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ALGORITHM FOR SOLVING TWO-LEVEL HIERARCHICAL MINIMAX PROGRAM CONTROL PROBLEM IN NONLINEAR DISCRETE-TIME DYNAMICAL SYSTEM

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Abstract
This paper discusses the discrete-time dynamical system consisting from two controlled objects and described by a nonlinear and linear recurrent vector equations in the presence of uncertain perturbations. For this dynamical system we propose a mathematical formalization in the form of solving two-level hierarchical minimax program control problem with incomplete information and propose the algorithm that has a form of a recurrent procedure of solving a linear programming and a finite optimization problems.

Key words
Hierarchical discrete-time dynamical system, minimax control.

1 Object's dynamics in the two-level hierarchical control system

On a given integer-valued time interval (simply interval) $\overline{0,T} = \{0, 1, \cdots, T\}$ $(T > 0,\ T \in \mathbb{N}$; where $\mathbb{N}$ is the set of all natural numbers) we consider a controlled multistep dynamical system which consists of the two objects. Dynamics of the object I (main object of the system) controlled by dominant player P, is described by a vector nonlinear discrete-time recurrent relation of the form

$$y(t+1) = f(t, y(t), u(t), v(t), \xi(t)),\ y(0) = y_0,\ (1)$$

and the dynamics of the object II (auxiliary object of the system) controlled by subordinate player E, is described by the linear relation:

$$z(t+1) = A(t)z(t) + B(t)u(t)+$$

$$+C(t)v(t) + D(t)\xi^{(1)}(t),\ z(0) = z_0, \qquad (2)$$

where $t \in \overline{0, T-1}$; $y(t) = (y_1(t), y_2(t), \ldots, y_r(t)) \in \mathbb{R}^r$ is a phase vector of the object I at the time moment t; $z(t) = (z_1(t), z_2(t), \ldots, z_s(t)) \in \mathbb{R}^s$ is a phase vector of the object II at the time moment t; $(r, s \in \mathbb{N}$; for $n \in \mathbb{N}$, $\mathbb{R}^n$ is an n-dimensional Euclidean vector space of column vectors); $u(t) = (u_1(t), u_2(t), \ldots, u_p(t)) \in \mathbb{R}^p$ is a vector of control action (control) of the dominant player P at the time moment t, that satisfies the given constraint:

$$u(t) \in \mathbf{U}_1(t) \subset \mathbb{R}^p, \qquad (3)$$

where $\mathbf{U}_1(t)$ for each time moment $t \in \overline{0, T-1}$ is a finite set of vectors in the space $\mathbb{R}^p$; $v(t) = (v_1(t), v_2(t), \ldots, v_q(t)) \in \mathbb{R}^q$ is a vector of control action (control) of the subordinate player E at the time moment t, which depends from admissible realization of the control $u(t) \in \mathbf{U}_1(t)$ of the player P and must be satisfy the given constraint:

$$v(t) \in \mathbf{V}_1(u(t)) \subset \mathbb{R}^q, \qquad (4)$$

where $\mathbf{V}_1(u(t))$ for each time moment $t \in \overline{0, T-1}$ and control $u(t) \in \mathbf{U}_1(t)$ of the player P is the finite set of vectors in the space $\mathbb{R}^q$.

In the equations (1) and (2) describing dynamics of the objects I and II, respectively, $\xi(t) = (\xi_1(t), \xi_2(t), \ldots, \xi_m(t)) \in \mathbb{R}^m$ and $\xi^{(1)}(t) = (\xi_1^{(1)}(t), \xi_2^{(1)}(t), \ldots, \xi_l^{(1)}(t)) \in \mathbb{R}^l$ are a perturbations vectors for these objects that at each time moment t $(t \in \overline{0, T-1})$ satisfies the given constraints:

$$\xi(t) \in \mathbf{\Xi}_1(t) \subset \mathbb{R}^m,\ \xi^{(1)}(t) \in \mathbf{\Xi}_1^{(1)}(t) \subset \mathbb{R}^l, \qquad (5)$$

where the sets $\Xi_1(t)$ and $\Xi_1^{(1)}(t)$ are convex, closed and bounded polyhedrons (with a finite number of vertices) in the spaces $\mathbb{R}^m$ and $\mathbb{R}^l$, respectively, and restrict admissible values of realizations of perturbations vectors of the objects I and II, respectively at the time moment t.

