DYNAMICS AND CONTROL OF A TWO-DEGREE-OF-FREEDOM VIBRATION-DRIVEN SYSTEM

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Abstract: A rectilinear motion of a system of two bodies connected by a spring on a rough horizontal plane is studied. The coefficient of friction between the bodies and the plane is assumed to be small. The motion of the system is excited by two identical unbalance rotors based on the respective bodies. Major attention is given to the steady-state velocity-periodic motion. A nearly-resonant excitation mode, for which the angular velocities of the rotor are close to the natural frequency of the system, is considered. It is shown that control of the steady-state motion can be provided by changing the phase shift between the rotations of the rotors and the sign of the resonant detuning measured by the difference between the angular velocity of the rotors and the natural frequency of the system. By varying the phase shift one can control the magnitude of the average velocity and varying the detuning enables one to change the direction of the motion.

Keywords: Vibration-driven Systems, Dry Friction, Dynamics, Control

1. INTRODUCTION

In (Zimmermann *et al.*, 2004; Zeidis *et al.*, 2007), the motion of a system of two identical point masses connected by a spring along a straight line on a rough horizontal plane is studied. Dry (Coulomb's) friction is assumed to act between the masses and the plane. The motion is excited by a harmonic force acting between the masses. The coefficient of friction is assumed to depend on the direction of motion of the respective mass. The characteristic of the spring is either linear (Zimmermann *et al.*, 2004) or nonlinear (cubic) (Zeidis *et al.*, 2007). Bolotnik *et al.* (2006) considered the rectilinear motion of one point mass acted upon by two harmonic forces. The mass moves on a horizontal rough plane. One of the forces acts along the line of motion and the other along the vertical. The excitation forces have the same frequency but can have different amplitudes and be shifted in phase. The coefficient of friction is independent of the direction of motion.

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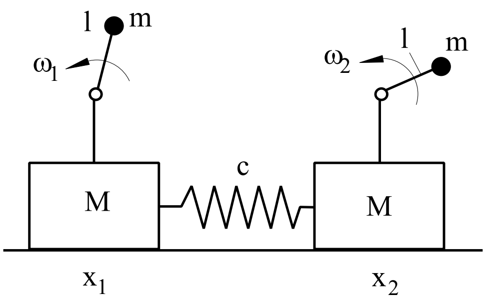

Fig. 1. The schematic of the system

The present paper deals with a system of two identical bodies connected by a linear spring. The rectilinear motion of this system on a horizontal rough plane is investigated. The motion is excited by two unbalance rotors attached to the respective bodies. The coefficient of friction is independent of the direction of motion. In the case of small friction, an approximate algebraic equation is obtained for the steady-state velocity of the entire system. The steady-state velocity is studied as a function of parameters of the system. The results obtained can form a theoretical basis for the design of vibration-driven microrobots.

2. MATHEMATICAL MODEL

Consider two bodies (modeled by particles), 1 and 2, that have the same mass M and are connected by a linear spring of stiffness c. The bodies are based on a rough horizontal plane and can move along the same straight line. There is Coulomb's friction acting between the bodies and the plane. The coefficient of friction k is independent of the direction of motion of the bodies. The motion of the system is excited by two identical unbalance rotors based on the respective bodies. The axes of rotation of the rotors are perpendicular to the vertical plane passing through the line of motion of the system and the centers of mass of the rotors lie in this plane. The schematic of the system is shown in Fig. 1. Let m and l denote the mass of each rotor and the distance between the axis of rotation and the center of mass, respectively. The rotors rotate in the same direction with the angular velocities ω_1 and ω_2. Let x_1 and x_2 be the displacements of bodies 1 and 2 relative to a fixed (inertial) reference frame. The zero points for the coordinates x_1 and x_2 are shifted by the length of the undeformed spring. Denote by φ_i the angle between the line of motion of the system and the perpendicular dropped from the center of mass of the ith rotor to its axis of rotation.

