SMOOTH INTERPOLATION ON ELLIPSOIDS VIA ROLLING MOTIONS

Krzysztof A. Krakowski
School of Science and Technology
University of New England, Armidale NSW, 2351
Australia and
Wydział Matematyczno-Przyrodniczy
Uniwersytet Kardynała Stefana Wyszyńskiego w Warszawie
Poland
kris@mcs.une.edu.au

Fátima Silva Leite
Department of Mathematics
University of Coimbra Apartado 3008, 3001-454
Coimbra and
Institute of Systems and Robotics
University of Coimbra - Pólo II
3030-290 Coimbra
Portugal
fleite@mat.uc.pt

Abstract
We present an algorithm to generate a smooth curve interpolating a set of data on an n-dimensional ellipsoid. This is inspired by an algorithm based on a rolling and wrapping technique, described in (Hüper and Silva Leite, 2007) for data on a general manifold embedded in Euclidean space. Since the ellipsoid can be embedded in an Euclidean space, this algorithm can be implemented, at least theoretically. However, one of the basic steps of that algorithm consists in rolling the ellipsoid, over its affine tangent space at a point, along a geodesic curve. This allows to project data from the ellipsoid to a space where interpolation problems can be easily solved. The major obstacle to implement the rolling part of that algorithm is due to the fact that explicit forms for Euclidean geodesics on the ellipsoid are not known. To overcome this problem and achieve our goal, we embed the ellipsoid and its affine tangent space on $\mathbb{R}^{n+1}$ equipped with an appropriate Riemannian metric, so that geodesics are given in explicit form and the kinematics of the rolling motion are easy to solve. By doing so, we can rewrite the algorithm to generate a smooth interpolating curve on the ellipsoid which is given in closed form.

Key words
Rolling, group of isometries, ellipsoid, kinematic equations, interpolation.

1 Introduction

There are several classical methods to generate smooth interpolating curves in Euclidean spaces. Cubic splines are possibly the most interesting from the point of view of applications, since they also minimize changes in velocity. However, if one requires the curve and data points to live on a curved space, the classical methods do not produce a reasonable answer. Interpolation problems on manifolds have been studied by several authors, starting with the pioneer work of Noakes, Heinzinger and Paden in (Noakes *et al.*, 1989). Following this, other authors developed the theory of geometric splines on manifolds, using a variational approach (see, for instance, (Crouch and Leite, 1991), (Camarinha, 1996), and (Crouch and Leite, 1995)), more recently a geometric approach was initiated in (Giambó *et al.*, 2002), but in both cases the results, although theoretically very interesting, are very difficult to implement except in trivial cases. A geometric algorithm, which generalizes the classical De Casteljau algorithm, was also developed in (Park and Ravani, 1995) and (Crouch *et al.*, 1999). However, the algorithm on non Euclidean spaces produces interpolating curves defined implicitly, which makes its implementation very hard. The main drawback in all these approaches is that they do not produce interpolating curves in closed form.

In the present paper we present an algorithm that generates interpolating curves on ellipsoids given in explicit form. This algorithm is based on a procedure to generate interpolating curves on manifolds embedded in Euclidean space, first described in (Jupp and Kent, 1987) for the 2-sphere, generalised in (Hüper and Silva Leite, 2002) for the n-sphere and in (Hüper *et al.*, 2007) for the rotation group and Grassmann manifolds. The algorithm is based on a rolling/unrolling and wrapping/unwrapping technique that will be fully described in the last two sections. To achieve our goal, we organize the previous Sections as follows. We formulate the interpolating problem in Section 2, describe general rolling maps in Section 3, present the appropriate geometry of the ellipsoid in Section 4, and, finally, in Section 5, we derive the kinematic equations for the rolling motion of an ellipsoid on its affine tangent space at a point.

2 Smooth interpolation on the ellipsoid $\mathcal{E}^n$

Let $d_1, d_2, \ldots, d_{n+1}$ be positive real numbers. The n-dimensional ellipsoid is defined as

$$\mathcal{E}^n := \Big\{ (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \\ : \frac{x_1^2}{d_1^2} + \frac{x_2^2}{d_2^2} + \cdots + \frac{x_{n+1}^2}{d_{n+1}^2} = 1 \Big\}.$$

2.1 Statement of the problem

Given a set of $k+1$ distinct points $p_i \in \mathcal{E}^n$, $i = 0, 1, \ldots, k$, vectors V_0 and V_k tangent to $\mathcal{E}^n$ at p_0 and p_k respectively, and fixed times t_i, where

$$0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = \tau,$$

we aim to solve the following problem:

Problem 1. *Find a C^2-smooth curve*

$$\gamma \colon [0, \tau] \to \mathcal{E}^n \tag{1}$$

satisfying interpolation conditions:

$$\gamma(t_i) = p_i, \qquad 1 \le i \le k-1, \tag{2}$$

and boundary conditions:

$$\begin{aligned} &\gamma(0) = p_0,\ \gamma(\tau) = p_k, \\ &\dot\gamma(0) = V_0,\ \dot\gamma(\tau) = V_k. \end{aligned} \tag{3}$$

In the last section of this paper we present an algorithm that solves this problem and is an adaptation of the algorithm presented in (Hüper and Silva Leite, 2007) for generating interpolating curves on manifolds embedded in Euclidean space. One important step in this algorithm is based on a rolling technique that consists on rolling the given manifold over its affine tangent space at a point, along geodesic curves. The main drawback when trying to use the algorithm in (Hüper and Silva Leite, 2007) is that if the ellipsoid is embedded in Euclidean space, the corresponding geodesics are very hard to compute. In order to overcome this problem, we embed the ellipsoid in another Riemannian manifold where geodesics can be expressed in closed form. Moreover, rolling motions along these geodesics can be described easily, following the ideas in (Hüper *et al.*, 2011) to describe rolling motions of manifolds embedded in arbitrary Riemannian manifolds. So, in order that we can follow the implementation of the algorithm, we dedicate the next section to rolling maps. After presenting the general definition, we derive the kinematic equations for rolling the ellipsoid $\mathcal{E}^n$ over its affine tangent space at a point.

3 Rolling maps

We use an extended Sharpe's definition (Sharpe, 1997) of a *rolling map* which is applicable to Riemannian manifolds and can be found in (Hüper *et al.*, 2011). Hereafter $I \subset \mathbb{R}$ denotes a closed interval.

Definition 2. *Let M_0 and M_1 be two n-manifolds isometrically embedded in an m-dimensional Riemannian manifold M and $\sigma_1 \colon I \to \mathrm{M}_1$ a smooth curve in M_1. A* rolling map *of M_1 on M_0 along the curve σ_1, without slipping or twisting, is a map $\boldsymbol{\chi} \colon I \to \mathsf{Isom}(\mathrm{M})$ satisfying the following conditions, for all $t \in I$:*

Rolling

(a) $\boldsymbol{\chi}(t)(\sigma_1(t)) \in \mathrm{M}_0$*;*
(b) $\mathrm{T}_{\boldsymbol{\chi}(t)(\sigma_1(t))}(\boldsymbol{\chi}(t)(\mathrm{M}_1)) = \mathrm{T}_{\boldsymbol{\chi}(t)(\sigma_1(t))}\mathrm{M}_0$.

