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LIMIT CYCLES OF A POPULATION DYNAMICS MODEL

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Abstract
In this paper, using the global bifurcation theory, we complete the qualitative analysis of a quartic dynamical system which models the dynamics of the populations of predators and their prey that use the group defense strategy in a given ecological system. In particular, we prove that such a system can have at most two limit cycles.

Key words
Quartic ecological dynamical system; limit cycle; field rotation parameter; bifurcation; Wintner–Perko termination principle.

1 Introduction

In [Gaiko, 2003] we have developed the global bifurcation theory of polynomial dynamical systems by means of which some new results in the qualitative theory of differential equations have been obtained. For instance, in [Gaiko, 2003; Gaiko, 2008, NA] some complete results on quadratic systems have been presented. In particular, it has been proved that for quadratic systems four is really the maximum number of limit cycles and $(3:1)$, i. e., three limit cycles around one focus and the only limit cycle around another focus, is their only possible distribution (this is a solution of Hilbert's Sixteenth Problem in the quadratic case of polynomial dynamical systems). In [Gaiko and van Horssen, 2004] some preliminary results on generalizing new ideas and methods of [Gaiko, 2003] to cubic dynamical systems have been established. In particular, a canonical cubic system of Kukles type has been constructed and the global qualitative analysis of its special case corresponding to a generalized Liénard equation has been carried out. It has been proved also that the foci of such a Liénard system can be at most of second order and that such system can have at most three limit cycles on the whole phase plane. Moreover, unlike all previous works on the Kukles-type systems, global bifurcations of limit and separatrix cycles using arbitrary (including as large as possible) field rotation parameters of the canonical system have been studied in [Gaiko and van Horssen, 2004]. As a result, the classification of all possible types of separatrix cycles for the generalized Liénard system has been obtained and all possible distributions of its limit cycles have been found. In [Gaiko, 2008, CUBO] a solution of Smale's Thirteenth Problem proving that the Liénard system with a polynomial of degree $2k+1$ can have at most k limit cycles has been presented. Besides, in [Gaiko, 2008, CUBO] the global qualitative analysis of a Liénard-type piecewise linear dynamical system which is well-known in radio-electronics has been completed. In [Botelho and Gaiko, 2006] we have established the global qualitative analysis of a cubic centrally symmetric dynamical system which can be used as a learning model of planar neural networks. All of these methods and results can be applied to quartic dynamical systems as well. In this paper, using [Botelho and Gaiko, 2006; Gaiko, 2003 – Gaiko and van Horssen, 2004], we will complete the global qualitative analysis of a quartic ecological model [Bazykin, 1998; Broer, Naudot, Roussarie, Saleh and Wagener, 2007; Holling, 1959 – Li and Xiao, 2007; Zhu, Campbell and Wolkowicz, 2002]. In particular, studying global bifurcations of limit cycles, we will prove that the corresponding dynamical system can have at most two limit cycles.

We will investigate the following quartic dynamical system which models the dynamics of the populations of predators and their prey that use the group defense strategy in a given ecological system and which is a variation on the classical Lotka–Volterra system:

$$\begin{aligned} \dot{x} &= \ \ x((1-\lambda x)(\alpha x^2+\beta x+1)-y) \equiv P, \\ \dot{y} &= -y((\delta+\mu y)(\alpha x^2+\beta x+1)-x) \equiv Q, \end{aligned} \tag{1.1}$$

where $\alpha \geq 0$, $\delta > 0$, $\lambda > 0$, $\mu \geq 0$ and $\beta > -2\sqrt{\alpha}$ are parameters. This quartic ecological model was studied earlier, for instance, in [Broer, Naudot, Roussarie, Saleh and Wagener, 2007; Li and Xiao, 2007; Zhu, Campbell and Wolkowicz, 2002]. However, the qualitative analysis was incomplete, since the global bifurcations of limit cycles could not be studied properly by means of the methods and techniques which were used earlier in the qualitative theory of dynamical systems.