The motion of the mechanical system under consideration is governed by the set of equations

$$\begin{aligned}&(m+M)\ddot{x}_1 + c(x_1 - x_2) = ml\omega_1^2 \cos\varphi_1 + R_1,\\&(m+M)\ddot{x}_2 + c(x_2 - x_1) = ml\omega_2^2 \cos\varphi_2 + R_2,\\&\varphi_1 = \omega_1 t, \quad \varphi_2 = \omega_2 t + \varphi_0,\end{aligned} \tag{1}$$

where R_i, $i = 1,\ 2$, is the Coulomb friction force acting on body i. According to Coulomb's law,

$$R_i = \begin{cases}-kN_i \mathrm{sgn}\, \dot{x}_i, \text{ if } \dot{x}_i \neq 0,\\ -F_i, \text{ if } \dot{x}_i = 0 \text{ and } |F_i| \leq kN_i,\\ -kN_i \mathrm{sgn}\, F_i, \text{ if } \dot{x}_i = 0 \text{ and } |F_i| > kN_i,\end{cases} \tag{2}$$

where

$$\begin{aligned}&F_1 = m\omega_1^2 \cos\varphi_1 - c(x_1 - x_2),\\&F_2 = m\omega_2^2 \cos\varphi_2 + c(x_1 - x_2),\\&N_i = (m+M)g - ml\omega_i^2 \sin\varphi_i, \quad i = 1,\ 2.\end{aligned} \tag{3}$$

Here g is the acceleration due to gravity.

The quantity N_i in (3) is the normal force exerted on body i by the supporting plane. Contact between the bodies and the supporting plate is a unilateral constraint, since the plane resists penetration but does not resist separation of the bodies from the plane. Therefore, $N_i \geq 0$ while body i has contact with the plane. To ensure this inequality, assume

$$\frac{ml\omega_i^2}{(m+M)g} \leq 1, \ \ i = 1,\ 2. \tag{4}$$

Introduce the dimensionless variables

$$\begin{aligned}&x_i^* = \frac{x_i}{l},\ t^* = t\sqrt{\frac{c}{M+m}},\ \nu_i = \omega_i \sqrt{\frac{M+m}{c}},\\&\varepsilon = \frac{(M+m)kg}{cl},\ \alpha = \frac{mcl}{(M+m)^2 g},\ \beta = \frac{\alpha}{k}.\end{aligned} \tag{5}$$

Proceed to the dimensionless variables in Eqs. (1)–(3) and then omit the asterisks identifying the variables x^* and t^* to arrive at the relations

$$\begin{aligned}&\ddot{x}_1 + x_1 - x_2 = \varepsilon\beta\nu_1^2 \cos\varphi_1 + r_1,\\&\ddot{x}_2 + x_2 - x_1 = \varepsilon\beta\nu_2^2 \cos\varphi_2 + r_2,\\&\varphi_1 = \nu_1 t, \quad \varphi_2 = \nu_2 t + \varphi_0,\end{aligned} \tag{6}$$

where

$$r_i = \begin{cases}-\varepsilon n_i \mathrm{sgn}\, \dot{x}_i, \quad \text{if } \dot{x}_i \neq 0,\\ -f_i, \quad \text{if } \dot{x}_i = 0 \text{ and } |f_i| \leq \varepsilon n_i,\\ -\varepsilon n_i \mathrm{sgn}\, f_i, \quad \text{if } \dot{x}_i = 0 \text{ and } |f_i| > \varepsilon n_i,\end{cases} \tag{7}$$

$$\begin{aligned}&f_1 = \varepsilon\beta\nu_1^2 \cos\varphi_1 - x_1 + x_2,\\&f_2 = \varepsilon\beta\nu_2^2 \cos\varphi_2 + x_1 - x_2,\\&n_i = 1 - \alpha\nu_i^2 \sin\varphi_i.\end{aligned} \tag{8}$$

The conditions of Eq. (4) become

$$\alpha\nu_1^2 \leq 1, \quad \alpha\nu_2^2 \leq 1. \tag{9}$$

In what follows we assume that $\varepsilon \ll 1$ and $\varepsilon\beta = O(\varepsilon)$ and consider the motions that do not involve stick-slip modes that may occur in mechanical systems subject to dry friction forces. In this case, one should set $r_i = -\varepsilon n_i \mathrm{sgn}\, \dot{x}_i$ in Eq. (6). Note that $\varepsilon\beta = m/(M+m)$ and, hence, the assumption of $\varepsilon\beta = O(\varepsilon)$ implies that the mass of the rotor is small as compared with the mass of the body on which this rotor is based.