The curve $\sigma_0 \colon I \to \mathrm{M}_0$ defined by $\sigma_0(t) := \boldsymbol{\chi}(t)(\sigma_1(t))$ is called the development curve *of σ_1.*

No-slip $\dot\sigma_0(t) = \boldsymbol{\chi}_*(t)(\dot\sigma_1(t))$, *where $\boldsymbol{\chi}_*$ is the push-forward of $\boldsymbol{\chi}$.*

No-twist

tangential: $\left(\dot{\boldsymbol{\chi}}(t) \circ \boldsymbol{\chi}(t)^{-1}\right)_*(\mathrm{T}_{\sigma_0(t)}\mathrm{M}_0) \subset \mathrm{T}_{\sigma_0(t)}\mathrm{M}_0^\perp$,
normal: $\left(\dot{\boldsymbol{\chi}}(t) \circ \boldsymbol{\chi}(t)^{-1}\right)_*(\mathrm{T}_{\sigma_0(t)}\mathrm{M}_0^\perp) \subset \mathrm{T}_{\sigma_0(t)}\mathrm{M}_0$,
where $\mathrm{T}_p\mathrm{M}_0^\perp$ denotes the normal *space at $p \in \mathrm{M}_0$.*

This definition can be extended to the situation when σ_1 is only piecewise smooth. In this case $\boldsymbol{\chi}$ is also piecewise smooth and the constraints of no-slip and no-twist are valid for almost all t.

4 The geometry of the ellipsoid

The Euclidean metric in $\mathbb{R}^{n+1}$ is denoted by $\langle \cdot, \cdot \rangle$. The positive definite matrix $\mathbf{D} = \operatorname{diag}(d_1, d_2, \ldots, d_{n+1}) \succ 0$ induces another metric on $\mathbb{R}^{n+1}$ defined by $(U, V) \mapsto \langle U, V \rangle_{\mathbf{D}^{-2}} := \langle U, \mathbf{D}^{-2}V \rangle$. This metric space will be denoted by $\mathrm{M} = \left(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle_{\mathbf{D}^{-2}}\right)$. Since

$$\langle \mathbf{D}U, \mathbf{D}V \rangle_{\mathbf{D}^{-2}} = \langle \mathbf{D}U, \mathbf{D}^{-2}\mathbf{D}V \rangle = \langle U, V \rangle,$$

then the mapping

$$\varphi \colon (\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle) \to \mathrm{M}, \tag{4}$$

given in the standard coordinates by $\boldsymbol{x} \mapsto \mathbf{D}\boldsymbol{x}$, is an isometry. M is an example of a space equipped with a left-invariant metric. Unlike the Euclidean space, groups of isometries acting on objects from the left hand side is different than from the right. The main reason for introducing M is that the ellipsoid $\mathcal{E}^n$ behaves like the sphere in Euclidean space with its standard metric.

4.1 The Riemannian connection

Let $\overline{\nabla}$ be the standard Riemannian connection on $\mathbb{R}^{n+1}$. Then $\varphi_* \overline{\nabla}$ is the Riemannian connection on M, *cf.* (Lee, 1997, Proposition 5.6). More precisely, $\forall X, Y \in \mathrm{TM}$, $\varphi_*\big(\overline{\nabla}_X Y\big) = \nabla_{\varphi_* X}\left(\varphi_* Y\right)$, hence

$$\nabla_X Y = \mathbf{D}\overline{\nabla}_{\mathbf{D}^{-1}X}\left(\mathbf{D}^{-1}Y\right) = \overline{\nabla}_{\mathbf{D}^{-1}X} Y \qquad (5)$$

is the Riemannian connection on M. We leave it to the reader to check that ∇ defined by (5) is indeed the Riemannian connection (or Levi-Civitta connection) on M, and it is compatible with respect to $\langle\cdot,\cdot\rangle_{\mathbf{D}^{-2}}$ and is torsion free.

Let $\gamma\colon I \to \mathrm{M}$ be curve in M and let $V\colon I \to \mathrm{TM}$ be a vector field along γ, i.e., $V(t) \in \mathrm{T}_{\gamma(t)}\mathrm{M}$, for all $t \in I$. Let $\bar{\gamma}$ and $\overline{V}$ be their isometric images in $\mathbb{R}^{n+1}$. By (5) and because $\mathbf{D}$ is constant, there is

$$\nabla_{\dot{\gamma}} V = \mathbf{D}\nabla_{\dot{\bar{\gamma}}}\overline{V} = \mathbf{D}\dot{\overline{V}} = \dot{V}.$$

Hence the covariant derivative of the vector field V does not depend on γ!

4.2 The group of isometries of M

Let $\mathsf{Isom}(\mathrm{M})$ denote the (Lie) group of isometries of M. Suppose that $\varphi\colon \mathrm{M} \to \mathrm{M}$ is an isometry. Therefore, for any $p \in \mathrm{M}$ and $U, V \in \mathrm{T}_p\mathrm{M}$ the following equality holds.

$$\big\langle U, V\big\rangle_{\mathbf{D}^{-2}} = \big\langle \varphi_* U, \varphi_* V\big\rangle_{\mathbf{D}^{-2}}$$

or, equivalently,

$$\big\langle U, \mathbf{D}^{-2}V\big\rangle = \big\langle \varphi_* U, \mathbf{D}^{-2}\varphi_* V\big\rangle = \big\langle U, \varphi_*^{\mathrm{T}}\mathbf{D}^{-2}\varphi_* V\big\rangle.$$

It follows now that $\mathbf{D}^{-2} = \varphi_*^{\mathrm{T}}\mathbf{D}^{-2}\varphi_*$. Then, $\varphi_* \in \mathcal{G}_{\mathbf{D}^{-2}}$, where $\mathcal{G}_{\mathbf{D}^{-2}}$ is the matrix quadratic Lie group defined as

$$\mathcal{G}_{\mathbf{D}^{-2}} := \left\{\, g \;:\; g^{\mathrm{T}}\mathbf{D}^{-2}g = \mathbf{D}^{-2} \,\right\}.$$

The Lie algebra of $\mathcal{G}_{\mathbf{D}^{-2}}$ is defined as:

$$\mathcal{L}_{\mathbf{D}^{-2}} := \left\{\, A \;:\; A^{\mathrm{T}}\mathbf{D}^{-2} = -\mathbf{D}^{-2}A \,\right\}.$$