Together with (1.1), we will also consider an auxiliary system [Bautin and Leontovich, 1990; Perko, 2002]

$$\dot{x} = P - \gamma Q, \qquad \dot{y} = Q + \gamma P, \tag{1.2}$$

applying to these systems new bifurcation methods and geometric approaches developed in [Botelho and Gaiko, 2006; Gaiko, 2003 – Gaiko and van Horssen, 2004] and completing the qualitative analysis of (1.1).

2 Bifurcations of Limit Cycles

Let us first formulate the Wintner–Perko termination principle [Perko, 2002] for the polynomial system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}), \tag{$2.1_{\boldsymbol{\mu}}$}$$

where $\boldsymbol{x} \in \mathbf{R}^2;\ \boldsymbol{\mu} \in \mathbf{R}^n;\ \boldsymbol{f} \in \mathbf{R}^2$ ($\boldsymbol{f}$ is a polynomial vector function).

Theorem 2.1 (Wintner–Perko termination principle). *Any one-parameter family of multiplicity-m limit cycles of relatively prime polynomial system $(2.1_{\boldsymbol{\mu}})$ can be extended in a unique way to a maximal one-parameter family of multiplicity-m limit cycles of $(2.1_{\boldsymbol{\mu}})$ which is either open or cyclic.*

If it is open, then it terminates either as the parameter or the limit cycles become unbounded; or, the family terminates either at a singular point of $(2.1_{\boldsymbol{\mu}})$, which is typically a fine focus of multiplicity m, or on a (compound) separatrix cycle of $(2.1_{\boldsymbol{\mu}})$, which is also typically of multiplicity m.

The proof of this principle for general polynomial system $(2.1_{\boldsymbol{\mu}})$ with a vector parameter $\boldsymbol{\mu} \in \mathbf{R}^n$ parallels the proof of the planar termination principle for the system

$$\dot{x} = P(x, y, \lambda), \quad \dot{y} = Q(x, y, \lambda) \tag{2.1_λ}$$

with a single parameter $\lambda \in \mathbf{R}$ [Gaiko, 2003; Perko, 2002], since there is no loss of generality in assuming that system $(2.1_{\boldsymbol{\mu}})$ is parameterized by a single parameter λ; i. e., we can assume that there exists an analytic mapping $\boldsymbol{\mu}(\lambda)$ of $\mathbf{R}$ into $\mathbf{R}^n$ such that $(2.1_{\boldsymbol{\mu}})$ can be written as $(2.1\ {\boldsymbol{\mu}(\lambda)})$ or even (2.1_λ) and then we can repeat everything, what had been done for system (2.1_λ) in [Perko, 2002]. In particular, if λ is a field rotation parameter of (2.1_λ), the following Perko's theorem on monotonic families of limit cycles is valid [Perko, 2002].

Theorem 2.2. *If L_0 is a nonsingular multiple limit cycle of (2.1_0), then L_0 belongs to a one-parameter family of limit cycles of (2.1_λ); furthermore*:

1) *if the multiplicity of L_0 is odd, then the family either expands or contracts monotonically as λ increases through λ_0;*

2) *if the multiplicity of L_0 is even, then L_0 bifurcates into a stable and an unstable limit cycle as λ varies from λ_0 in one sense and L_0 disappears as λ varies from λ_0 in the opposite sense; i. e., there is a fold bifurcation at λ_0.*

Applying the definition of a field rotation parameter [Bautin and Leontovich, 1990; Gaiko, 2003; Perko, 2002], i. e., a parameter which rotates the field in one direction, to system (1.1), let us calculate the corresponding determinants for the parameters α and β, respectively,

$$\begin{aligned} \Delta_\alpha &= PQ'_\alpha - QP'_\alpha \\ &= x^3 y(y(\delta+\mu y) - x(1-\lambda x)), \end{aligned} \tag{2.2}$$

$$\begin{aligned} \Delta_\beta &= PQ'_\beta - QP'_\beta \\ &= x^2 y(y(\delta+\mu y) - x(1-\lambda x)). \end{aligned} \tag{2.3}$$

It follows from (2.2) and (2.3) that on increasing α or β the vector field of (1.1) in the first quadrant is rotated in positive direction (counterclockwise) only on the outside of the ellipse

$$y(\delta+\mu y) - x(1-\lambda x) = 0. \tag{2.4}$$

Therefore, to study limit cycle bifurcations of system (1.1), it makes sense together with (1.1) to consider also an auxiliary system (1.2) with a field rotation parameter γ:

$$\Delta_\gamma = P^2 + Q^2 \geq 0. \tag{2.5}$$

Using system (1.2) and applying Perko's results, we will prove the following theorem.

