ON THE MOMENTUM OF ELASTIC WAVES AND ITS FORCE ON THE OBSTACLE

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Abstract
It's illustrated with simple examples that for the calculation of the momentum of waves and the force exerted by the wave reflected from an obstacle it is necessary to consider nonlinear factor in motion equations of elastic systems and in boundary conditions. It is shown that the method using concepts of "wave momentum" and "wave pressure" for solution this problems is unreasonable.

Key words
elastic medium, momentum, pressure, wave

1. Interest to influence of waves on the reflecting obstacle has appeared long ago and is related with the assumption that the reflection of any physical wave exercises the nonzero pressure upon the reflector like the pressure of electromagnetic waves on their surface. There are many different points of view concerning the question about the influence of waves on the system boundary. The concept "wave momentum" [Vesnitsky, 2001; Vesnitsky, Kaplan and Utkin, 1983] is often used to answer this question. The change of this quantity caused by interaction of wave with obstacle explains the pressure existence on boundary which is called "wave pressure" as well.

Let $L = \int_a^b \lambda dx$ be a Lagrange functional of a one-dimensional elastic system, where $\lambda = \lambda(x,t,u(x,t),u_t(x,t),u_x(x,t))$ is the Lagrange function density and $u(x,t)$ - the shift of system points. The density of wave momentum is defined as

$$p^W(x,t) = -u_x \frac{\partial \lambda}{\partial u_t} \quad (1).$$

The differential law of change p^W and T^W provided that outside forces are absent is given by $\frac{\partial p^W}{\partial t} + \frac{\partial T^W}{\partial x} = \lambda_x$, where $T^W = \lambda - u_x \frac{\partial \lambda}{\partial u_x}$ - is so-called the wave pressure force in arbitrary elastic system profile x.

To all appearance in [Leech, 1961] wave characteristics concerned have been introduced for the first time. It's emphasized the quantity defined by (1) differs from $\frac{\partial \lambda}{\partial u_t}$ which is real medium momentum density. "It is new differential quantity", which is proposed to term as "wave momentum density, because it's not equal to zero only in wave motion when $u_x \neq 0$". At [Ostrovsky and Potapov, 2003] it's stated that "wave momentum is one of the general physical characteristics of wave processes and is true for any type of waves". The difference of generalized momentum and wave momentum is accented at that. "The first quantity is a linear function of variable u_t and is used in discrete systems as well as distributed systems. The second one is a new field characteristic of dynamical process and is reasonable only for distributed systems".

For string waves the transport equation of "wave momentum" is a result of multiplication the equation of string vibrations $\rho u_{tt} - N u_{xx} = 0$ by u_x:

$$\frac{\partial}{\partial t}\left(-\rho u_t u_x\right) + \frac{\partial}{\partial x}\left[\frac{1}{2}\left(\rho u_t^2 + N u_x^2\right)\right] = 0 \quad (2)$$

where ρ – density on the unit of length, N - tighting force.

Here $p^W = -\rho u_t u_x$ is the "wave momentum" density on the unit of length, $T^W = \frac{1}{2}\left[\rho u_t^2 + N u_x^2\right]$ - the force of "wave pressure". In particular, for harmonic wave $u = a\sin(\omega t - kx)$ we get $p^W = \rho a^2 k\omega \cos^2(\omega t - kx)$. "Wave momentum" is proportional to the square of amplitude and has the same direction with wave motion. Therefore "wave pressure" is positive, that is an obstacle is pushed forward. The average value of mechanical momentum density for harmonic wave is zero: $\bar{p} = \rho \bar{u}_t = 0$, whereas the average "wave momentum" $\bar{p}^W = \frac{1}{2}\rho\omega k a^2 \neq 0$.

Some questions occur about physical meaning of "wave momentum" and "wave pressure", about necessity and reasonability of this quantities introduction for studying the dynamics of distributed systems, equations of which is following from Newton's equations and for which classical motion characteristics - momentum and pressure - are enough.