It can be easily seen that, for any $g \in \mathcal{G}_{\mathbf{D}^{-2}}$, there exists exactly one $R \in \mathbb{SO}(n+1)$ such that $g = \mathbf{D}R\mathbf{D}^{-1}$. Therefore $\mathcal{G}_{\mathbf{D}^{-2}} = \mathbf{D}\mathbb{SO}(n+1)\mathbf{D}^{-1}$ and the two groups are isomorphic, i.e., $\mathcal{G}_{\mathbf{D}^{-2}} \cong \mathbb{SO}(n+1)$. Also, for any $\Omega \in \mathcal{L}_{\mathbf{D}^{-2}}$, there exists exactly one $A \in \mathfrak{so}(n+1)$ such that $\Omega = \mathbf{D}A\mathbf{D}^{-1}$. At the same time, we established that $\mathsf{Isom}(\mathrm{M}) = \mathcal{G}_{\mathbf{D}^{-2}} \ltimes \mathbb{R}^{n+1} \cong \mathbb{SE}(m)$. In the reminder of this paper elements of $\mathsf{Isom}(\mathrm{M})$ will be denoted as pairs (g, s), with $g \in \mathcal{G}_{\mathbf{D}^{-2}}$ and $s \in \mathbb{R}^{n+1}$. The group operations in $\mathcal{G}_{\mathbf{D}^{-2}} \ltimes \mathbb{R}^{n+1}$ are defined as: $(g,s)^{-1} = \left(g^{-1}, -g^{-1}s\right)$ and $(g_1, s_1)\cdot(g_2, s_2) = (g_1 g_2, g_1 s_2 + s_1)$.

4.3 The ellipsoid as the unit sphere in M

The ellipsoid $\mathcal{E}^n$ is the unit sphere in M, i.e., it is defined by

$$\mathcal{E}^n := \left\{\, x \in \mathrm{M} \;:\; |x|_{\mathbf{D}^{-2}} = 1 \,\right\}. \qquad (6)$$

For $\varepsilon > 0$, let $\gamma\colon\ (-\varepsilon, \varepsilon) \to \mathcal{E}^n$ be any differentiable curve with $\gamma(0) = p$ and $\dot{\gamma}(0) = V$. Differentiating the condition $|\gamma|_{\mathbf{D}^{-2}} = 1$ with respect to t yields

$$0 = \frac{d}{dt}\,|\gamma|^2_{\mathbf{D}^{-2}} = 2\big\langle\dot{\gamma}, \mathbf{D}^{-2}\gamma\big\rangle = 2\big\langle\dot{\gamma}, \gamma\big\rangle_{\mathbf{D}^{-2}}.$$

At $t = 0$, the above equality yields $\big\langle V, p\big\rangle_{\mathbf{D}^{-2}} = 0$. Henceforth the tangent space $\mathrm{T}_p\mathcal{E}^n$ is the subspace orthogonal to p in M with respect to its metric $\langle\cdot,\cdot\rangle_{\mathbf{D}^{-2}}$. The unit normal vector $\Lambda \in (\mathrm{T}_p\mathcal{E}^n)^{\perp}$ is given by $\Lambda = p/|p|_{\mathbf{D}^{-2}} = p$. Hence, the Weingarten map Ξ_Λ at $p \in \mathcal{E}^n$ is minus the identity, i.e., $\Xi_\Lambda = -\mathsf{id}$. The *scalar second fundamental form* h can be easily derived from the Weingarten equation $\big\langle\Xi_\Lambda(X), Y\big\rangle_{\mathbf{D}^{-2}} = -\big\langle X, Y\big\rangle_{\mathbf{D}^{-2}} = -h(X,Y)$. Hence the *second fundamental form* is $\mathrm{II}(X,Y) = \big\langle X, Y\big\rangle_{\mathbf{D}^{-2}}p$.

The tangent space may be defined in terms of $\mathbf{D}$ as:

$$\mathrm{T}_p\mathcal{E}^n := \left\{\, \mathbf{D}A\mathbf{D}^{-1}p \;:\; A \in \mathfrak{so}(n+1) \,\right\}. \qquad (7)$$

4.4 Geodesics on the ellipsoid

Given a point $p_0 \in \mathcal{E}^n$ and a vector $V_0 \in T_{p_0}\mathcal{E}^n$, there exists unique geodesic $t \mapsto \gamma(t)$ satisfying $\gamma(0) = p_0$, $\dot{\gamma}(0) = V_0$. This geodesic is defined by

$$\gamma(t) = p_0\,\cos(t\,|V_0|) + V_0\frac{\sin(t\,|V_0|)}{|V_0|}. \qquad (8)$$

The algorithm to be presented in the last section depends on the implementation of geodesic arcs that join two points on the ellipsoid. So, at this stage we also present an explicit formula to compute the geodesic arc $t \mapsto \gamma(t)$ on $\left(\mathcal{E}^n, \langle\cdot,\cdot\rangle_{\mathbf{D}^{-2}}\right)$, joining the points p_i (at $t = t_i$) and p_{i+1} (at $t = t_{i+1}$) (with $p_i \neq \pm p_{i+1}$):

$$\begin{aligned}\gamma(t) = &\frac{1}{\sin\theta_i}\left\{\sin\!\Big(\frac{\theta_i}{t_{i+1}-t_i}\,(t_{i+1}-t)\Big)\,p_i\right.\\ &\left.+\sin\!\Big(\frac{\theta_i}{t_{i+1}-t_i}\,(t-t_i)\Big)p_{i+1}\right\},\end{aligned} \qquad (9)$$

where $\theta_i = \arccos\big\langle p_i, p_{i+1}\big\rangle_{\mathbf{D}^{-2}}$.

This can be easily checked by computing $\ddot{\gamma}(t)$, to conclude that $\ddot{\gamma}(t) = -\theta_i^2\,\gamma(t)$, so $\ddot{\gamma}(t)$ belongs to $\big(\mathrm{T}_{\gamma(t)}\mathcal{E}^n\big)^{\perp}$ in M.

5 Rolling the ellipsoid

We aim to write kinematic equations for the ellipsoid $\mathcal{E}^n$ rolling upon its affine tangent plane, when both are embedded in $\mathrm{M} = \left(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle_{\mathbf{D}^{-2}}\right)$. We derive the equations in a few steps, starting with the distribution of the rolling map.