We assume, that for all fixed $t \in \overline{0, T-1}$ the vector-function $f : \overline{0, T-1} \times \mathbb{R}^r \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \longrightarrow \mathbb{R}^r$ is continuous by collection of the variables $(y(t), u(t), v(t), \xi(t))$; for all fixed time moment $t \in \overline{0, T-1}$, and convex set $Y_* \subset \mathbb{R}^r$, and controls $u_*(t) \in \mathbf{U}_1(t)$ and $v_*(t) \in \mathbf{V}_1(u_*(t))$, the set $f(t, Y_*, u_*(t), v_*(t), \Xi_1) = \{f(t, y(t), u_*(t), v_*(t), \xi(t)),\ y(t) \in Y_*,\ \xi(t) \in \Xi_1\}$ is convex set of the space $\mathbb{R}^r$; matrixes $A(t)$, $B(t)$, $C(t)$, and $D(t)$ in a vector recurrent equation (2), describing dynamics of the object II, are real matrices of dimensions $(s \times s)$, $(s \times p)$, $(s \times q)$, and $(s \times l)$, respectively.

2 Information conditions for the players in the control systems

The control process in discrete-time dynamical system (1)–(5) are realized in the presence of the following information conditions.

It is assumed that in the field of interests of the player P are both possible terminal (final) states $y(T)$ of the object I and possible states $z(T)$ of the object II, and for any considered time interval $\overline{\tau, T} \subseteq \overline{0, T}$ $(\tau < T)$ the player P also knows a future realization of the program control $v(\cdot) = \{v(t)\}_{t \in \overline{\tau, T-1}}$ $(\forall\, t \in \overline{\tau, T-1} :$ $v(t) \in \mathbf{V}_1(u(t)),\ u(t) \in \mathbf{U}_1(t))$ of the player E at this time interval which communicate to him, and he can use its for constructing his program control $u(\cdot) = \{u(t)\}_{t \in \overline{\tau, T-1}}$ $(\forall\, t \in \overline{\tau, T-1} :\ u(t) \in \mathbf{U}_1(t))$.

We assumed that in the field of interests of the player E are only possible terminal states $z(T)$ of the object II and for any considered time interval $\overline{\tau, T} \subseteq \overline{0, T}$ $(\tau < T)$ he also knows a future realization of the control $u(\cdot) = \{u(t)\}_{t \in \overline{\tau, T-1}}$ $(\forall\, t \in \overline{\tau, T-1} :$ $u(t) \in \mathbf{U}_1(t))$ of the player P at this time interval, which communicate to him, and he can use its for constructing his program control $v(\cdot) = \{v(t)\}_{t \in \overline{\tau, T-1}}$ $(\forall\, t \in \overline{\tau, T-1} :\ v(t) \in \mathbf{V}_1(u(t)),\ u(t) \in \mathbf{U}_1(t))$. Therefore, the behavior of player E explicitly depends on the behavior of player P.

It is also assumed that in the considered control process for every instant $t \in \overline{0, T}$ players P and E knows all relations and constraints (1)–(5).

3 Definitions and criterions of quality for the control process

For a strict mathematical formulation the two-level hierarchical minimax program control problem by a final states phase vectors in discrete-time dynamical system (1)–(5) with perturbation we introduce some definitions.

For a fixed number $k \in \mathbb{N}$ and an integer-valued interval $\overline{\tau, \vartheta} \subseteq \overline{0, T}$ $(\tau \leq \vartheta)$, similarly as in the work [Shorikov, 1997], we denote by $\mathbf{S}_k(\overline{\tau, \vartheta})$ the metric space of functions $\varphi : \overline{\tau, \vartheta} \longrightarrow \mathbb{R}^k$ of an integer argument t and by $\mathrm{comp}(\mathbf{S}_k(\overline{\tau, \vartheta}))$ we denote the set of all nonempty and compact subsets of the space $\mathbf{S}_k(\overline{\tau, \vartheta})$.