The equation (1) looks like a conservation law for wave momentum $-\rho u_t u_x$ and wave pressure $\frac{1}{2}(\rho u_t^2 + N u_x^2)$:

$$\frac{\partial a}{\partial t} + \frac{\partial c_k}{\partial x_k} = 0 . \qquad (3)$$

It can be construed as the balance equation for the rate of quantity *a* change and its flux across unit area. But in general this ratio doesn't determine single-valued quantities. The fact is that *a*, c_k can be added by quantities $a^{(1)}$, $c_k^{(1)}$ satisfying an identity $\frac{\partial a^{(1)}}{\partial t} + \frac{\partial c_k^{(1)}}{\partial x_k} = 0$ which doesn't change the form of initial equation (3) but changes included quantities. It is a well-known fact, in particular, in the theory of electromagnetic field. Such conversions are called calibration.

Thus the notions of "wave momentum" and "wave pressure" are the result of a mathematical reasoning. The question about their meaning and validity of use for studying wave motions in medium remains open. It is known [Denisov, 1994; Denisov, 2001] that formal application of these concepts and equations as (3) can lead to wrong results. It is essential that considered wave characteristics are proportional to the square of deformation amplitude but are usually used in linear model. Let demonstrate some originating contradictions with the particular example of the wave on a rod.

2. Let's consider the one-dimensional case of small vibrations of the rod bounded from one side. In the second approximation of the perturbation theory, the motion equation of the rod is

$$u_{tt} - c_0^2 u_{xx} + \beta u_x u_{xx} = 0 , \qquad (4)$$

where *u(x,t)* – the longitudinal shift, β – the parameter described nonlinear system properties. The boundary condition at *x=0* is *u(0,t)=0*.

Let's represent a solution as the sum of the quantities of the first and second infinitesimal order: $u=u_1+u_2$. In that case the task (4) is reduced to the next combined equations:

$$\begin{aligned} &u_{1tt} - c_0^2 u_{1xx} = 0 \\ &u_{2tt} - c_0^2 u_{2xx} + \beta u_{1x} u_{1xx} = 0 \end{aligned} . \qquad (5)$$

Consider the wave traveling to the left (in boundary direction) and characterizing to a linear approximation by $u_1(x,t) = f(x + c_0 t)$, a continuous function specified in the range $x_0 - c_0 t < x < x_0 - c_0 t + a$. At initial instant of time *t=0* $f(x_0) = f(x_0 + a) = 0$. At *t=0* the wave is entirely on the right of limiter ($x_0 > 0$), at *t>0* it moves to a fixed end point of the rod. At $x = x_0 - c_0 t$ ($x_0 > 0$) there is a leading wave front and $x = x_0 - c_0 t + a$ corresponds to the rear front. By $t_1 = \frac{x_0}{c_0}$ the leading front reaches the limiter and the wave begins to reflect. At time $t_2 = \frac{x_0 + a}{c_0}$ the rear front reaches the fixed end point too, that is at $t \ge t_2$ the wave reflected fully.

Let's determine the pressure produced by the wave at the fixed end point of the rod. For that let's find the solution of the equation at the second approximation

$$u_{2tt} - c_0^2 u_{2xx} = -\frac{\beta}{2}\left[f_x^2(x + c_0 t)\right]_x .$$

By the d'Alembert's formula, using zero initial conditions we obtain

$$\begin{aligned} u_2(x,t) &= -\frac{\beta}{4c_0}\int_0^t dt' \int_{x-c_0(t-t')}^{x+c_0(t-t')} \left[f_\xi^2(\xi + c_0 t')\right]_\xi d\xi = \\ &= \frac{\beta}{4c_0}\int_0^t \left[f_x^2(x - c_0 t + 2c_0 t') - f_x^2(x + c_0 t)\right] dt' \end{aligned} .$$

The equation (4) can be written as

$$\frac{\partial p}{\partial t} + \frac{\partial T}{\partial x} = 0 , \qquad (6)$$

where $p = \rho u_t$ - generalized momentum density, $T = \rho\left(-c_0^2 u_x + \frac{\beta}{2} u_x^2\right)$ – the force in profile *x*.