5.1 The configuration space and the distribution

For $\mathrm{M}_1 = \mathcal{E}^n$ choose an initial point of contact p_0 that, without loss of generality, to be the "south pole" of the ellipsoid. Then $p_0 := -\mathbf{D}\mathbf{e}_{n+1} = -d_{n+1}\mathbf{e}_{n+1} \in \mathbf{S}^n_{\mathbf{D}}$. The affine tangent space at p_0 is defined by

$$\mathrm{M}_0 = \mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n := \left\{ x \in \mathrm{M} \ : \ x = p_0 + (p_0)^{\perp} \right\},$$

where $(p_0)^{\perp}$ denotes the set of vectors in M that are normal to p_0 with respect to the metric $\mathbf{D}^{-2}$. The configuration space $\mathcal{Q} \subset \mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n \times \mathcal{G}_{\mathbf{D}^{-2}} \times \mathcal{E}^n$ of the rolling map $\boldsymbol{\chi}$ is the space of all possible positions of the unit sphere $\mathcal{E}^n$ tangent to its affine tangent plane. Namely

$$\begin{aligned} \mathcal{Q} =& \left\{ \left(p, g, q\right) \in \mathrm{M}_0 \times \mathcal{G}_{\mathbf{D}^{-2}} \times \mathcal{E}^n \right. \\ & \left. \ : \ g(\mathrm{T}_q\mathcal{E}^n) = \mathrm{T}_p\mathrm{M}_0 \right\}. \end{aligned}$$

The distribution of the rolling map is defined by the following set of three differential equations:

(1) $\dot{p} = g\dot{q}$,
(2) $\dot{g}g^{-1}V = -g\mathrm{II}^1(g^{-1}\dot{p}, g^{-1}V)$, for all $V \in \mathrm{T}_p\mathrm{M}_0$, and
(3) $\dot{g}g^{-1}\Lambda = -g\Xi^1(g^{-1}\dot{p}, g^{-1}\Lambda)$, for all $\Lambda \in (\mathrm{T}_p\mathrm{M}_0)^{\perp}$.

These equations, correspond to the no-slip and no-twist constraints in Definition 2. Let $\sigma_1(t) = g^{-1}(t)(p_0)$, where $g\colon I \to \mathcal{G}_{\mathbf{D}^{-2}}$ satisfies $g(0) = id$. Since the metric $\langle \cdot, \cdot \rangle_{\mathbf{D}^{-2}}$ is left-invariant with respect to $\mathcal{G}_{\mathbf{D}^{-2}}$ and $\mathcal{G}_{\mathbf{D}^{-2}}$ acts transitively on $\mathcal{E}^n$ any curve can be parameterised in this way. For any fixed $t \in I$ assign $\sigma_0(t) = p$ and $\sigma_1(t) = q$. Then the first condition above reads $\dot{\sigma}_0 = g\dot{\sigma}_1$. This is the no-slip condition. Equation (3) is redundant because the ellipsoid has codimension one, and equation (2) becomes

$$\begin{aligned} \dot{g}g^{-1}V =& - g\langle g^{-1}\dot{\sigma}_0, g^{-1}V\rangle_{\mathbf{D}^{-2}}\sigma_1 \\ =& - \langle \dot{\sigma}_0, V\rangle_{\mathbf{D}^{-2}}p_0, \end{aligned}$$

$\forall V \in \mathrm{T}_p\mathrm{M}_0$. Let $\boldsymbol{A} = \dot{R}R^{-1}$. Then $\dot{g}g^{-1} = \mathbf{D}\boldsymbol{A}\mathbf{D}^{-1}$ and the above equality becomes

$$\begin{aligned} \boldsymbol{A}\mathbf{D}^{-1}V &= -\langle \dot{\sigma}_0, V\rangle_{\mathbf{D}^{-2}}\mathbf{D}^{-1}p_0 \\ &= \langle \mathbf{D}^{-1}\dot{\sigma}_0, \mathbf{D}^{-1}V\rangle \mathbf{e}_{n+1}. \end{aligned} \tag{10}$$

In the standard coordinates, the entries of the matrix $\boldsymbol{A}$ can be found using $A^j{}_i = \langle \mathbf{e}_j, \boldsymbol{A}\mathbf{e}_i\rangle$. By (10), for any $1 \le i \le n$

$$\boldsymbol{A}\mathbf{e}_i = \boldsymbol{A}\mathbf{D}^{-1}\mathbf{D}\mathbf{e}_i = \langle \mathbf{D}^{-1}\dot{\sigma}_0, \mathbf{e}_i\rangle \mathbf{e}_{n+1}.$$

Hence, $A^j{}_i = d_i^{-1}\dot{\sigma}_0^i$, for $1 \le i < j = n+1$, and $A^j{}_i = 0$, otherwise. Since $\boldsymbol{A}$ is skew-symmetric then

$$\boldsymbol{A} = \begin{pmatrix} 0 & \dots & 0 & -u_1 \\ 0 & \dots & 0 & -u_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & -u_n \\ u_1 & \dots & u_n & 0 \end{pmatrix} = -u\,\mathbf{e}_{n+1}^{\mathrm{T}} + \mathbf{e}_{n+1}\,u^{\mathrm{T}},$$

where $u = (u_1, \dots, u_n, 0)^{\mathrm{T}} = -\mathbf{D}^{-1}\dot{\sigma}_0$. In general

$$\boldsymbol{A}(t) = u(t)\,p_0^{\mathrm{T}}\,\mathbf{D}^{-1} - \mathbf{D}^{-1}\,p_0\,u^{\mathrm{T}}(t). \tag{11}$$

Proposition 3. *Let $R\colon I \to \mathbb{SO}(n+1)$ and $s\colon I \to \mathbb{R}^{n+1}$ be solutions to the following set of equations*

$$\begin{cases} \dot{s}(t) = -\mathbf{D}\boldsymbol{A}(t)\mathbf{D}^{-1}p_0 \\ \dot{R}(t) = \boldsymbol{A}(t)R(t) \end{cases}, \tag{12}$$

with $R(0) = I$ and $s(0) = 0$. Then, $\boldsymbol{\chi}\colon I \to \mathcal{G}_{\mathbf{D}^{-2}} \ltimes \mathbb{R}^{n+1}$ given by

$$\boldsymbol{\chi}(t) = (g(t), s(t)) = \left(\mathbf{D}R(t)\mathbf{D}^{-1}, s(t)\right) \tag{13}$$

is a rolling map of the ellipsoid rolling on its affine tangent space in M, with rolling curve $\sigma_1(t) = \mathbf{D}R^{-1}\mathbf{D}^{-1}p_0$ and its development $\sigma_0(t) = s(t) + p_0$.

Proof. This is just a matter of checking that all the conditions of Definition 2 hold.

Rolling It is easy to verify that since $\dot{s}(t)$ is normal to p_0 in M then equality (a) holds:

$$\begin{aligned} \boldsymbol{\chi}(t)(\sigma_1(t)) &= \left(\mathbf{D}R(t)\mathbf{D}^{-1}\right)\left(\mathbf{D}R(t)^{\mathrm{T}}\mathbf{D}^{-1}\right)p_0 + s(t) \\ &= p_0 + s(t) \in \mathrm{M}_0. \end{aligned}$$

To verify (b) it is enough to see that since the metric on M is left invariant with respect to $\mathcal{G}_{\mathbf{D}^{-2}}$, this group sends the unit sphere to itself. Also, since the normal spaces of $\mathcal{E}^n$ and $\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n$ coincide at the point of contact, so do the tangent spaces.

