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HAMILTONIAN CONSTRUCTIONS IN INVERSE PROBLEMS OF NAVIGATION

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Abstract
Landing process on the Moon is under consideration. It is assumed that information about real motions is known with errors. A new algorithm for solving online dynamic reconstruction problems for controls of the navigation system is created. Key elements of the constructions are solutions of corresponding hamiltonian (characteristic) systems in auxiliary optimization problems.

Key words
Navigation, dynamic control reconstruction, Hamiltonian systems.

1 Introduction

Dynamic reconstruction problems for controls during the process of Moon landing are under consideration. It is assumed that current measurements of the real motion are inaccurate, with known error estimates. A new algorithm for solving the online dynamic reconstruction problems for controls of the navigation system is created. To solve the problems we suggest to introduce the cost functional of the discrepancy with measurements and the corresponding variation problems for the navigation systems minimizing the functional. We consider solutions of the auxiliary calculus of variation problems as approximations of the solution of online control reconstruction problems for the navigation system.

There is a well-known approach solving these inverse problems that was proposed in the studies by Osipov and Kryazhimskii [Kryazhimskii and Osipov, 1983] and [Osipov and Kryazhimskii, 1995]. The proposed method involves a regularized procedure of extremal aiming at the dynamics of a guiding system similar to the navigation one. The construction uses the couple system of the double state variables. This approach appels to the optimal feedback theory developed in N.N. Krasovskii school [Krasovskii, 1968] and [Krasovskii and Subbotin, 1988].

We introduces and discuss the new method which is close to this approach. In contrast with it, we introduce auxiliary calculus of variation problems with a regularized integral discrepancy functional and the measured fixed initial state and speed. We apply necessary optimality conditions in the terms of hamiltonian system for state and conjugate variables of the navigation system. So, our construction of solution uses the coupled system of the state and conjugate variables. A distinctive feature of the new method is that the negative discrepancies [Subbotina, Tokmantsev, and Krupennikov, 2015] can be used.

Both above–mentioned approaches to solving inverse problems of the dynamics of control systems can be regarded as variants of Tikhonov's regularization method [Tikhonov, 1963].

In this paper, we present results of applications of the new method for solving of control reconstruction problems for landing on the Moon [Leitmann, 1962], [Letov, 1969], [Michel, 1977], [Liu et al., 2008].

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2 Statement of the Control Reconstruction Problem

We consider the following mathematical model of navigation [Letov, 1969].

We are watching the last stage of moving of the ship before landing on the Moon. We assume that:
- the trajectory of the ship is a straight line orthogonal to the surface of the Moon at the landing point;

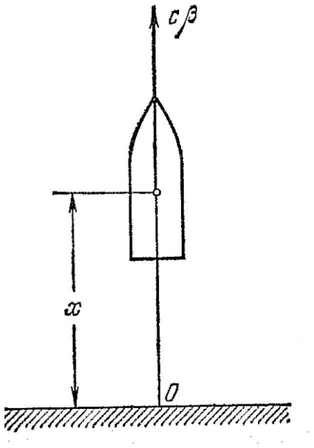

Figure 1. Landing Process on the Moon

- the Moon is stationary and it is flat in the neighborhood of the landing point;
- there are no aerodynamic forces;
- the ship is powered by gravity mg and traction force (braking) $T = cu$;
- here m is fuel mass, $m \geq m_0 > 0$, $g = 1,622$ m/sec^2 is Moon acceleration of gravity, velocity of gas outflow from the nozzle $c = 500$ m/sec is constant, fuel consumption (control) u is limited from above by the constant β.

So, dynamics of the ship is describe by the equation

$$\ddot{x} = \frac{cu}{m} - g, \qquad 0 \leq u \leq \beta, \tag{1}$$

or by the system

$$\begin{aligned} \frac{dx_1}{dt} &= x_2; \\ \frac{dx_2}{dt} &= \frac{cu}{x_3} - g; \\ \frac{dx_3}{dt} &= -u; \end{aligned} \tag{2}$$

with the restrictions on controls

$$u \in U = \{0 \leq u \leq \beta\}. \tag{3}$$

We denote by $\mathbf{U}[t_0, T]$ the set of measurable functions $u(\cdot)\colon [t_0, T] \to U$, $t_0 \in [0, T)$, i.e., the set of admissible controls.

In what follows, the symbol L_2 will denote the space $L_2[0, T]$.