The force on the boundary is defined by *T*: $R = -T\big|_{x=0}$. On the other side, the integration of the equation (6) over time at first and then over coordinate gives:

$$\int_0^t \left[T(x_2,t) - T(x_1,t)\right] dt = -\int_{x_1}^{x_2} \left[p_{omp}(x,t_2) - p_{na\partial}(x,t_1)\right] dx .$$

This implies that the total force on the rod of the distance x_2-x_1 for the time *t* is equal to the difference between momentums of an incident wave and reflected from the limiter wave. Taking an interest in the force at the fixed end point set $x_1=0$ and suppose x_2 as such point which hasn't been reached by disturbance yet, that is $T(x_2,t) = 0$. Then the left side of the obtained rate is written as: $-\int_{x_0/c_0}^{(x_0+a)/c_0} T(0,t)dt$. The limits of integration are the time instants of the beginning and ending of wave interaction with the limiter. The right side of the rate is equal to $\left(\int_{x_0-c_0t_1}^{x_0-c_0t_1+a} p_{na\partial}(x,t_1)dx - \int_{-x_0-a+c_0t_2}^{-x_0+c_0t_2} p_{omp}(x,t_2)dx\right)$, where $t_1 \le \frac{x_0}{c_0}$ - time when wave hasn't reached the limiter yet, $t_2 \ge \frac{x_0 + a}{c_0}$ - time when the reflection has already completed.

Let's calculate momentum *P* and force on boundary *R* in linear model. The momentum of an incident wave $P_{na\partial}(t_1)$ is given by:

$$\begin{aligned} P_{na\partial}(t_1) &= \int_{x_0-c_0t_1}^{x_0-c_0t_1+a} \rho u_{1t}(x,t_1)dx = \rho \int_{x_0-c_0t_1}^{x_0+a-c_0t_1} c_0 f_x(x + c_0 t_1)dx = \\ &= \rho c_0\left(f(x_0 + a) - f(x_0)\right) = 0 \end{aligned} .$$

For the momentum evaluation of a reflected wave $P_{omp}(t_2) = \int_{-x_0-a+c_0t_2}^{-x_0+c_0t_2} \rho u_{1t}(x,t_2)dx$ we find its expression $u_{1omp}(x,t)$. Consider, following [Tikhonov and

Samarsky, 1977], a boundless rod with condition *u(0,t)=0*. On the rod two waves $u_+(x+c_0t)+u_-(-x+c_0t)$ run towards to each other. In the profile *x=0* $u_-(c_0t)=-u_+(c_0t)$. Taking $u(x,t)=u_+(x+c_0t)-u_-(-x+c_0t)$ as a solution and considering it at $x\ge 0$, we obtain the solution of the original problem. The reflected wave will be $u_{1omp}(x,t)=-f(-x+c_0t)$, where $t\ge\frac{x_0+a}{c_0}$. The momentum of this wave

$$P_{omp}(t_2)=\int_{-x_0-a+c_0t_2}^{-x_0+c_0t_2}\rho u_{1t}^{omp}(x,t_2)dx=0.$$

The disturbance of the first approximation doesn't carry momentum under arbitrary function f which equals zero at the ends of interval. In that case the force on the boundary is equal to:

$$R|_{x=0}=P_{na\partial}(t_1)-P_{omp}(t_2)=0.$$

The same result for the force at the fixed end point is given by the expression $-\int_{x_0/c_0}^{(x_0+a)/c_0}T(0,t)dt$, where

$$T(0,t)=-\rho c_0^2u_{1x}(0,t)=-\rho c_0^2\left(u_{1x}^{na\partial}(0,t)+u_{1x}^{omp}(0,t)\right)= \\ =-2\rho c_0^2 f_x(c_0t)$$

