CYBERNETICS AND PHYSICS, VOL. 5, NO. 4, 2016 , 111–115

STOCHASTIC SENSITIVITY AND STABILIZATION OF OPERATION MODE FOR RANDOMLY FORCED SEMICONDUCTOR IMAGE CONVERTER

Irina Bashkirtseva
Institute of Mathematics and Computer Sciences
Ural Federal University
Russia
irina.bashkirtseva@urfu.ru

Abstract
A problem of stochastic excitability analysis and suppression of undesirable random fluctuations is considered. This problem is studied on the example of the nonlinear dynamical model describing operation of a semiconductor–gas–discharge image converter. We show that the stochastic excitability of this system can be explained by the high stochastic sensitivity of its equilibrium. For the stabilization of the operating mode of the stochastic gas discharge system, we suggest a new constructive approach based on the idea to reduce the stochastic sensitivity of the equilibrium by the appropriate feedback regulator. A mathematical background of the analysis and control of the stochastic sensitivity is presented. We show that using this control approach, one can suppress large-amplitude oscillations and provide a proper operation of the considered engineering device.

Key words
Stochastic sensitivity, random fluctuations, semiconductorgas-discharge image converter, feedback regulator.

1 Introduction

Many engineering devices and manufacturing processes are modelled by nonlinear dynamic systems. Required operating modes are typically associated with the corresponding stable equilibria. However, in some cases, the deterministic stability of the equilibrium is insufficient. Inevitable random disturbances can drastically change the dynamics of the system. The interaction of nonlinearity and stochasticity can cause many unexpected noise-induced phenomena [Horsthemke and Lefever, 1984; Lai and Tel, 2011; Lindner et al, 2004]. In excitable systems, even weak noise results in the generation of large-amplitude stochastic oscillations that are unacceptable from an engineering point of view. As a rule, similar effects are due to the high stochastic sensitivity [Bashkirtseva et al, 2012; Bashkirtseva et al, 2013] of such systems.

Along with the analysis of nonlinear stochastic systems, control problems are also widely studied [Kushner, 1967; Astrom, 1970; Sun, 2006; Guo and Wang, 2010]. In present paper, we suggest a mathematical approach to the analysis of the underlying reasons of the stochastic excitability, and to the synthesis of the systems with desired probabilistic properties. A general mathematical background of our approach is shortly presented and discussed in Section 2.

A typical example illustrating the developed theory is the stabilization of the operating mode in the stochastic semiconductorgas-discharge image converter [Astrov, 1988; Astrov et al, 2008]. It was shown that even small parametric noise can generate large-amplitude oscillations in current and cause the spontaneous interruption of the discharge process in the gap. This transformation from the conductive to dielectric state results in the destruction of the operating mode of this device. In Section 3, it is shown that this destruction is due to the high stochastic sensitivity of the equilibrium of considered model of this converter.

It is of practical importance to investigate whether it is possible to provide a proper operation of this system. We show that reducing the stochastic sensitivity by corresponding regulator, one can return the system to normal operating mode with acceptable low-amplitude oscillations.

2 Control of Stochastic Sensitivity of Equilibrium

Consider a nonlinear stochastic system with control

$$dx = f(x, u(x))dt + \varepsilon\sigma(x, u(x))dw(t), \qquad (1)$$

where x is n-dimensional state, u is l-dimensional control, $f(x, u)$ is a continuously differentiable n-vectorfunction, $w(t)$ is a m-dimensional standard Wiener process, $\sigma(x, u)$ is a $n \times m$-matrix-function characterizing a dependence of disturbances on state and control, ε is a scalar parameter of the noise intensity.

Assume that unperturbed and uncontrolled system (1) (with $\varepsilon = 0$, $u = 0$) has an equilibrium $\bar{x}$. The stability of $\bar{x}$ is not supposed.