For $\varepsilon = 0$, the general solution of Eq. (6) can be represented in the form

$$x_1 = A + Vt - a\cos(\sqrt{2}t + b),\\ x_2 = A + Vt + a\cos(\sqrt{2}t + b), \tag{10}$$

where A, V, a, and b are arbitrary constants to be determined by initial conditions. The motion of the system corresponding to this solution is the superposition of the uniform motion of the point $X = (x_1 + x_2)/2$, the middle point of the segment between bodies (particles) 1 and 2, and harmonic oscillations of the distance between these bodies. The oscillations have a frequency of $\sqrt{2}$ and the point X moves at the velocity V. In what follows we will study the motion of the system in the case of nearly-resonant excitation, when the excitation frequency is close to the frequency of natural oscillations of bodies 1 and 2 connected by a spring, i.e.,

$$\nu_1 = \sqrt{2} + \varepsilon\Delta_1, \quad \nu_2 = \sqrt{2} + \varepsilon\Delta_2, \tag{11}$$

where Δ_1 and Δ_2 are constant parameters.

Proceed from the variables x_1, x_2, φ_1, and φ_2 of Eq. (6) to the variables X, a, ξ_1, and ξ_2 introduced by the relations

$$x_1 = X - a\cos\varphi, \quad x_2 = X + a\cos\varphi,\\ \xi_1 = \varphi_1 - \varphi, \quad \xi_2 = \varphi_2 - \varphi, \quad V = \dot{X}. \tag{12}$$

The variable $X = (x_1 + x_2)/2$ in (12) is the coordinate of the middle point of the segment between bodies 1 and 2. The point X will be used as a representative point to characterize the progressive motion of the entire system.

Using this change of variables, one can derive the standard system of differential equations for the slow variables, V, a, ξ_1 and ξ_2,

$$\begin{aligned}
\dot{V} &= \frac{\varepsilon}{2}\Big\{2\beta[\cos(\varphi+\xi_2) + \cos(\varphi+\xi_1)] \\
&- \Big[(1 - 2\alpha\sin(\varphi+\xi_1))\mathrm{sgn}(V + a\sqrt{2}\sin\varphi) \\
&+ (1 - 2\alpha\sin(\varphi+\xi_2))\mathrm{sgn}(V - a\sqrt{2}\sin\varphi)\Big]\Big\}, \\
\dot{a} &= -\frac{\xi}{2\sqrt{2}}\sin\varphi\Big\{2\beta[\cos(\varphi+\xi_2) - \cos(\varphi+\xi_1)] \\
&+ \Big[(1 - 2\alpha\sin(\varphi+\xi_1))\mathrm{sgn}(V + a\sqrt{2}\sin\varphi) \\
&- (1 - 2\alpha\sin(\varphi+\xi_2))\mathrm{sgn}(V - a\sqrt{2}\sin\varphi)\Big]\Big\}, \qquad (13) \\
\dot{\xi}_1 &= \frac{\varepsilon}{2\sqrt{2}}\frac{1}{a}\cos\varphi\Big\{2\beta[\cos(\varphi+\xi_2) - \cos(\varphi+\xi_1)] \\
&+ \Big[(1 - 2\alpha\sin(\varphi+\xi_1))\mathrm{sgn}(V + a\sqrt{2}\sin\varphi) \\
&- (1 - 2\alpha\sin(\varphi+\xi_2))\mathrm{sgn}(V - a\sqrt{2}\sin\varphi)\Big]\Big\} + \varepsilon\Delta_1, \\
\dot{\xi}_2 &= \frac{\varepsilon}{2\sqrt{2}}\frac{1}{a}\cos\varphi\Big\{2\beta[\cos(\varphi+\xi_2) - \cos(\varphi+\xi_1)] \\
&+ \Big[(1 - 2\alpha\sin(\varphi+\xi_1))\mathrm{sgn}(V + a\sqrt{2}\sin\varphi) \\
&- (1 - 2\alpha\sin(\varphi+\xi_2))\mathrm{sgn}(V - a\sqrt{2}\sin\varphi)\Big]\Big\} + \varepsilon\Delta_2.
\end{aligned}$$

The motion of the mechanical system under consideration will be studied on the basis of the method of averaging (Bolgolyubov and Mitropolskii, 1961) applied to Eq. (13). Major attention will be given to the steady-state motion with constant average velocity V and constant amplitude of oscillations a.