Let's evaluate the waves of the second approximation. For the momentum at the time $t_1\le\frac{x_0}{c_0}$ we have $P(t_1)=\int_{x_0-c_0t_1}^{x_0-c_0t_1+a}\rho u_{2t}(x,t_1)dx$, where

$$u_{2t}(x,t_1)=-\frac{\beta}{8c_0}\Big[f_x^2(x+c_0t_1)-f_x^2(x-c_0t_1)+ \\ +2c_0t_1\left[f_x^2(x+c_0t_1)\right]_x\Big].$$

Notice that, because of nonlinearity the activated wave running to the left breaks up into the group of waves traveling to the same direction and one wave traveling to the right.

Considering waves interactive with the limiter we get:

$$P_{na\partial}(t_1)=-\frac{\beta\rho}{8c_0}\int_{x_0-c_0t_1}^{x_0-c_0t_1+a}f_x^2(x+c_0t_1)dx- \\ -\frac{\beta\rho}{4c_0}c_0t_1\int_{x_0-c_0t_1}^{x_0-c_0t_1+a}\left[f_x^2(x+c_0t_1)\right]_x dx=-\frac{\beta\rho}{8c_0}\int_{x_0}^{x_0+a}f_x^2(x)dx$$

The final identity is written provided the assumption $f_x(x_0)=f_x(x_0+a)=0$.

The momentum *P(t₂)* at $t_2\ge\frac{x_0+a}{c_0}$ is given by:

$$P(t_2)=\int_{-x_0-a+c_0t_2}^{-x_0+c_0t_2}\rho u_{2t}(x,t_2)dx\text{, where}$$

$$u_{2t}^{omp}(x,t_2)=\frac{\beta}{8c_0}\left[f_x^2(-x+c_0t_2)+2c_0t_2\left[f_x^2(-x+c_0t_2)\right]_x\right].$$

Integrating we obtain

$$P_{omp}(t_2)=\frac{\beta\rho}{8c_0}\int_{x_0}^{x_0+a}f_x^2(x)dx=-P_{na\partial}(t_1).$$

The force exerted by the wave on the boundary is

$$R|_{x=0}=P_{na\partial}(t_1)-P_{omp}(t_2)=-\frac{\beta\rho}{4c_0}\int_{x_0}^{x_0+a}f_x^2(x)dx.$$

We get the same result in another way:

$$R|_{x=0}=-\int_{x_0/c_0}^{(x_0+a)/c_0}T(0,t)dt,$$

where $T(0,t)=-\rho c_0^2u_{2x}(0,t)+\frac{\rho\beta}{2}u_{1x}^2(0,t)$.

However, the first member in the expression

$$u_{2x}(x,t)=-\frac{\beta}{8c_0^2}\Big[f_x^2(x-c_0t)-f_x^2(x+c_0t)+ \\ +2c_0t\left[f_x^2(x+c_0t)\right]_x\Big]$$

doesn't influence on the limiter. So we don't take it into account. The wave $F(x+c_0t)$ incident on the limiter generate the reflected wave $-F(-x+c_0t)$. At that $u(0,t)=F(c_0t)-F(c_0t)=0$, the time derivative $u_t(0,t)=0$, but $u_x(0,t)=2F_x(c_0t)$. That's why we consider coefficient 2 and not consider the first member in the expression for u_{2x} in the formula for finding R. Provided that we obtain: $R|_{x=0}=-\frac{\beta\rho}{4c_0}\int_{x_0}^{x_0+a}f_x^2(x)dx$.

Thus, the problem solution of the force exerted by the wave on the elastic system boundary is different in the frame of linear and nonlinear model. The pressure appears only in the presence of nonlinearity and can be positive as well as negative according to the sign of the nonlinearity coefficient *β*. The same is also concerned with the momentum of wave. The momentum isn't connected with the wave traveling direction. The sign of momentum depends on the nonlinearity coefficient.