We select a stabilizing regulator from the class $\mathcal{U}$ of admissible feedbacks $u = u(x)$, satisfying the following conditions:

(a) a function $u(x)$ is continuously differentiable and $u(\bar{x}) = 0$;

(b) a feedback $u(x)$ provides an exponential stability of the equilibrium $\bar{x}$ for the closed-loop deterministic system

$$dx = f(x, u(x))dt \tag{2}$$

in the neighbourhood of $\bar{x}$.

Here, the first condition (a) means that $\bar{x}$ remains an equilibrium of the system (2) for any $u \in \mathcal{U}$.

The first approximation system for the deviation $z(t) = x(t) - \bar{x}$ of states $x(t)$ of system (2) from the equilibrium $\bar{x}$ is as follows:

$$dz = (F + BK)zdt, \tag{3}$$

where

$$F = \frac{\partial f}{\partial x}(\bar{x}, 0), \quad B = \frac{\partial f}{\partial u}(\bar{x}, 0).$$

Note that the second condition (b) is equivalent to the exponential stability of a trivial solution of the system (3).

Here, without loss of generality, we restrict our consideration by regulators in the following linear feedback form:

$$u(x) = K(x - \bar{x}). \tag{4}$$

Consider a set $\mathbf{K}$ of matrices K which provide an exponential stability for the trivial solution of the system (3)

$$\mathbf{K} = \{K| \, Re\lambda_i(F + BK) < 0\}.$$

Here, $\lambda_i(F + BK)$ are the eigenvalues of the matrix $F + BK$. Suppose the pair (F, B) is stabilizable. It means that the set $\mathbf{K}$ and class $\mathcal{U}$ are not empty.

Under the small random disturbances ($\varepsilon \neq 0$), the trajectories $x^\varepsilon(t)$ of the stochastic system (1) leave the equilibrium $\bar{x}$. The feedback (4) with a matrix $K \in \mathbf{K}$ providing the exponential stability of the equilibrium, allows us to localize random states of the system (1),(4) in the neighbourhood of the equilibrium $\bar{x}$, and to form a stationary distributed solution $\bar{x}^\varepsilon(t)$.

The dynamics of small deviations $z(t) = x^\varepsilon(t) - \bar{x}$ is governed by the following first approximation stochastic system

$$dz = (F + BK)zdt + \varepsilon G dw, \; G = \sigma(\bar{x}, 0). \tag{5}$$

The sensitivity of the system (5) solution to noise with intensity ε is characterized by a variable $y = \dfrac{z}{\varepsilon}$.

For the covariance matrix $V(t) = cov(y(t), y(t))$, the following equation holds

$$\dot{V} = (F + BK)V + V(F + BK)^\top + S, \quad S = GG^\top. \tag{6}$$

For any $K \in \mathbf{K}$, this equation has a unique stationary solution W which satisfies the matrix algebraic equation

$$(F + BK)W + W(F + BK)^\top + S = 0. \tag{7}$$

For non-singular noises ($\det S \neq 0$), the solution W of equation (7) is positive definite.

Any solution $V(t)$ of the system (6) converges to the corresponding solution W of the system (7)

$$\lim_{t \to \infty} V(t) = W.$$

For the matrix W it holds that

$$cov(\bar{x}^\varepsilon(t), \bar{x}^\varepsilon(t)) \approx \varepsilon^2 W,$$

where $cov(\bar{x}^\varepsilon(t), \bar{x}^\varepsilon(t))$ is a covariance matrix of the solutions $\bar{x}^\varepsilon$ of the system (1). So, the matrix W is a simple quantitative characteristic of a response of the nonlinear system (1) to the small noise with intensity ε. The matrix W is called a stochastic sensitivity matrix of the equilibrium $\bar{x}$.

In many real processes, the deterministic stability of the equilibrium is insufficient for the proper operation. In excitable systems, noise can cause large-amplitude stochastic oscillations unacceptable from an engineering point of view. In these circumstances, it is very important to take into account the stochastic sensitivity of the equilibrium and provide a small dispersion of random states by reducing the stochastic sensitivity.