No-slip From the above calculations of the development curve it follows that

$$\dot{\sigma}_0 = \dot{s} = -gg^{-1}\,\dot{g}g^{-1}p_0 = -\dot{g}g^{-1}p_0 = -\mathbf{D}\boldsymbol{A}\mathbf{D}^{-1}p_0.$$

No-twist It is enough to verify the tangential part because the normal one follows immediately from (12). For any vector $V \in \mathrm{T}_{\sigma_0(t)}\mathrm{M}_0$.

$$\begin{aligned}
(\dot{g}g^{-1})(V) &= \mathbf{D}\boldsymbol{A}\mathbf{D}^{-1}V \\
&= \mathbf{D}\big(up_0^{\mathrm{T}}\mathbf{D}^{-1} - \mathbf{D}^{-1}p_0u^{\mathrm{T}}\big)\mathbf{D}^{-1}V \\
&= \mathbf{D}up_0^{\mathrm{T}}\mathbf{D}^{-2}V - \mathbf{D}\mathbf{D}^{-1}p_0u^{\mathrm{T}}\mathbf{D}^{-1}V \\
&= \mathbf{D}u\langle p_0, V\rangle_{\mathbf{D}^{-2}} - p_0\langle u, \mathbf{D}^{-1}V\rangle \\
&= -\langle u, \mathbf{D}^{-1}V\rangle p_0 \in \mathrm{T}_{p_0}^{\perp}\mathcal{E}^n.
\end{aligned}$$

The proof is now complete. □

(One can find an alternative proof of Proposition 3 in (Hüper *et al.*, 2011).) In general, the kinematic equations (12) may be hard to solve. However, when $\boldsymbol{A}(t) = \boldsymbol{A}$ is constant, explicit solutions can be found. This corresponds to rolling motions along geodesics.

Corollary 4. *For the special situation when $\boldsymbol{A}(t) = \boldsymbol{A}$ is constant, the solution of the kinematic equations is given by*

$$R(t) = \exp(t\boldsymbol{A}) \quad \textit{and} \quad s(t) = -t\,\mathbf{D}\boldsymbol{A}\mathbf{D}^{-1}p_0,$$

the rolling curve and its development, given respectively by

$$\begin{aligned}
\sigma_1(t) &= g^{-1}(t)p_0 = \mathbf{D}\exp(-t\boldsymbol{A})\,\mathbf{D}^{-1}p_0, \\
\sigma_0(t) &= p_0 + s(t) = p_0 - t\,\mathbf{D}\boldsymbol{A}\mathbf{D}^{-1}p_0,
\end{aligned} \tag{14}$$

are geodesics on the ellipsoid $\big(\mathcal{E}^n, \langle\cdot,\cdot\rangle_{\mathbf{D}^{-2}}\big)$ and on its affine tangent space $\big(\mathrm{T}_{p_0}^{\mathrm{aff}}\mathcal{E}^n, \langle\cdot,\cdot\rangle_{\mathbf{D}^{-2}}\big)$.

Proof. The only statement that requires a computation is that σ_1 is a geodesic on the ellipsoid for the metric induced by $\mathbf{D}^{-2}$. This is easily checked by computing its second derivative and comparing with (7) as follows:

$$\begin{aligned}
\dot{\sigma}_1(t) &= -\mathbf{D}\boldsymbol{A}\exp(-t\boldsymbol{A})\,\mathbf{D}^{-1}p_0 = -\mathbf{D}\boldsymbol{A}\mathbf{D}^{-1}\sigma_1(t), \\
\ddot{\sigma}_1(t) &= \mathbf{D}\boldsymbol{A}^2\mathbf{D}^{-1}\sigma_1(t) = -|u|^2\,\sigma_1(t) \in (\mathrm{T}_{\sigma_1(t)}\mathcal{E}^n)^{\perp}.
\end{aligned}$$

The last equality can be verified by noting that $\langle u, \mathbf{D}^{-1}p_0\rangle = 0$ and, because $\boldsymbol{A}^2\exp(-t\boldsymbol{A})\,\mathbf{D}^{-1}p_0 = \exp(-t\boldsymbol{A})\,\boldsymbol{A}^2\mathbf{D}^{-1}p_0$, then it follows from (11) that

$$\boldsymbol{A}^2\mathbf{D}^{-1}p_0 = \boldsymbol{A}\left(\boldsymbol{A}\mathbf{D}^{-1}p_0\right) = \boldsymbol{A}u = -|u|^2\,\mathbf{D}^{-1}p_0.$$

What was to show. □

Remark 5. *Expanding the power series*

$$\begin{aligned}
\exp(-t\boldsymbol{A})\,\mathbf{D}^{-1}p_0 &= \sum_{i=0}^{\infty}\frac{(-t)^i}{i!}\boldsymbol{A}^i\,\mathbf{D}^{-1}p_0 \\
&= \sum_{k=0}^{\infty}(-1)^k\,\frac{t^{2k}}{(2k)!}\,|u|^{2k}\;\mathbf{D}^{-1}p_0 \\
&\quad - \sum_{k=0}^{\infty}(-1)^k\,\frac{t^{2k+1}}{(2k+1)!}\,|u|^{2k}\,u
\end{aligned}$$

yields the expression for the geometric exponential map. *So,*

$$\sigma_1(t) = p_0\,\cos(t\,|u|) - \mathbf{D}u\frac{\sin(t\,|u|)}{|u|}, \tag{15}$$

and $-\mathbf{D}\,u \in \mathrm{T}_{p_0}\mathcal{E}^n$ is the initial velocity vector of the geodesic σ_1. This agrees with the formula (8) *and gives a geometric interpretation of the control vector u in* (11).

From the point of view of Control Theory, the ellipsoid rolling on its affine tangent space is controllable. This is a direct consequence of the positivity of the Gaussian curvature of the ellipsoid. In turn, one can steer the ellipsoid from an admissible configuration (any configuration in which the ellipsoid is tangent to the affine tangent space at a point) to any other admissible configuration, only by rolling without twist and without slip. Interested reader is referred to (Krakowski and Silva Leite, 2012) for more details.

6 Algorithm to generate an interpolating curve on the ellipsoid $\mathcal{E}^n$

This algorithm is based on a procedure to generate interpolating curves on some manifolds embedded in Euclidean space, first described in (Jupp and Kent, 1987) for the 2-sphere, generalised in (Hüper and Silva Leite, 2002) for the n-sphere and in (Hüper *et al.*, 2007) for the rotation group and Grassmann manifolds. Here we show how this algorithm can be extended to the ellipsoid $\mathcal{E}^n$ to generate an interpolating curve, given in closed form, that solves the Problem 1 stated in Subsection 2.1. We also implement the algorithm for the 2-dimensional ellipsoid.