We observe the real landing process $(x_1^*(t), x_2^*(t), x_3^*(t))$ and get online inaccurate discrete state information $(y_1(t_j), y_3(t_j))$:

$$\|y_1(t_j) - x_1^*(t_j)\| \leq \delta, \quad \|y_3(t_j) - x_3^*(t_j)\| \leq \delta,$$

$t_0 = 0 < t_1 < t_2, \ldots, < t_N = T$, $\delta > 0$ with small delay $\Delta t_j = t_j - t_{j-1} \leq \delta$.

We construct smooth continuous approximations $(y_1(t), y_2(t), y_3(t))$ of the measurements. We assume that

$$\begin{gathered} \|y_1(t) - x_1^*(t)\| \leq \delta, \\ \|y_3(t) - x_3^*(t)\| \leq \delta, \\ \delta > 0, \\ y_2(t) = \dot{y}_1(t). \end{gathered}$$

We denote by $\mathbb{U}$ (where $\mathbb{U} \subset \mathbf{U}[0, T]$) the subset of admissible controls generating the real trajectory $(x_1^*(t), x_2^*(t), x_3^*(t))$.

Using properties of the set $\mathbb{U}$ and the strong convexity of the norm $\|u(\cdot)\|$ in L_2, we can provide the following proposition like it was done in the paper [Subbotina and Tokmantsev, 2015].

Proposition 1. *The set $\mathbb{U}$ is nonempty, convex, and bounded in L_2. Moreover, there exists a unique element $u^*(\cdot) \in \mathbb{U}$ with minimum norm in L_2.*

The solution $u^*(\cdot) \in \mathbb{U}$ is called the *normal solution* to the inverse problem of dynamical reconstruction of control generating the real trajectory $(x_1^*(t), x_2^*(t), x_3^*(t))$. In general case, the set $\mathbb{U}$ contains more than one element.

Our aim is to reconstruct the normal control $u^*(t)$ generated the observing landing process $(x_1^*(t), x_2^*(t), x_3^*(t))$. It means the following: for any known $\delta > 0$, $y_1(\cdot)$, $y_2(\cdot)$, $y_3(\cdot)$ we consider the inverse problem consists in constructing a control $u^\delta(\cdot)\colon [0, T] \to U$ and the trajectory $x^\delta(\cdot)\colon [0, T] \to \mathbb{R}^n$ system (2), (3) generated by this control such that

$$\|x^\delta(\cdot) - x^*(\cdot)\|_C = \max_{t \in [0,T]} \|x^\delta(t) - x^*(t)\| \to 0,$$

$$\|u^\delta(\cdot) - u^*(\cdot)\|_{L_2}^2 = \int_0^T \|u^\delta(t) - u^*(t)\|^2 dt \to 0$$

hold as $\delta \to 0$. Here, $\|\cdot\|_C$ is the norm in the space of continuous functions and $\|\cdot\|_{L_2}$ is the norm in the space L_2.

3 Solution of the Reconstruction Problem

To solve the inverse problem we consider the following auxiliary variation problem.

3.1 Auxiliary Calculus of Variations Problems

We introduce additional controls u_1, u_2, u_3 and controlled system of the form:‘

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 + u_1; \\ \frac{dx_2}{dt} &= \frac{c}{x_3}u_3 - g + u_2; \\ \frac{dx_3}{dt} &= -u_3; \end{aligned} \tag{4}$$

$$0 \le u_3 \le \beta. \tag{5}$$

Let us consider the following cost functional:

$$I(u(\cdot), x(\cdot)) = \int_0^T -\frac{(x_1(t) - y_1(t))^2}{2} - \frac{(x_2(t) - y_2(t))^2}{2} - \frac{(x_3(t) - y_3(t))^2}{2} + \frac{\alpha^2}{2}[u_1(t)^2 + u_2(t)^2 + u_3(t)^2]dt. \tag{6}$$

Here $\alpha > 0$ is a small regularizing parameter.

We need to minimize cost functional (6) over the set of continuously differential functions

$$\begin{aligned} x(\cdot) &= (x_1(\cdot), x_2(\cdot), x_3(\cdot)) : \ [0, T] \to R^3, \\ u(\cdot) &= (u_1(\cdot), u_2(\cdot), u_3(\cdot)) : \ [0, T] \to R^3, \end{aligned}$$

which satisfy the fixed initial conditions

$$x_i(0) = y_i(0), \quad \dot{x}_i(0) = \dot{y}_i(0), \quad i = 1, 2, 3, \tag{7}$$

and dynamic relations (5).