Theorem 2.3. *System* (1.1) *can have at most two limit cycles.*

Proof. First let us prove that system (1.1) can have at least two limit cycles.

Let the parameters α, β vanish and consider first the quadratic system

$$\begin{aligned} \dot{x} &= \ \ x(1 - \lambda x - y), \\ \dot{y} &= -y(\delta + \mu y - x). \end{aligned} \tag{2.6}$$

It is clear that such a system, with two invariant straight lines, cannot have limit cycles at all [Gaiko, 2003].

Inputting a negative parameter β into this system, the vector field of the cubic system

$$\begin{aligned} \dot{x} &= \ \ x((1 - \lambda x)(\beta x + 1) - y), \\ \dot{y} &= -y((\delta + \mu y)(\beta x + 1) - x) \end{aligned} \tag{2.7}$$

will be rotated in negative direction (clockwise) at infinity, the structure and the character of stability of infinite singularities will be changed, and an unstable limit, Γ_1, will appear immediately from infinity in this case. This cycle will surround a stable antisaddle (a node or a focus), A_1, which is in the first quadrant of system (2.7).

Inputting a positive parameter α into system (2.7), the vector field of quartic system (1.1) will be rotated in positive direction (counterclockwise) at infinity, the structure and the character of stability of infinite singularities will be changed again, and a stable limit, Γ_2, surrounding Γ_1 will appear immediately from infinity in this case. On further increasing the parameter α, the limit cycles Γ_1 and Γ_2 combine a semi-stable limit, Γ_{12}, which then disappears in a "trajectory concentration" [Bautin and Leontovich, 1990; Gaiko, 2003].

On further increasing α, two other singular points, a saddle S and an antisaddle A_2, will appear in the first quadrant in system (1.1). We can fix the parameter α, fixing simultaneously the positions of the finite singularities A_1, S, A_2, and consider system (1.2) with a positive parameter γ which acts like a positive parameter α of system (1.1), but on the whole phase plane.

So, consider system (1.2) with a positive parameter γ. On increasing this parameter, the stable nodes A_1 and A_2 becomes first stable foci, then they change the character of their stability, becoming unstable foci. At these Andronov–Hopf bifurcations [Bautin and Leontovich, 1990; Gaiko, 2003], stable limit cycles will appear from the foci A_1 and A_2. On further increasing γ, the limit cycles will expand and will disappear in small separatrix loops of the saddle S. If these loops are formed simultaneously, we will have a so-called eight-loop separatrix cycle. In this case, a big stable limit surrounding three singular points, A_1, S, and A_2, will appear from the eight-loop separatrix cycle after its destruction, expanding to infinity on increasing γ. If a small loop is formed earlier, for example, around the point A_1 (A_2), then, on increasing γ, a big loop formed by two lower (upper) adjoining separatrices of the saddle S and surrounding the points A_1 and A_2 will appear. After its destruction, we will have simultaneously a big limit cycle surrounding three singular points, A_1, S, A_2, and a small limit cycle surrounding the point A_2 (A_1). Thus, we have proved that system (1.1) can have at least two limit cycles, see also [Broer, Naudot, Roussarie, Saleh and Wagener, 2007; Li and Xiao, 2007; Zhu, Campbell and Wolkowicz, 2002].