The basic idea behind the algorithm is to project the data from $\mathcal{E}^n$ to $\mathrm{T}_{p_0}^{\mathrm{aff}}\mathcal{E}^n$, solve an interpolation problem in this affine space and, finally, projecting back to $\mathcal{E}^n$ the interpolating curve on the affine space. The projection uses a mixed technique of rolling/unrolling and unwrapping/wrapping, performed by an appropriate rolling map and a convenient diffeomorphism. These two maps must satisfy some conditions, as follows.

1. The rolling map (to perform the rolling/unrolling):

Choose a rolling map $\chi = (\mathbf{D}R\mathbf{D}^{-1}, s)\colon [0,\tau] \to \mathcal{G}_{\mathbf{D}^{-2}} \ltimes \mathbb{R}^{n+1}$ of $\mathcal{E}^n$ on $\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n$, along a smooth curve α_1 that joins p_0 (at $t=0$) to p_k (at $t=\tau$), with development α_0.

2. The local diffeomorphism (to perform the unwrapping/wrapping):
Choose a suitable local diffeomorphism, on an open neighbourhood U of p_0,

$$\Phi\colon U \subset \mathcal{E}^n \to \mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n, \tag{16}$$

so that

$$\Phi(p_0) = p_0 \quad \text{and} \quad \partial\Phi^{-1}(p_0) = I_{n+1}, \tag{17}$$

where $\partial\Phi$ denotes the Jacobian matrix of Φ.

6.1 The Algorithm

The algorithm consists essentially of five steps.

Step 1. Compute the rolling curve

$$\alpha_1\colon [0,\tau] \to \mathcal{E}^n, \tag{18}$$

connecting p_0 with p_k, i.e., such that

$$\alpha_1(0) = p_0 \quad \text{and} \quad \alpha_1(\tau) = p_k. \tag{19}$$

Step 2. Unwrap the boundary data by rolling $\mathcal{E}^n$ along α_1, so that:

$$\begin{aligned} p_0 &\mapsto \chi(0)p_0 := q_0 = p_0 \in \mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n,\\ p_k &\mapsto \chi(\tau)p_k := q_k \in \mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n, \end{aligned} \tag{20}$$

as well as

$$\begin{aligned} V_0 &\mapsto \chi_*(0)V_0 := W_0 = V_0 \in \mathrm{T}_{q_0}(\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n),\\ V_k &\mapsto \chi_*(\tau)V_k := W_k \in \mathrm{T}_{q_k}(\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n). \end{aligned} \tag{21}$$

Step 3. Unwrap the remaining interpolating points p_i, $i = 1,\dots,k-1$ at t_i, from $\mathcal{E}^n$ to $\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n$, using the diffeomorphism Φ and the time dependent rolling map χ, so that

$$\begin{aligned} p_i \mapsto \Phi\Big(&\chi(t_i)p_i - \alpha_0(t_i) + p_0\Big)\\ &+ \alpha_0(t_i) - p_0 =: q_i. \end{aligned} \tag{22}$$

Step 4. Solve the interpolating problem on $\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n$ for the projected data $\{q_0,\dots,q_k; W_0, W_k\}$, to generate a C^2-smooth curve

$$\beta\colon [0,\tau] \to \mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n \tag{23}$$

satisfying

$$\begin{aligned} &\beta(0) = p_0 = q_0,\\ &\beta(t_i) = q_i,\\ &\beta(\tau) = q_k,\\ &\dot\beta(0) = V_0 = W_0,\\ &\dot\beta(\tau) = W_k. \end{aligned} \tag{24}$$

Step 5. Wrap $\beta([0,\tau])$ back onto the ellipsoid using Φ^{-1}, while unrolling along α_1, to produce a curve γ, defined by the following explicit formula.

$$\begin{aligned} \gamma(t) := \chi(t)^{-1}\Big(\Phi^{-1}\Big(&\beta(t) - \alpha_0(t) + p_0\Big)\\ &+ \alpha_0(t) - p_0\Big) \end{aligned} \tag{25}$$

Theorem 6. *The curve $\gamma\colon [0,\tau] \mapsto \mathcal{E}^n$ defined by* (25) *solves Problem 1.*

Proof. Recall that $s(t) = \alpha_0(t) - p_0$, $\chi = \big(\mathbf{D}R\mathbf{D}^{-1}, s\big)$ and $\chi^{-1} = \big(\mathbf{D}R^{-1}\mathbf{D}^{-1}, -\mathbf{D}R^{-1}\mathbf{D}^{-1}s\big)$. A simple calculation shows that

$$\gamma(t) = \mathbf{D}R^{-1}(t)\mathbf{D}^{-1}\Big(\Phi^{-1}\big(\beta(t) - s(t)\big)\Big),$$

$$\begin{aligned} \dot\gamma(t) = \mathbf{D}\dot R^{-1}(t)\mathbf{D}^{-1}\Big(\Phi^{-1}\Big(\beta(t) - s(t)\Big)\Big) +\\ \mathbf{D}R^{-1}(t)\mathbf{D}^{-1}\Big(\partial\Phi^{-1}\Big(\beta(t) - s(t)\Big)\Big(\dot\beta(t) - \dot s(t)\Big)\Big). \end{aligned}$$

To compute the boundary conditions, note that $R(0) = I$, $s(0) = 0$ and $\beta(0) = p_0$. So,

$$\gamma(0) = \Phi^{-1}\Big(\beta(0) - s(0)\Big) = \Phi^{-1}(p_0) = p_0. \tag{26}$$

Also, $\beta(\tau) = \alpha_0(\tau)$, which implies

$$\begin{aligned} \gamma(\tau) &= \mathbf{D}R^{-1}(\tau)\mathbf{D}^{-1}\Big(\Phi^{-1}(p_0)\Big)\\ &= \mathbf{D}R^{-1}(\tau)\mathbf{D}^{-1}(p_0) = \alpha_1(\tau) = p_k. \end{aligned} \tag{27}$$