3.2 Solution of the Calculus of Variations Problem

Necessary optimality conditions [Elsgolc, 1962] in the problem (4), (5), (6) have the form

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 - \frac{s_1}{\alpha^2}; \\ \frac{dx_2}{dt} &= -g - \frac{s_2}{\alpha^2} + \frac{c}{x_3}u_3^0; \\ \frac{dx_3}{dt} &= -u_3^0; \\ \frac{ds_1}{dt} &= x_1 - y_1(t); \\ \frac{ds_2}{dt} &= x_2 - y_2(t) - s_1; \\ \frac{ds_3}{dt} &= x_3 - y_3(t) + \frac{cs_2}{x_3^2}u_3^0. \end{aligned} \tag{8}$$

where

$$u_3^0 = \begin{cases} -\frac{cs_2}{\alpha^2 x_3} + \frac{s_3}{\alpha^2}, & \text{if } 0 < -\frac{cs_2}{\alpha^2 x_3} + \frac{s_3}{\alpha^2} < \beta; \\ \beta, & \text{if } -\frac{cs_2}{\alpha^2 x_3} + \frac{s_3}{\alpha^2} \ge \beta; \\ 0, & \text{if } -\frac{cs_2}{\alpha^2 x_3} + \frac{s_3}{\alpha^2} \le 0. \end{cases} \tag{9}$$

State variables $x_1(\cdot), x_2(\cdot), x_3(\cdot)$ of the solution of the system with initial conditions (7) are called state characteristics. Conjugate variables $s_1(\cdot), s_2(\cdot), s_3(\cdot)$ of the solution of the system with initial conditions (7) are called impulse characteristics.

Note that the conditions are also sufficient for optimality because of the uniqueness of the solution of the system (9) with the conditions (7).

3.3 Solution of the Reconstruction Problem

We consider the control $u_3^0(t)$ as the approximation of the normal control in the online control reconstruction problem:

$$\begin{aligned} \frac{dx_1}{dt} &= x_2; \\ \frac{dx_2}{dt} &= \frac{c}{x_3}u_3^0(t) - g; \\ \frac{dx_3}{dt} &= -u_3^0(t); \end{aligned}$$

The following proposition can be proved using the scheme of the proof in the paper [Subbotina, Tokmantsev, and Krupennikov, 2015].

Proposition 2. Parameters α, δ in the problem (4), (5), (6) can be concorded in such a way that

$$\begin{aligned} &\|x^0(\cdot) - x^*(\cdot)\|_C \to 0, \\ &\|u_3^0(\cdot) - u^*(\cdot)\|_{L_2} \to 0 \end{aligned}$$

as α, δ $\to 0$.

4 Numerical Examples

We watch the real landing process $(x_1^*(t), x_2^*(t), x_3^*(t))$ and get online inaccurate discrete state information $(y_1(t_j), y_2(t_j), y_3(t_j))$:

$$\|y_1(t_j) - x_1(t_j)\| \le \delta, \quad \|y_3(t_j) - x_3^*(t_j)\| \le \delta,$$

$t_0 = 0 < t_1 < t_2, \ldots, < t_N = T$, $\delta > 0$. We construct smooth continuous approximations $(y_1(t), y_2(t), y_3(t))$ of the measurements and apply the above presented method to get the approximation $u^0(t)$ of the reconstructing normal control $u^*(t)$ with a small delay.

Parameters are: $t \in [0, 10]$, $\beta = 2$, $\alpha = 0.05$, $g = 1.622$, $c = 500$, $x_0 = (162, -16.22, 150)$, $s_0 = (0, 0, 0)$.

Pictures 2–5 described results of the numerical solution for the case of the real control of the form:

$$u(t) = \begin{cases} 0, 0 \leq t < 5, \\ 2, 5 \leq t \leq 10. \end{cases}$$

Black lines on the pictures below are smooth approximations of trajectories, blue lines are the reconstructed control and the reconstructed trajectories.