So, the control of the dispersion of random states can be implemented by means of synthesis of an assigned stochastic sensitivity matrix W.

Let $\mathbf{M}$ be a set of symmetric and positive definite $n \times n$-matrices. For any $K \in \mathbf{K}$ the regulator (4) forms a corresponding equilibrium of the system (1) with stochastic sensitivity matrix W_K. This matrix is a solution of the equation (7). Consider the following control problem.

Problem of stochastic sensitivity synthesis

For the assigned matrix $W \in \mathbf{M}$, it is necessary to find a matrix $K \in \mathbf{K}$ guaranteeing the equality $W_K = W$, where W_K is a solution of the equation (7).

In some cases, this problem is unsolvable, and so we introduce notions of the attainability and the stochastic controllability.

Definition 1.
The element $W \in \mathbf{M}$ is said to be attainable if the equality $W_K = W$ is true for some $K \in \mathbf{K}$.
A set of all attainable elements

$$\mathbf{W} = \{W \in \mathbf{M} \mid \exists K \in \mathbf{K} \quad W_K = W\}$$

is called attainability set.

Definition 2.
An equilibrium $\bar{x}$ is completely stochastic controllable if

$$\forall\, W \in \mathbf{M} \quad \exists\, K \in \mathbf{K} \;:\quad W_K \equiv W.$$

The equality $\mathbf{W} = \mathbf{M}$ is a condition of complete stochastic controllability of the equilibrium $\bar{x}$.

Let us describe the attainability set. The connection between the assigned matrix W and the feedback coefficient K follows from the equation (7) which can be rewritten in the form:

$$\begin{aligned} &BKW + WK^\top B^\top + H(W) = 0,\\ &H(W) = FW + WF^\top + S \end{aligned} \tag{8}$$

Solution of the problem of the synthesis of the assigned stochastic sensitivity matrix W is given by the following theorem [Ryashko and Bashkirtseva, 2008].

Theorem.
Let noise be non-singular ($\mathrm{det} S \neq 0$).
(a) If the matrix B is quadratic and non-singular then $\mathbf{W} = \mathbf{M}$ and for any matrix $W \in \mathbf{M}$

$$\begin{aligned} &K = \bar{K} + B^{-1} Z W^{-1} \in \mathbf{K},\\ &\bar{K} = -B^{-1}\left(\tfrac{1}{2} S W^{-1} + F\right) \end{aligned}$$

where Z is an arbitrary skew-symmetric $n \times n$-matrix.

(b) If $\mathrm{rank}(B) < n$ then the element $W \in \mathbf{M}$ is attainable if and only if the matrix W is a solution of the equation

$$P_2 H(W) P_2 = 0. \tag{9}$$

Under these conditions for any matrix $W \in \mathbf{M}$

$$K = \bar{K} + C \in \mathbf{K}, \quad \bar{K} = B^+ H(W) \left(\frac{1}{2} P_1 - I\right) W^{-1},$$

where C is an arbitrary $l \times n$-matrix satisfying the condition

$$BCW + WC^\top B^\top = 0.$$

Here, the sign "+" means a pseudo-inversion, $P_1 = BB^+$ and $P_2 = I - P_1$ are projective matrices.

Note that if $\mathrm{rank} B = 1$, the equation (8) has a unique solution $K = \bar{K}$.

In the next Section, we will apply this theory to the solution of the stabilization problem for the stochastic gas discharge system.