Now, it follows from the kinematic equations that $\dot R^{-1}(0) = -A(0)$, $\dot s(t) = -\mathbf{D}A(t)\mathbf{D}^{-1}p_0$, and $\dot R^{-1}(\tau) = -R^{-1}(\tau)A(\tau)$. Also, since $\dot\beta(\tau) = \dot\alpha_0(\tau) = \chi_*(\tau)\dot\alpha_1(\tau)$, we have $\dot\beta(\tau) = \mathbf{D}R(\tau)\mathbf{D}^{-1}V_k$. As a consequence,

$$\begin{aligned} \dot\gamma(0) =& -\mathbf{D}A(0)\mathbf{D}^{-1}\Big(\Phi^{-1}(p_0)\Big)\\ &+ \mathbf{D}R^{-1}(0)\mathbf{D}^{-1}\Big(\partial\Phi^{-1}(p_0)\Big)\Big(\dot\beta(0) - \dot s(0)\Big)\\ =&\dot s(0) + (\dot\beta(0) - \dot s(0)) = V_0, \end{aligned}$$

$$
\begin{aligned}
\dot{\gamma}(\tau) =& \mathbf{D}\dot{R}^{-1}(\tau)\mathbf{D}^{-1}\Big(\Phi^{-1}(\beta(\tau)-s(\tau))\Big) \\
&+ \mathbf{D}R^{-1}(\tau)\mathbf{D}^{-1}\Big(\partial\Phi^{-1}(\beta(\tau)-s(\tau))(\dot{\beta}(\tau)-\dot{s}(\tau)\Big) \\
&= -\mathbf{D}R^{-1}(\tau)A(\tau)\mathbf{D}^{-1}p_0 \\
&\quad + \mathbf{D}R^{-1}(\tau)\mathbf{D}^{-1}\Big(\dot{\beta}(\tau)+\mathbf{D}A(\tau)\mathbf{D}^{-1}p_0\Big) \\
&= \mathbf{D}R^{-1}(\tau)\mathbf{D}^{-1}\dot{\beta}(\tau) \\
&= \mathbf{D}R^{-1}(\tau)\mathbf{D}^{-1}\mathbf{D}R(\tau)\mathbf{D}^{-1}V_k = V_k.
\end{aligned}
$$

Finally, looking at the expression of $\gamma(t_i)$ and using the expression

$$
\beta(t_i) = \Phi\Big(\boldsymbol{\chi}(t_i)p_i - \alpha_0(t_i) + p_0\Big) + \alpha_0(t_i) - p_0,
$$

that comes from (22), since $\beta(t_i) = q_i$, we obtain after simplifications

$$
\gamma(t_i) = p_i.
$$

The resulting curve is C^2-smooth by construction, since Φ and $\boldsymbol{\chi}$ are smooth and β is C^2-smooth. This concludes the proof. □

Remark 7. *At this point it is important to point out that step 4. can be easily implemented, although performed on a non-Euclidean submanifold. This is due to the fact that geodesics, and other polynomial curves are the same on* $\left(\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n, \langle\cdot,\cdot\rangle_{\mathbf{D}^{-2}}\right)$ *and* $\left(\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n, \langle\cdot,\cdot\rangle\right)$*. Indeed, the Euler Lagrange equation is the same for the two problems*

$$
\min_x \int_0^\tau \left\langle x^{(k)}(t), x^{(k)}(t)\right\rangle dt
$$

and

$$
\min_x \int_0^\tau \left\langle x^{(k)}(t), x^{(k)}(t)\right\rangle_{\mathbf{D}^{-2}} dt
$$

and is given by $x^{(2k)} = 0$*. In particular, for* $k = 2$*, cubic polynomials (cubic splines) in* $\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^n$ *may be generated by the classical De Casteljau algorithm.*

7 Implementation of the algorithm on $\mathcal{E}^2$

In order to implement the algorithm on $\mathcal{E}^2$, we have to choose the rolling map so that the corresponding kinematic equations can be solved explicitly. For that reason, we choose $\boldsymbol{\chi}\colon [0,\tau] \to \mathcal{G}_{\mathbf{D}^{-2}} \ltimes \mathbb{R}^{n+1}$ to be the the rolling map of $\mathcal{E}^2$ on $\mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^2$, along the geodesic α_1 that joins p_0 (at $t=0$) to p_k (at $t=\tau$), with development α_0. Our choice for the local diffeomorphism Φ is the stereographic projection from the "north pole". Before we proceed with the implementation of the algorithm, we give details about this projection.

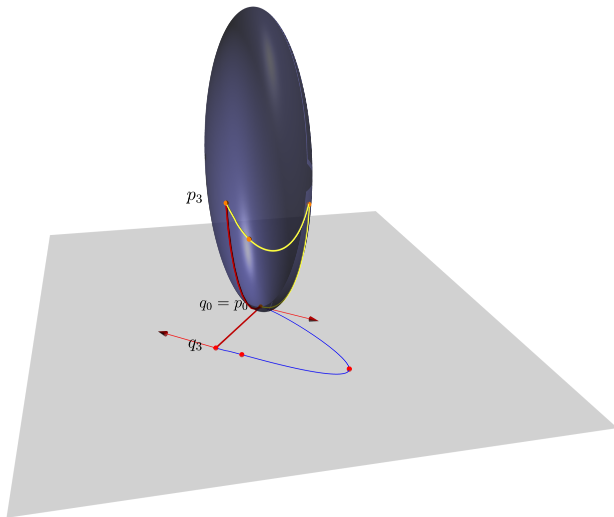

Figure 1. Smooth interpolation on the ellipsoid.

7.1 Stereographic projection of $\mathcal{E}^2$

The stereographic projection from the "north pole" of the ellipsoid to the tangent plane at the "south pole" $p_0 = [0,0,-d_3]^\top \in \mathcal{E}^2$ is given by:

$$
\begin{aligned}
\Phi\colon \mathcal{E}^2 \setminus \{[0,0,d_3]^\top\} &\to \quad \mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^2 \\
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &\mapsto \begin{pmatrix} \dfrac{2d_3x_1}{d_3-x_3} \\[2ex] \dfrac{2d_3x_2}{d_3-x_3} \\[2ex] -d_3 \end{pmatrix},
\end{aligned}
\tag{28}
$$

with inverse

$$
\begin{aligned}
\Phi^{-1}\colon \mathrm{T}^{\mathrm{aff}}_{p_0}\mathcal{E}^2 &\to \quad \mathcal{E}^2 \setminus \{[0,0,d_3]^\top\} \\
\begin{pmatrix} \xi_1 \\ \xi_2 \\ -d_3 \end{pmatrix} &\mapsto \begin{pmatrix} \dfrac{4d_1^2d_2^2\xi_1}{d_2^2\xi_1^2+d_1^2\xi_2^2+4d_1^2d_2^2} \\[2ex] \dfrac{4d_1^2d_2^2\xi_2}{d_2^2\xi_1^2+d_1^2\xi_2^2+4d_1^2d_2^2} \\[2ex] \dfrac{(d_2^2\xi_1^2+d_1^2\xi_2^2-4d_1^2d_2^2)d_3}{d_2^2\xi_1^2+d_1^2\xi_2^2+4d_1^2d_2^2} \end{pmatrix}.
\end{aligned}
$$

Remark 8. *It can easily be shown that* Φ *satisfies the following:*

$$
\Phi(p_0) = p_0, \qquad \partial\Phi^{-1}(p_0) = I_3, \tag{29}
$$

where $\partial\Phi^{-1}$ *denotes the Jacobian matrix of the differentiable map* Φ^{-1}*.*