Based on constraint (3), and similarly as in the work [Shorikov, 1997], we define the set $\mathbf{U}(\overline{\tau, \vartheta}) \in \mathrm{comp}(\mathbf{S}_p(\overline{\tau, \vartheta - 1}))$ of all admissible program controls $u(\cdot) = \{u(t)\}_{t \in \overline{\tau, \vartheta-1}}$ of the player P on the interval $\overline{\tau, \vartheta} \subseteq \overline{0, T}$ $(\tau < \vartheta)$. And for a fixed program control $u(\cdot) \in \mathbf{U}(\overline{\tau, \vartheta})$ of the player P according to constraint (4) we define the finite set $\mathbf{V}(\overline{\tau, \vartheta}; u(\cdot))$ of all admissible program controls of player E on the interval $\overline{\tau, \vartheta} \subseteq \overline{0, \mathrm{T}}$ $(\tau < \vartheta)$ of the corresponding $u(\cdot)$. According to constraints (5) we define the sets $\Xi(\overline{\tau, \vartheta})$ and $\Xi^{(1)}(\overline{\tau, \vartheta}; u(\cdot))$ of all admissible program perturbations vectors that respectively affect on the dynamics of the objects I and II on the interval $\overline{\tau, \vartheta} \subseteq \overline{0, \mathrm{T}}$ $(\tau < \vartheta)$.

Let for instant $\tau \in \overline{0, T}$ the set $\mathbf{W}(\tau) = \overline{0, T} \times \mathbb{R}^r \times \mathbb{R}^s$ is the set of all admissible τ-positions $w(\tau) = \{0, y(\tau), z(\tau)\} \in \overline{0, T} \times \mathbb{R}^r \times \mathbb{R}^s$ of the player P $(\mathbf{W}(0) = \{w(0)\} = \mathbf{W}_0 = \{w_0\},\ w(0) = w_0 = \{0, y_0, z_0\})$ on level I of the control process.

Then we define the following convex terminal functional

$$\alpha :\ \mathbf{W}(\tau) \times \mathbf{U}(\overline{\tau, T}) \times \hat{\mathbf{V}}(\overline{\tau, T}) \times \Xi(\overline{\tau, T}) \times$$

$$\times \Xi^{(1)}(\overline{\tau, T}) = \mathbf{\Gamma}(\overline{\tau, T}, \alpha) \longrightarrow \mathbf{E} =]-\infty, +\infty[, \quad (6)$$

and its value for every collection $(w(\tau), u(\cdot), v(\cdot), \xi(\cdot), \xi^{(1)}(\cdot)) \in \mathbf{W}(\tau) \times \mathbf{U}(\overline{\tau, T}) \times \hat{\mathbf{V}}(\overline{\tau, T}) \times \Xi(\overline{\tau, T}) \times \Xi^{(1)}(\overline{\tau, T})$ is defined by the following relation

$$\alpha(w(\tau), u(\cdot), v(\cdot), \xi(\cdot), \xi^{(1)}(\cdot)) = \hat{\alpha}(y(T), z(T)) =$$

$$= \mu \hat{\gamma}(y(T)) + \mu^{(1)} < e^{(1)}, z(T) >_s . \quad (7)$$

Where $\hat{\mathbf{V}}(\overline{\tau, T}) = \{\mathbf{V}(\overline{\tau, T}; u(\cdot)),\ u(\cdot) \in \mathbf{U}(\overline{\tau, T})\}$; by $y(T) = y_T(\overline{\tau, T}, y(\tau), u(\cdot), v(\cdot), \xi(\cdot))$, and by $z(T) = z_T(\overline{\tau, T}, z(\tau), u(\cdot),\ v(\cdot), \xi^{(1)}(\cdot))$ we denote the sections of motions of object I and object II, respectively at final (terminal) instant T on the interval $\overline{\tau, T}$; $\hat{\alpha} :\ \mathbb{R}^r \times \mathbb{R}^s \rightarrow \mathbb{R}^1$ is convex terminal functional; $\hat{\gamma} :\ \mathbb{R}^r \rightarrow \mathbb{R}^1$ is convex terminal functional; $e^{(1)} \in \mathbb{R}^s$ is fixed vector; here and below, for each $k \in \mathbb{N}$, $a \in \mathbb{R}^k$ and $b \in \mathbb{R}^k$ will be denoted by the symbol $< a, b >_k$ scalar product of vectors a and b of the space $\mathbb{R}^k$; $\mu \in \mathbb{R}^1$ and $\mu^{(1)} \in \mathbb{R}^1$ are fixed numerical parameters which satisfying the following conditions:

$$\mu \geq 0;\ \mu^{(1)} \geq 0;\ \mu + \mu^{(1)} = 1. \quad (8)$$

We denote by $\mathbf{W}^{(1)}(\tau) = \overline{0,T} \times \mathbb{R}^s$ the set of all admissible τ-positions $w^{(1)}(\tau) = \{\tau, z(\tau)\} \in \overline{0,T} \times \mathbb{R}^s$ of the player E ($\mathbf{W}^{(1)}(0) = \{w^{(1)}(0)\} = \mathbf{W}_0^{(1)} = \{w_0^{(1)}\},\ w^{(1)}(0) = w_0^{(1)} = \{0, z_0\}$) on level II of the control process.