Let us prove now that this system can have at most two limit cycles. The proof is carried out by contradiction applying Catastrophe Theory, see [Gaiko, 2003; Perko, 2002]. Consider system (1.2) with three parameters: α, β, and γ (the parameters δ, λ, and μ can be fixed, since they do not generate limit cycles). Suppose that (1.2) has three limit cycles surrounding the only point, A_1, in the first quadrant. Then we get into some domain of the parameters α, β, and γ being restricted by definite conditions on three other parameters, δ, λ, and μ. This domain is bounded by two fold bifurcation surfaces forming a cusp bifurcation surface of multiplicity-three limit cycles in the space of the parameters α, β, and γ [Gaiko, 2003; Perko, 2002].

The corresponding maximal one-parameter family of multiplicity-three limit cycles cannot be cyclic, otherwise there will be at least one point corresponding to the limit cycle of multiplicity four (or even higher) in the parameter space. Extending the bifurcation curve of multiplicity-four limit cycles through this point and parameterizing the corresponding maximal one-parameter family of multiplicity-four limit cycles by the field rotation parameter, γ, according to Theorem 2.2, we will obtain two monotonic curves of multiplicity-three and one, respectively, which, by the Wintner–Perko termination principle (Theorem 2.1), terminate either at the point A_1 or on a separatrix cycle surrounding this point. Since we know at least the cyclicity of the singular point which is equal to two [Broer, Naudot, Roussarie, Saleh and Wagener, 2007; Li and Xiao, 2007; Zhu, Campbell and Wolkowicz, 2002], we have got a contradiction with the termination principle stating that the multiplicity of limit cycles cannot be higher than the multiplicity (cyclicity) of the singular point in which they terminate. If the maximal one-parameter family of multiplicity-three limit cycles is not cyclic, using the same principle (Theorem 2.1), this again contradicts the cyclicity of A_1 [Broer, Naudot, Roussarie, Saleh and Wagener, 2007; Li and Xiao, 2007; Zhu, Campbell and Wolkowicz, 2002] not admitting the multiplicity of limit cycles to be higher than two. This contradiction completes the proof in the case of one singular point in the first quadrant.

Suppose that system (1.2) with three finite singularities, A_1, S, and A_2, has two small limit cycles around, for example, the point A_1 (the case when limit cycles surround the point A_2 is considered in a similar way). Then we get into some domain in the space of the parameters α, β, and γ which is bounded by a fold bifurcation surface of multiplicity-two limit cycles [Gaiko, 2003; Perko, 2002].

The corresponding maximal one-parameter family of multiplicity-two limit cycles cannot be cyclic, otherwise there will be at least one point corresponding to the limit cycle of multiplicity three (or even higher) in the parameter space. Extending the bifurcation curve of multiplicity-three limit cycles through this point and parameterizing the corresponding maximal one-parameter family of multiplicity-three limit cycles by the field rotation parameter, γ, according to Theorem 2.2, we will obtain a monotonic curve which, by the Wintner–Perko termination principle (Theorem 2.1), terminates either at the point A_1 or on some separatrix cycle surrounding this point. Since we know at least the cyclicity of the singular point which is equal to one in this case [Broer, Naudot, Roussarie, Saleh and Wagener, 2007; Li and Xiao, 2007; Zhu, Campbell and Wolkowicz, 2002], we have got a contradiction with the termination principle (Theorem 2.1). If the maximal one-parameter family of multiplicity-two limit cycles is not cyclic, using the same principle (Theorem 2.1), this again contradicts the cyclicity of A_1 [Broer, Naudot, Roussarie, Saleh and Wagener, 2007; Li and Xiao, 2007; Zhu, Campbell and Wolkowicz, 2002] not admitting the multiplicity of limit cycles higher than one. Moreover, it also follows from the termination principle that either an ordinary (small) separatrix loop or a big loop, or an eight-loop cannot have the multiplicity (cyclicity) higher than one in this case. Therefore, according to the same principle, there are no more than one limit cycle in the exterior domain surrounding all three finite singularities, A_1, S, and A_2.

Thus, taking into account all other possibilities for limit cycle bifurcations [Broer, Naudot, Roussarie, Saleh and Wagener, 2007; Li and Xiao, 2007; Zhu, Campbell and Wolkowicz, 2002], we conclude that system (1.1) cannot have either a multiplicity-three limit cycle or more than two limit cycles in any configuration. The theorem is proved.

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