3 Stabilization of the Semiconductor–Gas–Discharge Image Converter

It is known that in excitable systems, noise can generate large-amplitude escapes from the area of the phase plane where the normal operation of the system is ensured. For example, this can take place in converters of optical images that are used to record high speed processes in the infrared range of light [Astrov et al, 2008]. These converters operate on the base of the structure "semiconductor–gas–discharge gap". In these devices it was found that weak noise can generate large-amplitude oscillations in current even at the small current density. These undesired oscillations can result in the interruption of the discharge process in the gap. So it is important to suppress these large-amplitude oscillations and return the system to normal operating mode with acceptable low-amplitude oscillations. In [Astrov et al, 2008], a control approach based on the speed-gradient method has been suggested.

In present paper, we suggest another approach based on the synthesis of stochastic sensitivity shortly presented above.

3.1 Deterministic Model

Consider the following gas discharge dynamic system [Astrov et al, 2008]:

$$\begin{aligned} &\dot{E} = a(E_m - E) - bNE\\ &\dot{N} = \frac{N}{\tau}\left(\frac{E}{E_c} - 1\right) \end{aligned} \quad , \tag{10}$$

where E is the electric field strength in the discharge gap, and N is the density of free charge carriers in this gap. The first equation describes the charging of the capacity of the discharge gap from a source of feeding voltage and its discharging due to the presence of free carriers in the gap. The characteristic time of the charging process is equal to $\frac{1}{a}$, b is a positive parameter. The value E_m is a maximal value of E that can be provided by a source of constant voltage.

The second equation describes dynamics of the density of free carriers in the gap. It is assumed that N

grows when E is larger than some critical electric field strength E_c. The positive parameter τ defines the rate of a variation of the charge carriers density when the electric field in the gap is not equal to the critical value E_c.

The system (10) has the equilibrium

$$\bar{E} = E_c, \qquad \bar{N} = \frac{a}{b}\left(\frac{E_m}{E_c} - 1\right)$$

corresponding to the normal operating mode of this gas discharge system.

This equilibrium is stable for $E_m > E_c$. But the deterministic stability of the equilibrium does not guarantee the normal operation of this system in the presence of the even small random variations of its parameters.

In [Astrov, 1988] it was shown that weak noise in the parameter E_c generates large-amplitude stochastic oscillations and causes the unwanted interruptions of electric current.

In present paper, following [Astrov et al, 2008], we fix parameters $a = 10^4,\ b = 5\cdot 10^{-3},\ \tau = 1.5\cdot 10^{-9},\ E_m = 8\cdot 10^4,\ E_c = 4\cdot 10^4$.

3.2 Analysis of Stochastic Sensitivity of the Model

Consider the model (10) forced by parametric noise. We replace the constant value E_c by $E_c(1+\varepsilon\xi(t))$ where $\xi(t)$ is a Gaussian white noise, ε is the noise intensity. Taking into account the first-order terms in the decomposition

$$\frac{1}{1+\varepsilon\xi} = 1 - \varepsilon\xi + (\varepsilon\xi)^2 + \dots$$

we get the following stochastic system

$$\begin{aligned} \dot{E} &= a(E_m - E) - bNE \\ \dot{N} &= \frac{N}{\tau}\left(\frac{E}{E_c} - 1\right) - \varepsilon\frac{NE}{\tau E_c}\xi(t) \end{aligned} \qquad (11)$$

Under the random disturbances, the random trajectory leaves the equilibrium and form some stochastic oscillations around it. In Fig. 1, time series $N(t)$ and phase trajectories of system (11) for three values of intensity ε are shown. As one can see, despite the deterministic stability of the equilibrium, even extremely small random perturbations generate large-amplitude oscillations that destroy normal operation of this system. We will show that the underlying reason for such a reaction is the high stochastic sensitivity of this equilibrium.