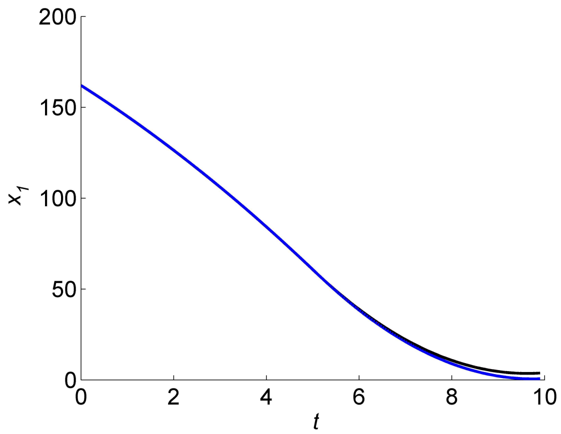

Figure 2. State information $y_1(\cdot)$ (black line) and reconstructed trajectory $x_1(\cdot)$ (blue line)

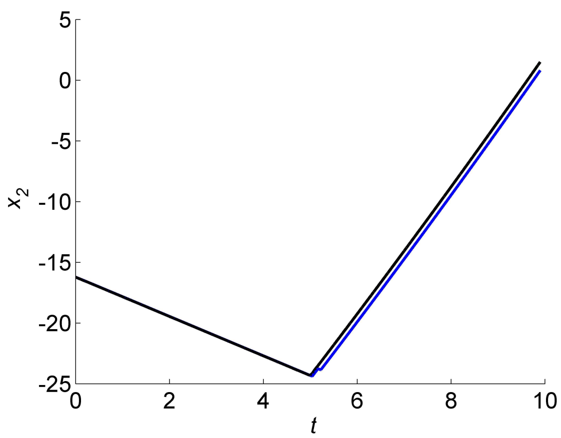

Figure 3. State information $y_2(\cdot)$ (black line) and reconstructed trajectory $x_2(\cdot)$ (blue line)

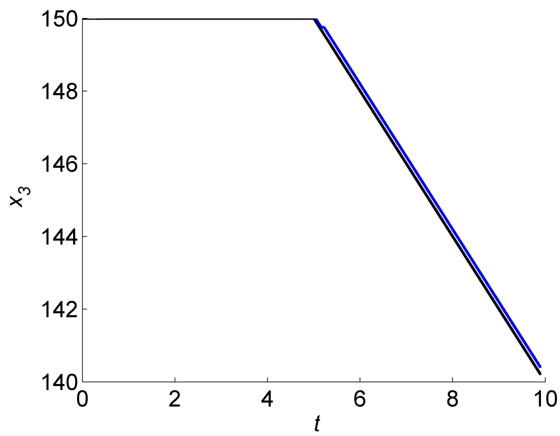

Figure 4. State information $y_3(\cdot)$ (black line) and reconstructed trajectory $x_3(\cdot)$ (blue line)

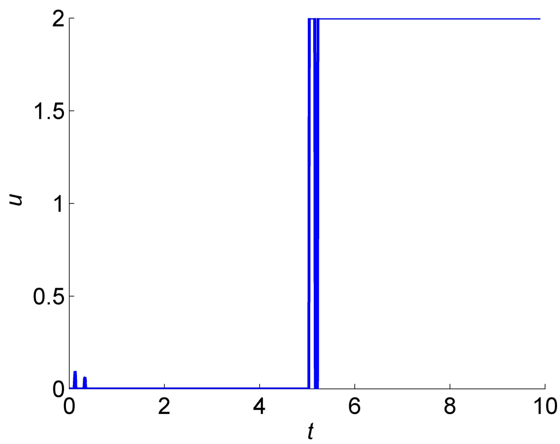

Figure 5. Reconstructed control $u_3^0(\cdot)$

4.1 Example 2

Parameters $t \in [0, 7.5]$, $\beta = 50$, $\alpha = 0.1$, $g = 1.622$, $c = 500$, $x_0 = (140, -20, 2500)$, $s_0 = (0, 0, 0)$.

Black lines on the pictures below are smooth approximations of trajectories, blue lines are the reconstructed control and the reconstructed trajectories.

Real control

$$u(t) = \begin{cases} 0, 0 \leq t < 4.22, \\ \beta, 5 \leq t \leq 7.5. \end{cases}$$

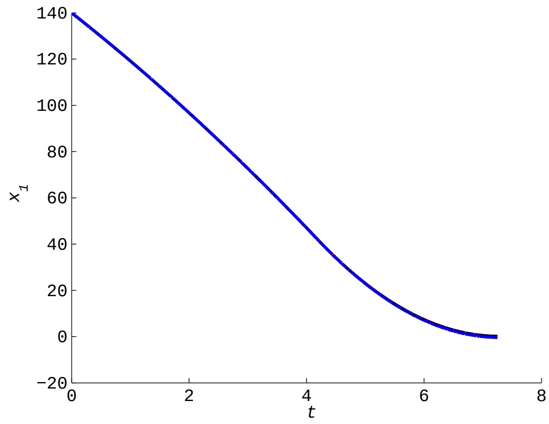

Figure 6. State information $y_1(\cdot)$ (black line) and reconstructed trajectory $x_1(\cdot)$ (blue line)