Then we define the following linear terminal functional

$$\beta:\ \mathbf{W}^{(1)}(\tau) \times \mathbf{U}(\overline{\tau,T}) \times \hat{\mathbf{V}}(\overline{\tau,T}) \times \mathbf{\Xi}^{(1)}(\overline{\tau,T}) =$$

$$= \mathbf{\Gamma}(\overline{\tau,T}, \beta) \longrightarrow \mathbf{E}, \tag{9}$$

which estimate for player E a quality of the final phase states of the object II, and its value for each collection $(w^{(1)}(\tau), u(\cdot), v(\cdot), \xi^{(1)}(\cdot)) \in \mathbf{W}^{(1)}(\tau) \times \mathbf{U}(\overline{\tau,T}) \times \hat{\mathbf{V}}(\overline{\tau,T}) \times \mathbf{\Xi}^{(1)}(\overline{\tau,T})$ is defined by the following relation

$$\beta(w^{(1)}(\tau), u(\cdot), v(\cdot), \xi^{(1)}(\cdot)) =$$

$$= \hat{\beta}(z(T)) =< e^{(1)}, z(T) >_s . \tag{10}$$

Where $\hat{\beta}\ :\ \mathbb{R}^s \rightarrow \mathbb{R}^1$ is linear terminal functional; $z(T) = z_T(\overline{\tau,T}, z(\tau), u(\cdot), v(\cdot), \xi^{(1)}(\cdot))$ is the section of motion of object II at final (terminal) instant T on the interval $\overline{\tau,T};\ e^{(1)} \in \mathbb{R}^s$ is fixed vector.

Let also, for any interval $\overline{\tau,T} \subset \overline{0,T}$, and admissible realizations of τ-position $w(\tau) \in \mathbf{W}(\tau)$, program controls $u(\cdot) \in \mathbf{U}(\overline{\tau,T})$ and $v(\cdot) \in \mathbf{V}(\overline{\tau,T}; u(\cdot))$, and program perturbation vector $\xi(\cdot) \in \mathbf{\Xi}(\overline{\tau,T})$ we shall consider the convex terminal functional

$$\gamma:\ \mathbf{W}(\tau) \times \mathbf{U}(\overline{\tau,T}) \times \hat{\mathbf{V}}(\overline{\tau,T}) \times \mathbf{\Xi}(\overline{\tau,T}) =$$

$$= \mathbf{\Gamma}(\overline{\tau,T}, \gamma) \longrightarrow \mathbf{E}, \tag{11}$$

which estimate for player P a quality of the final phase states of the object I, and its value for each collection $(w(\tau), u(\cdot), v(\cdot), \xi(\cdot)) \in \mathbf{W}(\tau) \times \mathbf{U}(\overline{\tau,T}) \times \hat{\mathbf{V}}(\overline{\tau,T}) \times \mathbf{\Xi}(\overline{\tau,T})$ is defined by the following relation

$$\gamma(w(\tau), u(\cdot), v(\cdot), \xi(\cdot)) = \hat{\gamma}(y(T)). \tag{12}$$

Where $\hat{\gamma}$ is convex terminal functional from (7); $y(T) = y_T(\overline{\tau,T}, y(\tau), u(\cdot),\ v(\cdot), \xi(\cdot))$ is the section of motion of object I at final (terminal) instant T on the interval $\overline{\tau,T}$.

4 Optimization problems for the control process

Then for realization the aim of the player E we can formulate the following minimax program control problem by a final state phase vector of the object II on the level II of the control in two-level hierarchical control process for dynamical system (1)–(5).