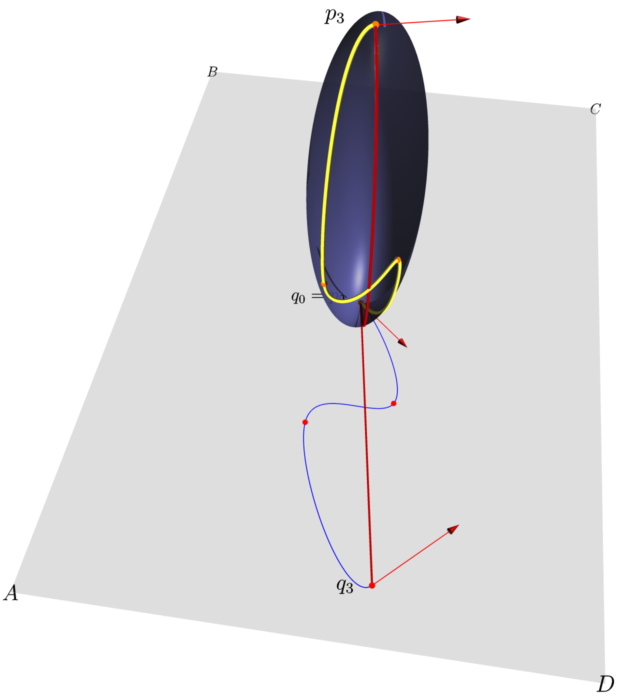

Figure 2. Comparison of interpolation on the ellipsoid through two geodesic segments between conjugate (antipodal) points.

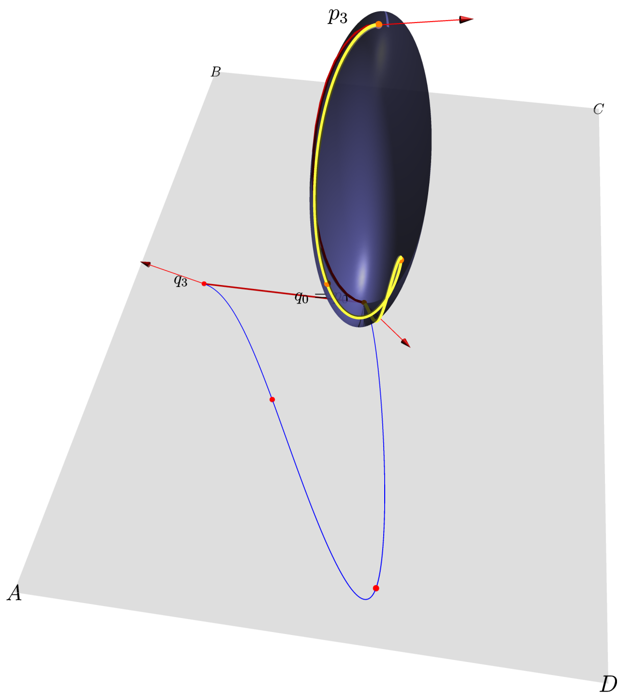

Figure 3. Comparison of interpolation on the ellipsoid through two geodesic segments between conjugate (antipodal) points.

We now have all the necessary ingredients to generate interpolating curves on $\mathcal{E}^2$. Figure 1 shows the main steps of the algorithm and the resulting interpolating curve.

Although Theorem 6 guarantees that a solution to the interpolation problem exists, it says nothing about its uniqueness. It is clear from (25) that the interpolating curve γ depends on the choice of a rolling curve α_1 and a diffeomorphism Φ. But even when the later is fixed and the rolling curve is chosen to be a geodesic arc joining the initial and the final points, there might be many solution curves for the interpolating problem. This occurs, and was already expected, when those points are antipodal since there are infinitely many geodesics joining them. Figures 2 and 3 illustrate what happens when two different geodesic segments joining antipodal points are used as rolling curves. It is worth noting different directions of the transformed ending vectors W_k in each case. This is a result of the curvature of the ellipsoid.

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References

Camarinha, M. (1996). *The Geometry of Cubic Polynomials on Riemannian Manifolds*. PhD Thesis. Departamento de Matemática, Universidade de Coimbra, Portugal.

Crouch, P. and F. Silva Leite (1991). Geometry and the Dynamic Interpolation Problem. *Proc. American Control Conference, Boston* pp. 1131–1137.

Crouch, P. and F. Silva Leite (1995). The Dynamic Interpolation Problem: on Riemannian Manifolds, Lie Groups and Symmetric Spaces. *Journal of Dynamical and Control Systems* **1**(2), 177–202.

Crouch, P., G. Kun and F. Silva Leite (1999). The De Casteljau Algorithm on Lie Groups and Spheres. *Journal of Dynamical and Control Systems* **5**(3), 397–429.

Giambó, R., Giannoni F. and Piccione P. (2002). Fitting smooth paths to spherical data. *IMA J. Math. Control Inform.* **19**, 445–460.

Hüper, K. and F. Silva Leite (2002). Smooth interpolating curves with applications to path planning. In *10th IEEE Mediterranean Conference on Control and Automation (MED 2002)*. Lisbon - Portugal.

Hüper, K., M. Kleinsteuber and F.S. Leite (2007). On the geometry of rolling maps and applications to robotics. *ROBOMAT 07* p. 117.

Hüper, Knut and Fátima Silva Leite (2007). On the geometry of rolling and interpolation curves on S^n, SO_n and Graßmann manifolds. *Journal of Dynamical and Control Systems* **13**(4), 467–502.

Hüper, Knut, Krzysztof Andrzej Krakowski and Fátima Silva Leite (2011). *Rolling Maps in a Riemannian Framework. Mathematical Papers in Honour of Fátima Silva Leite*. pp. 15–30. Vol. 43 of *Textos de Matemática*. Department of Mathematics, University of Coimbra, Portugal.

Jupp, P. E. and J. T. Kent (1987). Fitting smooth paths to spherical data. *Applied Statist.* **36**, 34–46.

Krakowski, Krzysztof Andrzej and Fátima Silva Leite (2012). Why controllability of rolling may fail: a few illustrative examples. In *Pré-Publicações do Departamento de Matemática*. pp. 1–30. Number 12-26. University of Coimbra.

Lee, John M. (1997). *Riemannian Manifolds: An Introduction to Curvature*. Number 176 in *Graduate Texts in Mathematics*. Springer-Verlag. New York.

Noakes, L., G. Heinzinger and B. Paden (1989). Cubic Splines on Curved Spaces. *IMA Journal of Mathematics Control and Information* **6**, 465–473.

Park, F. . and B. Ravani (1995). Optimal control of the sphere S^n rolling on E^n. *ASME J. Mech. Design* **117**, 36–40.

Sharpe, Richard W. (1997). *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program*. Number 166 in *Graduate Texts in Mathematics*. Springer-Verlag. New York.