3. ANALYSIS OF THE STEADY-STATE MOTION

Average the right-hand sides of the relations of Eq. (13) with respect to the fast variable φ in accordance with Bolgolyubov and Mitropolskii (1961) to arrive at the approximate (averaged) system of equations

$$\dot{V} = \begin{cases} \varepsilon, & V < -a\sqrt{2}, \\ -\dfrac{2\varepsilon}{\pi}\Big[\arcsin\dfrac{V}{a\sqrt{2}} + \alpha(\cos\xi_2 \\ \quad - \cos\xi_1)\sqrt{1 - \dfrac{V^2}{2a^2}}\Big], & |V| \le a\sqrt{2}, \\ -\varepsilon, & V > a\sqrt{2}, \end{cases} \tag{14}$$

$$\dot{a} = \begin{cases} \dfrac{\varepsilon}{2\sqrt{2}}\Big[\beta(\sin\xi_2 - \sin\xi_1) \\ \quad +\alpha(\cos\xi_2 - \cos\xi_1)\Big], & V < -a\sqrt{2}, \\ -\dfrac{\varepsilon}{2\sqrt{2}\pi}\Big[4\sqrt{1 - \dfrac{V^2}{2a^2}} - \pi\beta(\sin\xi_2 - \sin\xi_1) \\ \quad +2\alpha(\cos\xi_2 - \cos\xi_1)\Big(\arcsin\dfrac{V}{a\sqrt{2}} \\ \quad -\dfrac{V}{a\sqrt{2}}\sqrt{1 - \dfrac{V^2}{2a^2}}\Big)\Big], & |V| \le a\sqrt{2}, \\ \dfrac{\varepsilon}{2\sqrt{2}}\Big[\beta(\sin\xi_2 - \sin\xi_1) \\ \quad -\alpha(\cos\xi_2 - \cos\xi_1)\Big], & V > a\sqrt{2}, \end{cases} \tag{15}$$

$$\dot{\xi}_1 = \begin{cases} \dfrac{\varepsilon}{2\sqrt{2}\pi}\dfrac{1}{a}\Big[\pi\beta(\cos\xi_2 - \cos\xi_1) - \alpha(\sin\xi_2 \\ \quad - \sin\xi_1)\Big] + \varepsilon\Delta_1, & V < -a\sqrt{2}, \\ \dfrac{\varepsilon}{2\sqrt{2}\pi}\dfrac{1}{a}\Big[\pi\beta(\cos\xi_2 - \cos\xi_1) + 2\alpha(\sin\xi_2 \\ \quad - \sin\xi_1)\Big(\arcsin\dfrac{V}{a\sqrt{2}} + \dfrac{V}{a\sqrt{2}}\sqrt{1 - \dfrac{V^2}{2a^2}}\Big)\Big] \\ \quad +\varepsilon\Delta_1, & |V| \le a\sqrt{2}, \\ \dfrac{\varepsilon}{2\sqrt{2}\pi}\dfrac{1}{a}\Big[\pi\beta(\cos\xi_2 - \cos\xi_1) + \alpha(\sin\xi_2 \\ \quad - \sin\xi_1)\Big] + \varepsilon\Delta_1, & V > a\sqrt{2}, \end{cases} \tag{16}$$