Notice that in linear model a momentum transfer is possible only if a longitudinal shifts function has a discontinuity at the disturbance area boundary. This condition is nonphysical. The analysis of wave motion should be specified by introduction nonlinearity to the model. In this case the local wave carries momentum on the condition of zero longitudinal displacements at the boundary too even if their first coordinate derivatives are zero.

Let's consider what the solution of the task would be if we used the notions of "wave momentum" and "wave pressure". Multiplying the first equation of (5) by u_x we get $\frac{\partial p^W}{\partial t}+\frac{\partial T^W}{\partial x}=0$, where $p^W=-\rho u_tu_x$ - the "wave momentum" density, $T^W=\frac{\rho}{2}\left(u_t^2+c_0^2u_x^2\right)$ - "wave pressure" in profile *x*.

"Wave momentum" of the incident wave $P_{na\partial}^W(t_1)$ for disturbance $u(x,t)=f(x+c_0t)$ is given by:

$$P_{na\partial}^W(t_1)=-\rho\int_{x_0-c_0t_1}^{x_0+a-c_0t_1}c_0f_x^2(x+c_0t_1)dx= \\ =-\rho c_0\int_{x_0}^{x_0+a}f_x^2(x)dx \qquad (7)$$

"Wave momentum" of the reflected wave $P_{omp}^W(t_2)$ is:

$$P_{omp}^{W}(t_2)=\rho\int_{-x_0-a+c_0t_2}^{-x_0+c_0t_2}c_0f_x^2(-x+c_0t_2)dx= \\ =\rho c_0\int_{x_0}^{x_0+a}f_x^2(x)dx=-P_{na\partial}^{W}(t_1) \quad (8)$$

In that case the force on the boundary is equal to:

$$R|_{x=0}=P_{na\partial}^{W}(t_1)-P_{omp}^{W}(t_2)=-2\rho c_0\int_{x_0}^{x_0+a}f_x^2(x)dx,$$

what coincides with the result of integration of "wave pressure" on time:

$$R|_{x=0}=-\int_{x_0/c_0}^{(x_0+a)/c_0}T^W(0,t)dt=-2\rho c_0\int_{x_0}^{x_0+a}f_x^2(x)dx. \quad (9)$$

Thus, "wave momentum" P^W is nonzero and has the same direction with a traveling wave direction. The change of "wave momentum" because of the reflection of wave from the limiter coincides with the positive "wave pressure" on the fixed end point of the rod. This result differs from the solution obtained by using classical concepts.

Consider the same task in the case of free end of the rod, that is on condition $u_x(0,t)=0$ at $x=0$. Arguments for finding the momentum of wave and its force on the boundary are analogous to the fixed end case. In the frame of a linear model the momentum $P_{na\partial}(t_1)$ of the incident wave $u_1(x,t)=f(x+c_0t)$ and the momentum $P_{omp}(t_2)$ of the reflected wave $u_{1omp}(x,t)=f(-x+c_0t),\ t\ge\frac{x_0+a}{c_0}$ are equal to zero. At that the force on the boundary is zero too.

Let's consider waves in the second approximation. As the previous case the momentum $P(t_1)$ at $t_1\le\frac{x_0}{c_0}$ is given by:

$$P(t_1)=\int_{x_0-c_0t_1}^{x_0-c_0t_1+a}\rho u_{2t}(x,t_1)dx, \text{ where}$$

$$u_{2t}(x,t_1)=-\frac{\beta}{8c_0}\left[f_x^2(x+c_0t_1)-f_x^2(x-c_0t_1)+ \\ +2c_0t_1\left[f_x^2(x+c_0t_1)\right]_x\right].$$

Proceeding on the assumption that $f_x(x_0)=f_x(x_0+a)=0$ for momentum of the incident wave we have

$$P_{na\partial}(t_1)=-\frac{\beta\rho}{8c_0}\int_{x_0}^{x_0+a}f_x^2(x)dx.$$

The momentum $P(t_2)$ at the time $t_2\ge\frac{x_0+a}{c_0}$ is:

$$P(t_2)=\int_{-x_0-a+c_0t_2}^{-x_0+c_0t_2}\rho u_{2t}(x,t_2)dx, \text{ where}$$

$$u_{2t}^{omp}(x,t_2)=-\frac{\beta}{8c_0}\left[f_x^2(-x+c_0t_2)+2c_0t_2\left[f_x^2(-x+c_0t_2)\right]_x\right].$$