Problem 1. For fixed interval $\overline{\tau,T} \subseteq \overline{0,T}$ ($\tau < T$), admissible on the level II in the two level hierarchical control system for dynamical system (1)–(5) realization τ-position $w^{(1)}(\tau) = \{\tau, z(\tau)\} \in \mathbf{W}^{(1)}(\tau)$ ($w^{(1)}(0) = w_0^{(1)} \in \mathbf{W}_0^{(1)}$) of the player E and every admissible realization of the program control $u(\cdot) \in \mathbf{U}(\overline{\tau,T})$ of the player P on the level I of this control process, it is required to find the set $\mathbf{V}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot)) \subseteq \mathbf{V}(\overline{\tau,T}; (u(\cdot))$ of minimax program controls $v^{(e)}(\cdot) \in \mathbf{V}(\overline{\tau,T}; u(\cdot))$ of the player E and his minimax result $c_\beta^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot))$ corresponding the control $u(\cdot)$ of the player P, and these elements are determines by the following relation:

$$\mathbf{V}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot)) = \{v^{(e)}(\cdot) :$$

$$v^{(e)}(\cdot) \in \mathbf{V}(\overline{\tau,T}; u(\cdot)),\ c_\beta^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot)) =$$

$$= \min_{v(\cdot) \in\ \mathbf{V}(\overline{\tau,T}; u(\cdot))} \ \max_{\xi^{(1)}(\cdot) \in\ \mathbf{\Xi}^{(1)}(\overline{\tau,T})} \{$$

$$\beta(w^{(1)}(\tau), v(\cdot), u(\cdot), \xi^{(1)}(\cdot))\}\}, \tag{13}$$

where the functional β is defined by the relations (9) and (10).

Below, for realization the aim of the player P corresponding by the level I of considered control process we formulate the following minimax program control problem by a final state phase vectors of the objects I and II on the level I of the control in two-level hierarchical control process for dynamical system (1)–(5).

Problem 2. For fixed time interval $\overline{\tau,T} \subseteq \overline{0,T}$ ($\tau < T$) and admissible on the level I of the two-level hierarchical dynamical system (1)–(5) of the realization τ-position $w(\tau) = \{\tau, y(\tau), z(\tau)\} \in \mathbf{W}(\tau)$ ($w(0) = \{0, y_0, z_0\} = w_0 \in \mathbf{W}_0$) of the player P it is required to find the set $\mathbf{U}^{(e)}(\overline{\tau,T}, w(\tau)) \subseteq \mathbf{U}(\overline{\tau,T})$ of the minimax program controls of the player P and his minimax result $c_\alpha^{(e)}(\overline{\tau,T}, w(\tau))$, and these elements are determines by the following relation

$$\mathbf{U}^{(e)}(\overline{\tau,\mathrm{T}}, w(\tau)) = \{u^{(e)}(\cdot) :\ u^{(e)}(\cdot) \in \mathbf{U}(\overline{\tau,T}),$$

$$c_\alpha^{(e)}(\overline{\tau,T}, w(\tau)) = \min_{u(\cdot) \in \mathbf{U}(\overline{\tau,T})} \{$$

$$\min_{v^{(e)}(\cdot)\in\mathbf{V}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot))} \max_{\substack{\xi(\cdot)\in\mathbf{\Xi}(\overline{\tau,T})\\ \xi^{(1)}(\cdot)\in\mathbf{\Xi}^{(1)}(\overline{\tau,T})}} \{$$

$$\alpha(w(\tau),u(\cdot),v^{(e)}(\cdot),\xi(\cdot),\xi^{(1)}(\cdot))\}\}\}. \qquad (14)$$

Based on the solutions of the problems 1 and 2 we consider the following problem.

Problem 3. For fixed time interval $\overline{\tau,T} \subseteq \overline{0,T}$ $(\tau < T)$ and admissible on the level I of the control in two-level hierarchical dynamical system (1)–(5) realization the τ-position $w(\tau) = \{\tau, y(\tau), z(\tau)\} \in \mathbf{W}(\tau)$ $(w(0) = \{0, y_0, z_0\} = w_0 \in \mathbf{W}_0)$ of the player P and admissible on the level II of the control process for this dynamical system the realization τ-position $w^{(1)}(\tau) \in \mathbf{W}^{(1)}(\tau)$ $(w^{(1)}(0) = w_0^{(1)} \in \mathbf{W}_0^{(1)})$ of the player E which formed due from the τ-position $w(\tau)$, and admissible realization of the program minimax control $u^{(e)}(\cdot) \in \mathbf{U}^{(e)}(\overline{\tau,T}, w(\tau))$ of the player P on the level I of it control process, which formed due from solutions of the problems 1 and 2 it is required to find the set $\hat{\mathbf{V}}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u^{(e)}(\cdot)) \subseteq \mathbf{V}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u^{(e)}(\cdot)) \subseteq \mathbf{V}(\overline{\tau,T}; u^{(e)}(\cdot))$ of the optimal minimax program controls $\hat{v}^{(e)}(\cdot) \in \mathbf{V}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u^{(e)}(\cdot))$ of the player E on level II of the control of this control process and the number $c_\beta^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u^{(e)}(\cdot))$ of optimal value of the result of the minimax program control for the player E on the level II of the control of this control process for considered dynamical system and corresponding the control $u^{(e)}(\cdot)$ to the player P and these determines by the following relations:

$$\hat{\mathbf{V}}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u^{(e)}(\cdot)) = \{\hat{v}^{(e)}(\cdot) : \ \hat{v}^{(e)}(\cdot) \in$$

$$\in \mathbf{V}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u^{(e)}(\cdot)), \ c_\alpha^{(e)}(\overline{\tau,T}, w(\tau)) =$$

$$= \min_{v^{(e)}(\cdot)\in\mathbf{V}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),u^{(e)}(\cdot))} \max_{\substack{\xi(\cdot)\in\mathbf{W}(\overline{\tau,T})\\ \xi^{(1)}(\cdot)\in\mathbf{W}^{(1)}(\overline{\tau,T})}} \{$$

$$\alpha(w(\tau),u^{(e)}(\cdot),v^{(e)}(\cdot),\xi(\cdot),\xi^{(1)}(\cdot))\}\}; \qquad (15)$$

$$c_\beta^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u^{(e)}(\cdot)) =$$

$$= \min_{v(\cdot)\in\mathbf{V}(\overline{\tau,T};u^{(e)}(\cdot))} \max_{\xi^{(1)}(\cdot)\in\mathbf{\Xi}^{(1)}(\overline{\tau,T})} \{$$

$$\beta(w^{(1)}(\tau), v(\cdot), u^{(e)}(\cdot), \xi^{(1)}(\cdot))\}. \qquad (16)$$

Note, that we can consider the solutions of formulated problems 1–3 which in union are determine the solution of the main problem of two-level hierarchical minimax program control by the final states of the objects I and II for the discrete-time dynamical system (1)–(5) in the presence of perturbations.

5 Algorithm of solving the Problems 1–3

Thus, for any fixed and admissible time interval $\overline{\tau,T} \subseteq \overline{0,T}$ $(\tau < T)$ and realization τ-position $w(\tau) = \{\tau, y(\tau), z(\tau)\} \in \mathbf{W}(\tau)$ $(w(0) = \{0, y_0, z_0\} = w_0 \in \mathbf{W}_0)$ of the player P on the level I of the control process and corresponding to it τ-position $w^{(1)}(\tau) = \{\tau, z(\tau)\} \in \mathbf{W}^{(1)}(\tau)$ $(w^{(1)}(0) = \{0, z_0\} = w_0^{(1)} \in \mathbf{W}_0^{(1)})$ of the player E on the level II of this two-level hierarchical control system for the discrete-time dynamical system (1)–(5) we can describe the algorithm for solving Problems 1–3 formulated above.

Then, for every admissible realization of the program control $u(\cdot) \in \mathbf{U}(\overline{\tau,T})$ of the player P on the level I of this control process, and on the basis of the above definitions and results of the works [Shorikov, 1997], [Shorikov, 2005] the procedure of the construction the solution of the Problem 1 can be represented as a sequence consisting from solving of the following three sub-problems:

1) constructing for every admissible control $v(\cdot) \in \mathbf{V}(\overline{\tau,T}; u(\cdot))$ of the player E of the reachable set $\mathbf{G}^{(1)}(\tau, z(\tau), u(\cdot), v(\cdot), T)$ (note, that this set can be construct by finding a solutions of a finite sequence a linear mathematical programming problems, and this set is convex, closed and bounded polyhedron (with a finite number of vertices) in the space $\mathbb{R}^s$ [Shorikov, 1997]);

2) maximizing of the linear terminal functional $\tilde{\beta}$ which is defined by the relations (9) and (10) on the set $\mathbf{G}^{(1)}(\tau, z(\tau), u(\cdot), v(\cdot), T)$, namely, the formation of the following number:

$$\kappa_\beta^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot), v(\cdot)) =$$

$$= \max_{z(T)\in\mathbf{G}^{(1)}(\tau,z(\tau),u(\cdot),v(\cdot),T)} < e^{(1)}, z(T) >_s =$$

$$= < e^{(1)}, \tilde{z}^{(e)}(T) >_s \qquad (17)$$

(note, that the solving of this problem is reduced to solving a linear mathematical programming problem [Shorikov, 1997]);