$$\dot{\xi}_2 = \begin{cases} \frac{\varepsilon}{2\sqrt{2}\pi}\frac{1}{a}\Big[\pi\beta(\cos\xi_2 - \cos\xi_1) - \alpha(\sin\xi_2 \\ \quad - \sin\xi_1)\Big] + \varepsilon\Delta_2, \qquad V < -a\sqrt{2}, \\ \frac{\varepsilon}{2\sqrt{2}\pi}\frac{1}{a}\Bigg[\pi\beta(\cos\xi_2 - \cos\xi_1) + 2\alpha(\sin\xi_2 \\ - \sin\xi_1)\left(\arcsin\frac{V}{a\sqrt{2}} + \frac{V}{a\sqrt{2}}\sqrt{1-\frac{V^2}{2a^2}}\right)\Bigg] \\ \quad + \varepsilon\Delta_2, \qquad |V| \le a\sqrt{2}, \\ \frac{\varepsilon}{2\sqrt{2}\pi}\frac{1}{a}\Big[\pi\beta(\cos\xi_2 - \cos\xi_1) + \alpha(\sin\xi_2 \\ \quad - \sin\xi_1)\Big] + \varepsilon\Delta_2, \qquad V > a\sqrt{2}. \end{cases} \tag{17}$$

Introduce the new variables

$$\psi = \frac{\xi_1 + \xi_2}{2}, \quad \chi = \frac{\xi_2 - \xi_1}{2} \tag{18}$$

to represent the relations of Eq. (14)–(18) in the form

$$\dot{V} = \begin{cases} \varepsilon, \qquad V < -a\sqrt{2}, \\ -\frac{2\varepsilon}{\pi}\left[\arcsin\frac{V}{a\sqrt{2}} \right. \\ \left. -2\alpha\sin\chi\,\sin\psi)\sqrt{1 - \frac{V^2}{2a^2}}\right], |V| \le a\sqrt{2}, \\ -\varepsilon, \qquad V > a\sqrt{2}, \end{cases} \tag{19}$$

$$\dot{a} = \begin{cases} \frac{\varepsilon}{\sqrt{2}}\Big[\beta\sin\chi\cos\psi + \alpha\sin\chi\sin\psi\Big), \\ \qquad V < -a\sqrt{2}, \\ -\frac{\varepsilon}{\sqrt{2}\pi}\Bigg[2\sqrt{1 - \frac{V^2}{2a^2}} - \pi\beta\sin\chi\cos\psi \\ \quad -2\alpha\sin\chi\sin\psi\Bigg(\arcsin\frac{V}{a\sqrt{2}} \\ \qquad -\frac{V}{a\sqrt{2}}\sqrt{1 - \frac{V^2}{2a^2}}\Bigg)\Bigg], |V| \le a\sqrt{2}, \\ -\frac{\varepsilon}{\sqrt{2}}\Big(-\beta\sin\chi\cos\psi + \alpha\sin\chi\sin\psi\Big), \\ \qquad V > a\sqrt{2}, \end{cases} \tag{20}$$

$$\dot{\psi} = \begin{cases} \frac{\varepsilon}{\sqrt{2}}\frac{1}{a}\Big[\beta\sin\chi\cos\psi + \alpha\sin\chi\sin\psi)\Big] \\ \quad +\frac{\varepsilon}{2}(\Delta_2 + \Delta_1), \qquad V < -a\sqrt{2}, \\ \frac{\varepsilon}{\sqrt{2}\pi}\frac{1}{a}\Bigg[\pi\beta\sin\chi\cos\psi - 2\alpha\sin\chi\sin\psi \\ \cdot\left(\arcsin\frac{V}{a\sqrt{2}} + \frac{V}{a\sqrt{2}}\sqrt{1 - \frac{V^2}{2a^2}}\right)\Bigg] \\ \quad +\frac{\varepsilon}{2}(\Delta_2 + \Delta_1), \qquad |V| \le a\sqrt{2}, \\ -\frac{\varepsilon}{\sqrt{2}}\frac{1}{a}\Big[-\beta\sin\chi\cos\psi + \alpha\sin\chi\sin\psi)\Big] \\ \quad +\frac{\varepsilon}{2}(\Delta_2 + \Delta_1), \qquad V > a\sqrt{2}, \end{cases} \tag{21}$$