After calculating we obtain

$$P_{omp}(t_2)=-\frac{\beta\rho}{8c_0}\int_{x_0}^{x_0+a}f_x^2(x)dx=P_{na\partial}(t_1).$$

Thus, the momentum of wave doesn't change its direction after the wave reflection from the free end of the rod and the total force on the boundary is equal to zero in that case:

$$R|_{x=0}=P_{na\partial}(t_1)-P_{omp}(t_2)=0.$$

However the solution of this task with the usage of wave characteristics gives (7)-(9) as before. So the "wave" method reduces to the false result and also doesn't make difference between cases of free end of the rod and fixed end point.

So the wave momentum property to change or conserve its direction with respect to wave propagation direction under the type of boundary condition exists. Note that it obtains in frame of linear model too though it can be found the assertion about the momentum transfer in the direction of wave propagation.

Let $f(x_0)\ne f(x_0+a)\ne 0$. The momentum of the incident wave $u_1(x,t)=f(x+c_0t)$ is equal to

$$P_{na\partial}(t_1)=\int_{x_0-c_0t_1}^{x_0-c_0t_1+a}\rho u_{1t}(x,t_1)dx=\rho c_0\left(f(x_0+a)-f(x_0)\right).$$

The momentum of the reflected wave $u_{1omp}(x,t)=f(-x+c_0t),\ t\ge\frac{x_0+a}{c_0}$ under free boundary condition $u_x(0,t)=0$ is given by

$$P_{omp}(t_2)=\int_{-x_0-a+c_0t_2}^{-x_0+c_0t_2}\rho u_{1t}^{omp}(x,t_2)dx=\rho c_0\left(f(x_0+a)-f(x_0)\right)$$

that is the momentum has the same value as well as direction though the wave propagation direction changes by the opposite one as the interaction result with rod boundary.

It's easy to check up that the wave momentum direction coincides with the wave propagation direction after reflection from the boundary under boundary condition $u(0,t)=0$.

3. The investigation of the transverse motions of elastic systems has another speciality which should be considered for finding the force exerted on the boundary of system. That is the necessity to take into account the nonlinear connection of transverse and longitudinal motions in motion equations as well as in boundary conditions.

In case of the string transverse vibrations the Lagrange function density is

$$\lambda=\frac{1}{2}\rho_0(1-u_x)\left(u_t^2+v_t^2\right)-\frac{1}{2}T_0\left(\sqrt{(1+B+u_x)^2+v_x^2}-1\right)^2 \quad (10)$$

where the longitudinal and transverse shifts of string points are denoted by $u(x,t)$ and $v(x,t)$ respectively, ρ_0 – unperturbed density value, T_0 – tighting force, $B=\frac{\partial u_0}{\partial x}$ - the initial constant string tension.

The motion equations in the second approximation of the perturbation theory are given by:

$$u_{tt}-a^2u_{xx}=u_xu_{tt}+2u_{xt}u_t+v_tv_{tx}+a^2(1-\gamma^2)^2v_xv_{xx} \\ v_{tt}-\gamma^2a^2v_{xx}=(u_xv_t)_t+a^2(1-\gamma^2)^2(v_xu_x)_x,$$

where $a^2=\frac{T_0}{\rho_0}$, $\gamma^2=\frac{B}{1+B}$.