Using stochastic sensitivity function technique [Bashkirtseva et al, 2012; Bashkirtseva et al, 2013] one can find the stochastic sensitivity matrix $W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}$ of the equilibrium. For system (11), elements of the stochastic sensitivity matrix can be found explicitly:

$$\begin{aligned} w_{11} &= \frac{E_c^2(E_m - E_c)}{2\tau E_m},\ w_{12} = \frac{a(E_c - E_m)}{2\tau b}, \\ w_{22} &= \frac{a(E_m - E_c)(\tau a E_m^2 + E_m E_c - E_c^2)}{2\tau^2 b^2 E_m E_c^2}. \end{aligned}$$

For the fixed set of parameters, this matrix has the following elements: $w_{11} = 2.6\cdot 10^{17},\ w_{12} = w_{21} = -2.6\cdot 10^{19},\ w_{22} = 4.4\cdot 10^{25}$. So, the generation of large-amplitude oscillations around the equilibrium can be explained by huge values of the elements of the stochastic sensitivity matrix.

To suppress these large-amplitude oscillations and return the system to normal operating mode, we will decrease a level of the stochastic sensitivity by the appropriate feedback regulator.

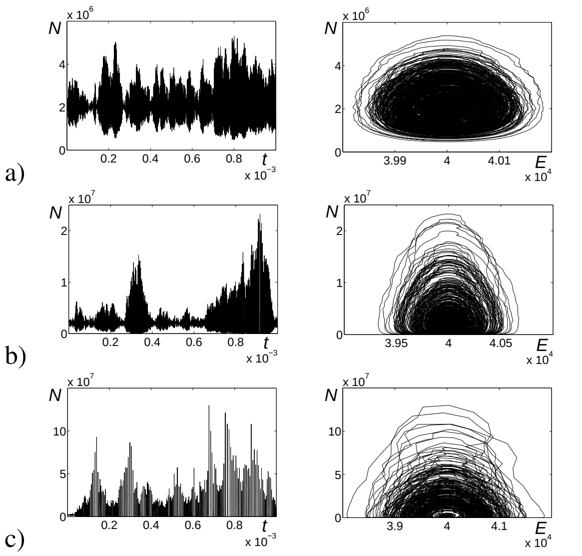

Figure 1. Time series (left) and phase trajectories (right) of stochastic system (11 with a) $\varepsilon = 1\cdot 10^{-11}$, b) $\varepsilon = 2\cdot 10^{-11}$, c) $\varepsilon = 3\cdot 10^{-11}$.

3.3 Stabilization of the Stochastic System

Consider the stochastic model

$$\begin{aligned} \dot{E} &= a(E_m + u - E) - bNE \\ \dot{N} &= \frac{N}{\tau}\left(\frac{E}{E_c} - 1\right) - \varepsilon\frac{NE}{\tau E_c}\xi(t) \end{aligned} \qquad (12)$$

with feedback regulator

$$u = k_1(E - \bar{E}) + k_2(N - \bar{N}). \qquad (13)$$

Here, $k_1,\ k_2$ are control parameters.

Consider a problem of the synthesis of the assigned stochastic sensitivity matrix W. In accordance with the Theorem, coefficients k_1 and k_2 of the feedback regulator (13) are connected (see (8) with elements of the matrix W by the following equalities:

$$\begin{aligned} &(\alpha + ak_1)w_{11} + (\beta + ak_2)w_{12} = 0 \\ &(\alpha + ak_1)w_{12} + (\beta + ak_2)w_{22} + \gamma w_{11} = 0\,, \\ &2\gamma w_{12} + \delta = 0 \end{aligned}$$

where

$$\alpha = -\frac{aE_m}{E_c},\ \beta = -bE_c,\ \gamma = \frac{\bar{N}}{\tau E_c},\ \delta = \left(\frac{\bar{N}}{\tau}\right)^2.$$

For the system (12), projective matrices are

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows from the attainability condition (9) that $w_{12} = -\frac{\delta}{2\gamma} = -2.6 \cdot 10^{19}$. So, we can not change the element w_{12} of the stochastic sensitivity matrix by our regulator (13).

The condition $w_{11}w_{22} - w_{12}^2 > 0$ allows us to assign $w_{11} = 10^{16}$, $w_{22} = 10^{23}$. It is worth noting that these values are reduced by one and two orders of magnitude than the corresponding values in the system without control.