$$\dot{\chi} = \frac{\varepsilon}{2}\left(\Delta_2 - \Delta_1\right). \tag{22}$$

From Eq. (19) it follows that for $\varepsilon \neq 0$, the identity $\dot{V} \equiv 0$ may hold only if $|V| \le a\sqrt{2}$. Equations (19) and (20) imply that $\dot{V} = \text{const}$ and $\dot{a} = \text{const}$ only if $\psi = \text{const}$. Then from Eq. (21) it follows that $\dot{\chi} = \text{const}$, and from Eq. (22) we find that $\Delta_1 = \Delta_2 = \Delta$. In this case, in accordance with Eq. (11), we obtain $\nu_1 = \nu_2 = \nu$. Then, in accordance with Eqs. (6), (12), and (18),

$$\chi = \frac{\xi_2 - \xi_1}{2} = \frac{\varphi_2 - \varphi_1}{2} = \frac{\varphi_0}{2}. \tag{23}$$

Denote

$$u = \frac{V}{a\sqrt{2}}, \quad \gamma = \sin\chi = \sin\frac{\varphi_0}{2} \tag{24}$$

and write the system of algebraic equations for steady-state solution of the system of Eqs. (19)–(22)

$$\arcsin u - 2\alpha\gamma\sin\psi\sqrt{1-u^2} = 0, \tag{25}$$

$$2\sqrt{1-u^2} - \pi\beta\gamma\cos\psi - 2\alpha\gamma\sin\psi\big(\arcsin u - u\sqrt{1-u^2}\big) = 0, \tag{26}$$

$$-\pi\beta\gamma\sin\psi + 2\alpha\gamma\cos\psi\big(\arcsin u + u\sqrt{1-u^2}\big) + \sqrt{2}\pi\Delta a = 0. \tag{27}$$

Eliminate ψ from these equations to obtain the expressions for γ and a as functions of u:

$$\gamma^2 = \frac{\arcsin^2 u}{4\alpha^2(1-u^2)} + \frac{\Big[2(1-u^2) - \arcsin u\big(\arcsin u - u\sqrt{1-u^2}\big)\Big]^2}{\pi^2\beta^2(1-u^2)}, \tag{28}$$

$$a = \frac{1}{\Delta}\frac{1}{\pi\sqrt{2(1-u^2)}}\Bigg\{\frac{\pi\beta}{2\alpha}\arcsin u - \frac{2\alpha}{\pi\beta}\big[(2+u^2)(1-u^2)\arcsin u + 2u(1-u^2)\sqrt{1-u^2}\arcsin^3 u\big]\Bigg\}. \tag{29}$$

Given $\gamma = \sin(\varphi_0/2)$, one can find u by solving Eq. (28) and substitute the resulting u into Eq. (29) to calculate a. Then the desired value of V is determined by the corresponding relation of Eq. (24): $V = ua\sqrt{2}$.

Although detailed analysis of the nonlinear system of Eqs. (28) and (29) is complicated, one can draw a number of important conclusions. Let (u_0, a_0) be a solution of the system of Eqs. (28) and (29) for a given set of parameters (α, β, γ, Δ). Then (-u_0, a_0) is a solution for the set of parameters (α, β, γ, $-\Delta$). Hence, one can control the direction of motion of the system by changing the detuning Δ in sign.

Consider an important particular case. We will find a solution of the system of Eqs. (28) and (29) such that

$$\xi_2 + \xi_1 = \pi + 2\delta \quad \left(\text{i.e., } \psi = \frac{\pi}{2} + \delta\right), \quad |\delta| \ll 1,$$
$$\xi_2 - \xi_1 = \pi, \quad \gamma = \sin\frac{\varphi_0}{2}. \tag{30}$$

In this case, Eq. (25) can be written, to within the terms of an order of δ^2, as

$$f(u) = \arcsin u - 2\alpha\sqrt{1-u^2} = 0. \tag{31}$$

Since $f(0) = -2\alpha < 0$, $f(1) = \pi/2 > 0$, and the derivative of the function $f(u)$ is positive for $0 < u < 1$,

$$f'_u(u) = \frac{1 + 2\alpha u}{\sqrt{1-u^2}}, \quad 0 < u < 1, \tag{32}$$

this function monotonically increases and has exactly one root on the interval $0 < u < 1$. For given u, from Eq. (26) we find (to within the terms of an order of δ^2)

$$\delta(u) = \frac{2}{\pi\beta}\left[\alpha\arcsin u - (\alpha u + 1)\sqrt{1-u^2}\right]. \tag{33}$$