Representing a solution as the sum of two quantities of the first and second infinitesimal order

$u = u_1 + u_2, \upsilon = \upsilon_1 + \upsilon_2$, in the first approximation we get the independent equations of the string transverse and longitudinal motions:

$$u_{1tt} - a^2 u_{1xx} = 0$$
$$\upsilon_{1tt} - \gamma^2 a^2 \upsilon_{1xx} = 0 \, .$$

The shifts u_2 and υ_2 are determined by the solution of the next system of equations:

$$u_{2tt} - a^2 u_{2xx} = u_{1x} u_{1tt} + 2u_{1xt} u_{1t} + \upsilon_{1t} \upsilon_{1tx} + a^2 (1-\gamma^2)^2 \upsilon_{1x} \upsilon_{1xx}$$
$$\upsilon_{2tt} - \gamma^2 a^2 \upsilon_{2xx} = (u_{1x} \upsilon_{1t})_t + a^2 (1-\gamma^2)^2 (\upsilon_{1x} u_{1x})_x$$

The transverse waves excitation in the first approximation leads to the longitudinal waves generation in the second one. However, usually "wave pressure" is calculated to get the solution of linear problem. The inaccuracy of this method has been illustrated with the example of the longitudinal motions in a one-dimensional solid system. Here it will be shown that the solution of the question concerned with the force exerted by the wave on an obstacle essentially depends on the obstacle type as well as on its nonlinear factors. We'll consider some static problems to show this and also to represent the obvious example.

Let's consider the string of length l_0 in undistorted state. At initial instant of time the string is tighten between two fixed points which are positioned on the distance $2l_1$ from each other. The force of the constant string tension is equal to $T_0 = k(2l_1 - l_0)$, where k is the elasticity modulus. At middle point the constant force F acts on the string (fig.1). The angle of the string deviation from horizontal line is α ($\alpha \ll 1$). At that the connection of u_x and υ_x with α follows from rates $du = dl \cos\alpha, d\upsilon = dl \sin\alpha$, where $dl = \sqrt{(1 + B + u_x)^2 + \upsilon_x^2}\, dx$ - the length of the string element after deformation. In static case it is $u_t = 0, \upsilon_t = 0$. So the calculation of the string force on each fixed end point can be carried out from the expression $R = -\left.\frac{\partial \lambda}{\partial u_x}\right|_{x-0}^{x+0}$ as it was in the second part but it is possible to use the statics rates.

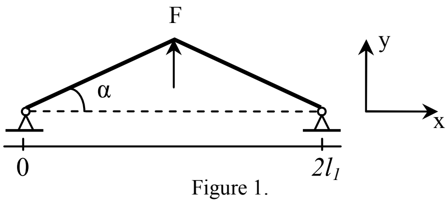

Figure 1.

Let's denote by R_{ij} the j-th component of the force exerted by the string on the boundary with number i ($i=1, 2; j=x, y$). At the left fixed point the next relations are valid:

$$R_{1x} = T\cos\alpha, R_{1y} = T\sin\alpha \, ,$$

where $T = k\left(\frac{2l_1}{\cos\alpha} - l_0\right)$ - tighting force.

From this, considering the members of the first and second infinitesimal order we obtain:

$$R_{1x} \approx T_0 + kl_0 \frac{\alpha^2}{2} \, , \; R_{1y} \approx T_0 \alpha \, .$$

In that case the string deviation leads to additional longitudinal force which is in the opposite direction from the boundary because $kl_0 \frac{\alpha^2}{2} > 0$.

Similarly we get the next force components for the right fixed point:

$$R_{2x} \approx -T_0 - kl_0 \frac{\alpha^2}{2} \, , \; R_{2y} \approx T_0 \alpha \, .$$

The additional longitudinal force is directed to the left tending to move the boundary to the left that is this force exerts the additional negative "pressure".

Thus, the string forces on boundaries in case of fixed end points are equal and opposite directed. After the angle α increasing, the force grows tending to join boundaries. Notice that in linear model the forces R_{1x}, R_{2x} are caused by the initial string tension: $R_{1x} = -R_{2x} = T_0$.

Let's consider the case of a fixed ended string as before but at point 1 and 2 there is a ring limiter that is the string is limited only in vertical displacement at these points (fig.2). In this example the initial tighting force is $T_0 = k(2l_1 + 2l_2 - l_0)$.