As it follows from Theorem (case b), the coefficients of the regulator which synthesizes this assigned stochastic sensitivity matrix W are $k_1 = -0.307 \cdot 10^3$, $k_2 = -0.113 \cdot 10^1$. Results of the control based on this stochastic sensitivity synthesis are shown in Fig. 2.

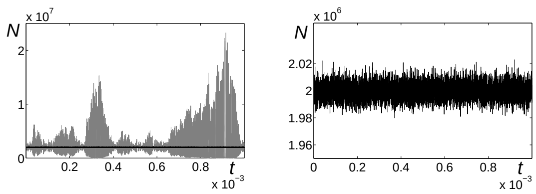

Figure 2. Time series of stochastic gas discharge system without control (grey) and with control (black) for $\varepsilon = 2 \cdot 10^{-11}$.

As one can see, in uncontrolled system, amplitudes of stochastic oscillations of the variable N exceed 10^7, whereas stochastic oscillations in the system with our regulator lie in the range of $4 \cdot 10^4$. It is worth noting that this result was achieved because of the essential decrease of the stochastic sensitivity of the equilibrium.

4 Conclusion

In present paper, we considered a problem of stochastic analysis and control of systems that demonstrate noise-induced large-amplitude oscillations. To suppress the undesirable random oscillations, we elaborated and discussed a new constructive control approach based on the reducing of the stochastic sensitivity by the appropriate feedback regulator. On the example of the nonlinear dynamical model of a semiconductor–gas–discharge image converter, we have shown that using this control approach, one can suppress unwanted large-amplitude oscillations and return the system to normal operation with acceptable low-amplitude oscillations. It is worth noting that the elaborated technique is readily applicable for the stabilization of operating modes of other, more complicated, devices.

Acknowledgements

This work was supported by RFBR (16-08-00388).

References

Astrom, K. J. (1970). *Introduction to the Stochastic Control Theory*. Academic Press. New York.

Astrov, Yu. A. (1988). Dynamic properties of discharge glow in a device with resistive electrode. *Ioffe Physico-Technical Institute Report*, **1255** (1988) (in Russian).

Astrov, Yu. A., Fradkov, A. L., and Guzenko, P. Yu. (2008). Control of a noise-induced transition in a nonlinear dynamical system. *Phys. Rev. E*, **77**, p. 026201.

Bashkirtseva, I., Chen, G., and Ryashko, L. (2012). Stochastic equilibria control and chaos suppression for 3D systems via stochastic sensitivity synthesis. *Commun. Nonlinear Sci. Numer. Simulat.*, **17**, pp. 3381–3389.

Bashkirtseva, I., Neiman, A. B., and Ryashko, L. (2013). Stochastic sensitivity analysis of the noise-induced excitability in a model of a hair bundle. *Phys. Rev. E*, **87**, p. 052711.

Guo, L., and Wang, H. (2010). *Stochastic Distribution Control System Design: a Convex Optimization Approach*. Springer-Verlag. New York.

Horsthemke, W. and Lefever, R. (1984). *Noise-Induced Transitions*. Springer. Berlin.

Kushner, H. J. (1967). *Stochastic Stability and Control*. Academic Press. New York.

Lai, Y.-C., and Tel, T. (2011). *Transient Chaos: Complex Dynamics on Finite Time Scales*. Springer. New York.

Lindner, B., Garcia-Ojalvo, J., Neiman, A., and Schimansky-Geier, L. (2004). Effects of noise in excitable systems. *Phys. Rep.*, **392**, pp. 321–424.

Ryashko, L. B., and Bashkirtseva, I. A. (2008). On control of stochastic sensitivity. *Automation and Remote Control*, **69**, pp. 1171–1180.

Sun, J.-Q. (2006). *Stochastic Dynamics and Control*. Elsevier.