Since $\delta(0) = -2/(\pi\beta) < 0$, $\delta(1) = \alpha/\beta = k > 0$, and the derivative of the function $\delta(u)$ is positive on the interval $0 < u < 1$,

$$\delta'_u(u) = \frac{2u(2\alpha u + 1)}{\pi\beta\sqrt{1-u^2}}, \quad 0 < u < 1, \tag{34}$$

the desired value of δ lies between $-2/(\pi\beta)$ and k. By adjusting appropriately the parameter β and the coefficient of friction k one can make δ reasonably small.

Having found u and δ, utilize Eq. (27) to calculate the amplitude a:

$$a = \frac{\pi\beta + 2\alpha\delta\left(\arcsin u + u\sqrt{1-u^2}\right)}{\pi\Delta\sqrt{2}}. \tag{35}$$

The amplitude a is positive if δ is nonnegative or negative with sufficiently small absolute value. In this case, the system moves forward, i.e., $V > 0$. By changing δ to $-\delta$ and Δ to $-\Delta$ we obtain the velocity equal in absolute value but negative in sign, which corresponds to the case of $\xi_2 - \xi_1 = \pi$ and $\xi_2 + \xi_1 = -\pi + 2\delta$, i.e., $\psi = -\pi/2 + \delta$ and $\gamma = \sin(\varphi_0/2) = 1$.

4. NUMERICAL RESULTS

The numerical calculations were performed for the experimental model of the vibration-driven system shown in Fig. 2. This model has been designed and constructed at the Technical University of Ilmenau and has the following parameters:

$$M = 0.1 \text{ kg}, \quad m = 0.03 \text{ kg}, \quad l = 0.03 \text{ m}, \quad \omega = 25 \text{ s}^{-1}, \quad k = 0.1, \quad \Delta = 1. \tag{36}$$

Then, in accordance with Eq. (5), $\varepsilon = 0.052$, $\alpha = 0.44$, and $\beta = 4.41$.

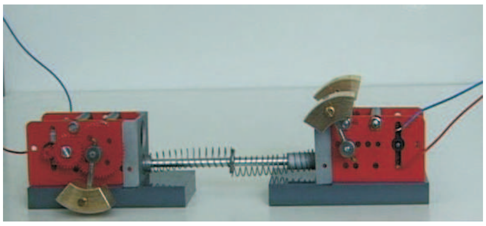

Fig. 2. Experimental model of the vibration-driven system

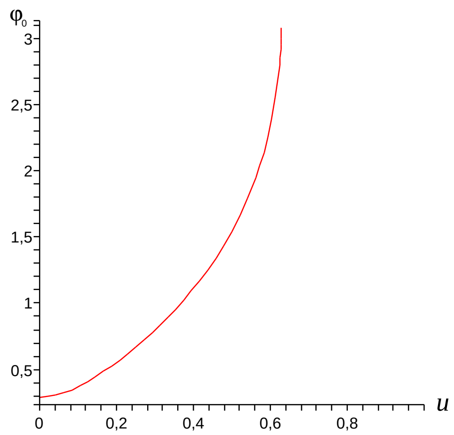

Fig. 3. Phase shift φ_0 as a function of u

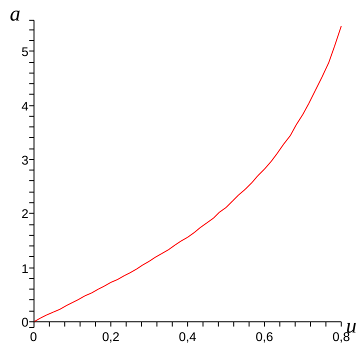

Fig. 4. Amplitude a as a function of u

Figure 3 shows φ_0 as a function of u. This plot was constructed on the basis of Eq. (28) and the relation $\gamma = \sin(\varphi_0/2)$ of Eq. (24). Figure 4 shows the graph of the function $a(u)$ defined by Eq. (29). The plot of Fig. 3 was used to calculate u for $\varphi_0 = \pi$ $(\gamma = 1)$, which yielded $u \approx 0.65$. Then Fig. 4 was used to obtain $a \approx 3.1$, after which the velocity $V = au\sqrt{2} \approx 2.84$ was calculated.

