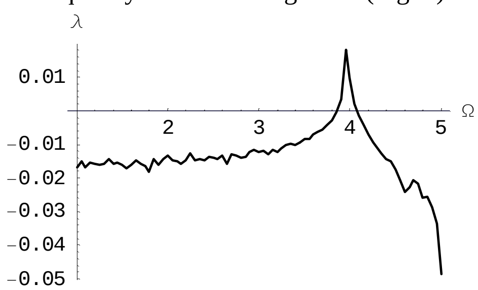

Figure 7. The largest Lapunov exponent λ.

3.2 Internal resonance

In that case the energy exchange between the modes of vibrations is clearly indicates. The time history for both co-ordinates with the same parameters and initial conditions as before is presented in Figure 8. Now frequencies are as follows: $\Omega = 5$, $\omega_1 = 4$, $\omega_2 = 2$. The other parameters are the same as before.

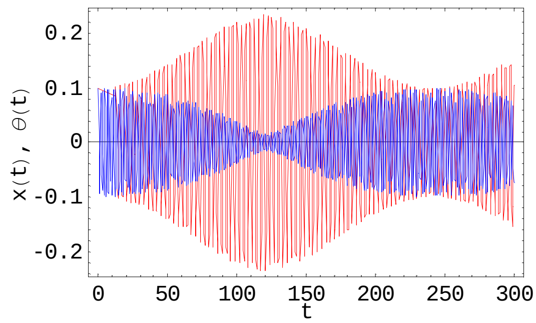

Figure 8. Time history of $x(t)$ —— and $\theta(t)$ —— (internal resonance).

3.3 Primary resonance

In Figure 9 the increase of amplitude appears due to same values of Ω and ω_1 ($\Omega = 3, \omega_1 = 3, \omega_2 = 2$). In the Figure 4 we can see the time history of generalized co-ordinates.

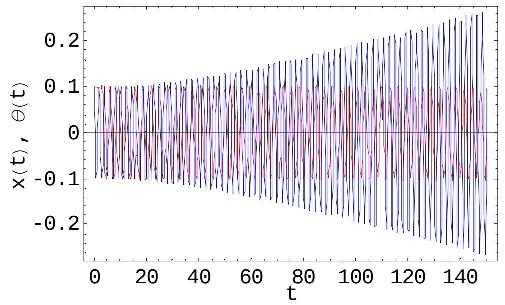

Figure 9. Time history of $x(t)$ —— and $\theta(t)$ —— (main resonance).

4. Conclusions

The transformations within the multiple-scale-method were carried out automatically with the use of a procedure elaborated in *Mathematica*. Analytical results obtained by the multi-scale perturbation method were confirmed numerically.

The results show that both quantitative and qualitative analyses of nonlinear dynamical systems can be made by the multiple-scale-method in time domain.

The method allow to recognize the parameters of the system with respect to the occurring resonance.

Stability of the motion is also carried out by the procedure written in *Mathematica*.

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Appendix

Coefficients of the characteristic equation (24)

$$\Gamma_1 = -b_0^2 \sin\eta_0 C_1 + a_0 \sin\eta_0 C_3 (2a_0 - 1) + C_2 \sin\xi_0$$

$$\begin{aligned}
&\Gamma_2 = 2a_0 b_0^2 \cos^2\eta_0 C_1 C_3 + a_0 \sin\eta_0 b_0^2 \sin\eta_0 C_1 C_3 - \\
&2b_0^2 \sin^2\eta_0 C_1 C_3 - 2a_0^3 \sin^2\eta_0 C_3^2 - a_0 \sin\eta_0 C_2 C_3 \sin\xi_0 + \\
&2a_0^2 \sin\eta_0 C_2 C_3 \sin\xi_0 + C_2 \cos\xi_0 \sigma_1 - \\
&b_0^2 \cos\eta_0 C_1 (4a_0 \cos\eta_0 C_3 + \sigma_2)
\end{aligned}$$

$$\begin{aligned}
&\Gamma_3 = -C_3\Big(C_2\big(2b_0^2 \sin\eta_0 \cos\eta_0 C_1 \cos\xi + 2b_0^2 \sin^2\eta_0 C_1 \sin\xi_0 \\
&+ a_0 \sin\eta_0 \cos\xi_0 \sigma_1\big) + 2a_0\big(4a_0 b_0^2 \cos^2\eta_0 \sin\eta_0 C_1 C_3 \\
&+ 2a_0 b_0 \sin^3\eta_0 C_1 C_3 - 2b_0^2 \cos\eta_0 C_1 C_2 \cos\xi_0 \\
&+ 2a_0^2 \sin\eta_0 C_2 C_3 \sin\xi - a_0 \sin\eta_0 C_2 \cos\xi_0 \sigma_1 + \\
&+ b_0^2 \cos\eta_0 \sin\eta_0 C_1 \sigma_2\big)\Big) + b_0 C_1\big(2b_0^3 \cos^2\eta_0 \sin\eta_0 C_1 C_3 \\
&+ bc(4a_0 \cos\eta_0 C_3 (a_0 \sin\eta_0 - C_2 \sin\xi_0) + a_0 \sin\eta_0 C_3 \sigma_2 \\
&- C_2 \sin\xi_0 (\sigma_1 + \sigma_2)) + b_0 \sin\eta_0 \big(2b_0^2 \sin^2\eta_0 C_1 C_3 \\
&- C_2 \cos\xi_0 (4a_0 \cos\eta_0 C_3 + \sigma_1 + \sigma_2)\big)\big)
\end{aligned}$$

$$\begin{aligned}
&\Gamma_4 = C_2 C_3 \big(b_0 C_1 (b_0 \sin\eta_0 \cos\xi + b_0 \cos\eta_0 \sin\xi_0) \\
&\big(2b_0^2 \sin(\eta_0 / 2) C_1 + a_0 \sin\eta_0 (4a_0 \cos\eta_0 C_3 + \sigma_1 + \sigma_2)\big) \\
&- 2a_0 \big(4a_0 b_0^2 \cos^2\eta_0 C_1 C_3 (\cos\eta_0 \cos\xi_0 + \sin\eta_0 \sin\xi_0) \\
&+ a_0 b_0^2 \sin^2\eta_0 C_1 C_3 (\cos\eta_0 \cos\xi_0 + \sin\eta_0 \sin\xi_0) \\
&+ a_0^2 \cos^2\eta_0 C_3 \cos\xi_0 \sigma_1 + b_0 \cos\eta_0 C_1 (b_0 \cos\eta_0 \cos\xi_0 \\
&(\sigma_2 - \sigma_1) + b_0 \sin\eta_0 \sin\xi_0 (\sigma_1 + \sigma_2))))
\end{aligned}$$