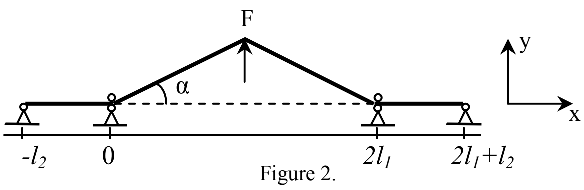

Figure 2.

The force exerted by the string on the limiter at point 1 has the next components:

$$R_{1x} = T\cos\alpha - T \, , \; R_{1y} = T\sin\alpha \, ,$$

and at point 2 the force components are:

$$R_{2x} = -T\cos\alpha + T \, , \; R_{2y} = T\sin\alpha \, .$$

Here the tighting force of the string is $T = k\left(\frac{2l_1}{\cos\alpha} + 2l_2 - l_0\right)$.

By the expansion procedure of the given expressions for force components in the second approximation we get:

$$R_{1x} \approx -T_0 \frac{\alpha^2}{2}, R_{1y} \approx T_0 \alpha \text{ and } R_{2x} \approx T_0 \frac{\alpha^2}{2}, R_{2y} \approx T_0 \alpha \, .$$

The force longitudinal component on the fixed point limiting only transverse shifts is nonlinear quantity about α. The force occurs in the presence of deviations only and is directed to limiters that is "presses down" on its. This essentially differs from the first case when at $\alpha \neq 0$ there is additional negative pressure.

Thus, the solution of the question about influence of elastic system on a boundary depends on the type of boundary conditions. The difference of force value as well as its sign becomes apparent only in the frame of nonlinear model.

In spite of the fact that "wave pressure" is proportional to the square of deformation amplitude it is usually calculated in linear models. In the static problems under consideration "wave" method gives equal force at fixed points without dependence on the type of limiter.

The Lagrange function density corresponding to linear transverse vibrations of the string follows from (10) and is given by $\lambda = \frac{1}{2}\rho_0 \upsilon_t^2 - \frac{1}{2}T_0 \upsilon_x^2$, where ρ_0 – density on the unit of string length, $\upsilon(x,t)$ - the transverse shift of string points, T_0 – tighting force.

In static case there is $\upsilon_t = 0$, $tg\alpha = \upsilon_x$. "Wave pressure" $T^W = \lambda - \frac{\partial \lambda}{\partial v_x}\upsilon_x$ is equal to $\frac{1}{2}T_0 \upsilon_x^2$ at that. The horizontal force at fixed point $x=x_1$ is determined by $R_x = -T^W \Big|_{x=x_1-0}^{x=x_1+0}$. From this it follows that the force on the boundary at point 1 is equal to $R_{1x} = -\frac{1}{2}T_0 \upsilon_x^2 \Big|_{x=-0}^{x=+0}$, at point 2 - $R_{2x} = -\frac{1}{2}T_0 \upsilon_x^2 \Big|_{x=2l_1-0}^{x=2l_1+0}$.

At the small angle α of deviation from horizontal line there is

$$R_{1x} = -\frac{1}{2}T_0 \alpha^2, \; R_{2x} = \frac{1}{2}T_0 \alpha^2.$$

In the frame of "wave" method this result is valid for the both considered types of fixed points 1 and 2 (fig.1, fig.2). Presented solution coincides with the previous result obtained by statics method in the case of the ring limiters at point 1 and 2 only. These results are different at fixed end points.

Thus, in the case of elastic solid wave motion the formally introduced concepts of "wave momentum" and "wave pressure" which are interpreted from doubtful analogies and dimensions as the specific characteristics of wave motion can lead to wrong results.

The problems of wave influence on obstacles require detailed analysis in every particular case with consideration of different nonlinear factors. First of all it is concerned those situations when the wave doesn't have momentum in linear model. The usage of "wave momentum" and "wave pressure" for the simplified solutions of questions about the dynamics of elastic systems is unreasonable.

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