A close result is obtained on the basis of the approximate system of Eqs. (31), (33), and (35). Using Eq. (31) we find $u \approx 0.63$, then use Eqs. (33) to calculate $\delta \approx -0.10$, and finally determine $a \approx 3.10$ on the basis of Eq. (35). The corresponding

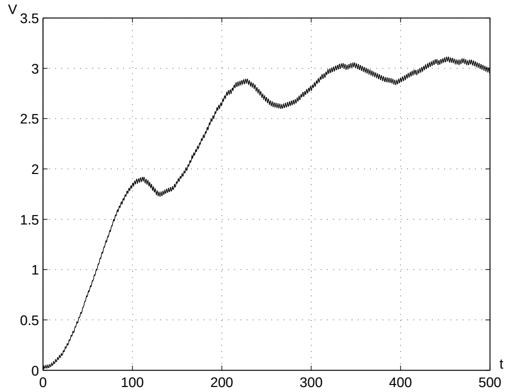

Fig. 5. Numerical solution result for $\varphi_0 = \pi$

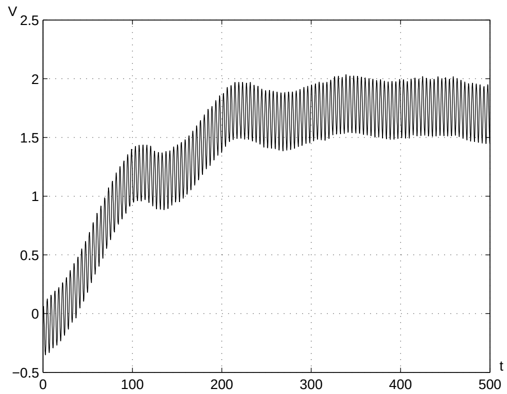

Fig. 6. Numerical solution result for $\varphi_0 = \pi/2$

value of the average velocity of the system is $V = au\sqrt{2} \approx 2.76$. It is apparent from Figs. 3 and 4 that this velocity, corresponding to $\varphi_0 = \pi$, is maximal.

Figure 5 has been constructed on the basis of the numerical solution of the exact equations of (6) subject to zero initial conditions ($x_1(0) = x_2(0) = 0$, $\dot{x}_1(0) = \dot{x}_2(0) = 0$), for the initial phase shift $\varphi_0 = \pi$. It is apparent from this figure that the average velocity of the steady-state motion of the system is close to 3, which demonstrates good agreement with the value $V \approx 2.84$ calculated on the basis of the averaged equations.

For $\varphi_0 = \pi/2$ ($\gamma = \sqrt{2}/2$), we use Eq. (3.15) to find $u = 0.5$, then use Eq. (29) to determine $a \approx 2.2$, and finally calculate $V = au\sqrt{2} \approx 1.55$.

Figure 6 presents the result of the numerical solution of the exact equations of (6) for $\varphi_0 = \pi/2$. It is apparent from the curve of Fig. 6 that the average steady-state velocity of motion of the entire system is about 1.7.

5. CONCLUSION

The motion of the system that consists of two bodies connected by a spring and is excited by two unbalance rotors attached to the respective bodies is studied. For small coefficients of friction between the bodies and the rough plane along which the system moves, a system of algebraic equations is obtained for determining an approximate value of the average steady-state velocity of the entire system. It is shown that the velocity can be controlled by changing the initial value of phase shift between the rotations of the rotors. The direction of the motion can be changed by changing the difference between the natural frequency of the system and the angular velocities of the rotors in sign. An experimental model of the vibration-driven system designed on the basis of the concept presented in the paper was designed and constructed.

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