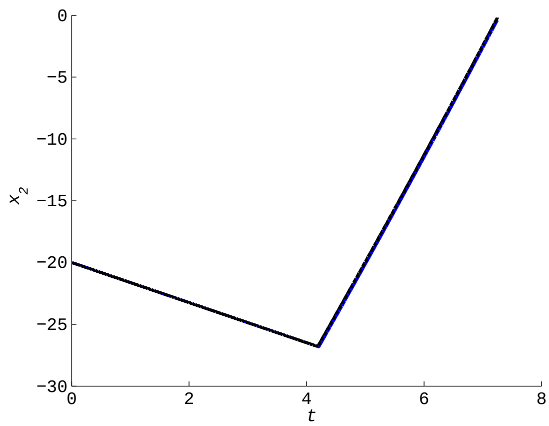

Figure 7. State information $y_2(\cdot)$ (black line) and reconstructed trajectory $x_2(\cdot)$ (blue line)

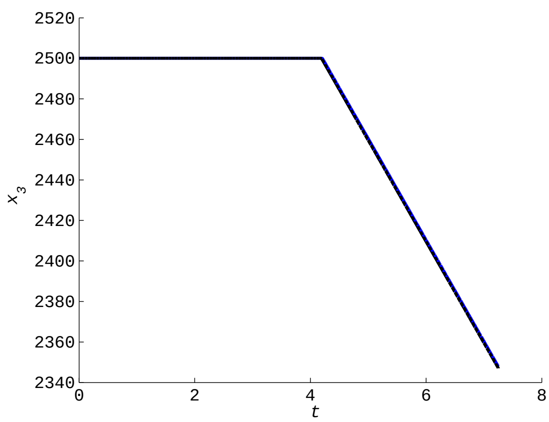

Figure 8. State information $y_3(\cdot)$ (black line) and reconstructed trajectory $x_3(\cdot)$ (blue line)

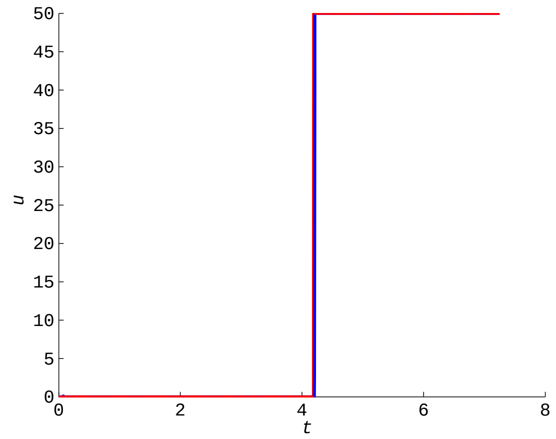

Figure 9. Real control $u(\cdot)$ (red line) and reconstructed control $u_3^0(\cdot)$ (blue line)

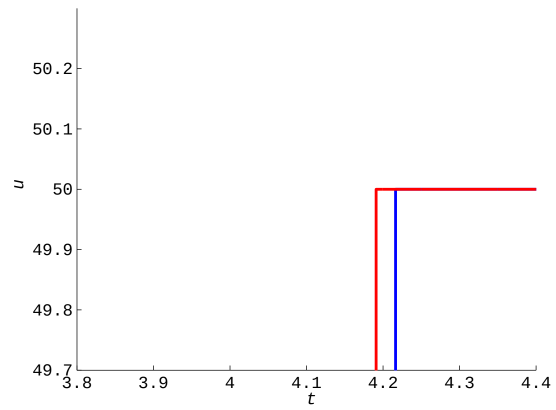

Figure 10. Real control $u(\cdot)$ and reconstructed control $u_3^0(\cdot)$ in large scale

5 Conclusion

In the paper dynamic reconstruction problems for controls of navigation systems in the process of the landing on the Moon are considered in assumption that online information about real motions is known with errors. A new method for solving the inverse problem is suggested on the base of solutions of auxiliary calculus of variations problems. A corresponding numerical algorithm is created. Results of simulations are exposed. The effective method will be developed for the navigation deterministic systems of general form and greater dimensions in the future papers.

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