3) constructing of the set $\tilde{\mathbf{V}}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot))$ and the number $\tilde{c}_\beta^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot))$ from solving

the following optimization problem:

$$\tilde{\mathbf{V}}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot))=\{\tilde{v}^{(e)}(\cdot):$$

$$\tilde{v}^{(e)}(\cdot)\in\mathbf{V}(\overline{\tau,T};u(\cdot)),\ \tilde{c}^{(e)}_{\beta}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot))=$$

$$=\kappa^{(e)}_{\beta}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot),\tilde{v}^{(e)}(\cdot))=\min_{v(\cdot)\in\mathbf{V}(\overline{\tau,T};u(\cdot))}\{$$

$$\kappa^{(e)}_{\beta}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot),v(\cdot))\}\} \qquad (18)$$

(note, that the set $\mathbf{V}(\overline{\tau,T};u(\cdot))$ is a finite set at the space $\mathbb{R}^q$, and then the solving of this problem is reduced to solving a finite discrete optimization problem).

Taking into consideration (9), (10), (13), (18), and the conditions stipulated for the system (1)–(5), one can prove (analogy as in the works [Shorikov, 1997], [Shorikov, 2005]), that the following assertion is valid.

Theorem 1. For fixed time interval $\overline{\tau,T}\subseteq\overline{0,T}$ ($\tau<T$), admissible on the level II in the two level hierarchical control system for the discrete-time dynamical system (1)–(5) realization τ-position $w^{(1)}(\tau)=\{\tau,z(\tau)\}\in\mathbf{W}^{(1)}(\tau)$ ($w^{(1)}(0)=w^{(1)}_0\in\mathbf{W}^{(1)}_0$) of the player E and for every admissible realization of the program control $u(\cdot)\in\mathbf{U}(\overline{\tau,T})$ of the player P on the level I of the control system, the set $\tilde{\mathbf{V}}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot))$ of the admissible program controls $\tilde{v}^{(e)}(\cdot)\in\mathbf{V}(\overline{\tau,T};u(\cdot))$ of the player E on the level II of the control system and the number $\tilde{c}^{(e)}_{\beta}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot))$ are constructed from a finite number procedures of solving the linear mathematical programming problems, and the finite discrete optimization problem, and the following equalities are true:

$$\tilde{\mathbf{V}}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot))=\mathbf{V}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot));$$

$$\tilde{c}^{(e)}_{\beta}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot))=c^{(e)}_{\beta}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot)), \qquad (19)$$

where the set $\mathbf{V}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot))$ and the number $c^{(e)}_{\beta}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot))$ determined by the relation (13).

Proof. Let $\tilde{v}^{(e)}(\cdot)\in\tilde{\mathbf{V}}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot))\subseteq\mathbf{V}(\overline{\tau,T};u(\cdot))$. Then from relations (17), (18), and properties of the reachable set $\mathbf{G}^{(1)}(\tau,z(\tau),u(\cdot),v(\cdot),T)$ for the dynamical system (2), (4), (5) (see [Shorikov, 1997]), and relations (9), (10), and (13), and taking into account that $\mathbf{V}(\overline{\tau,T};u(\cdot))$ is finite set, the following equality is true

$$\tilde{c}^{(e)}_{\beta}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot))=$$

$$=\kappa^{(e)}_{\beta}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot),\tilde{v}^{(e)}(\cdot))=\min_{v(\cdot)\in\mathbf{V}(\overline{\tau,T};u(\cdot))}\{$$

$$\max_{z(T)\in\mathbf{G}^{(1)}(\tau,z(\tau),u(\cdot),v(\cdot),T)}<e^{(1)},z(T)>_s\}=$$

$$=<e^{(1)},\tilde{z}^{(e)}(T)>_s=$$

$$=\max_{\xi^{(1)}(\cdot)\in\ \Xi^{(1)}(\overline{\tau,T})}\beta(w^{(1)}(\tau),\tilde{v}(\cdot),u(\cdot),\xi^{(1)}(\cdot))=$$

$$=\min_{v(\cdot)\in\ \mathbf{V}(\overline{\tau,T};u(\cdot))}\ \max_{\xi^{(1)}(\cdot)\in\ \Xi^{(1)}(\overline{\tau,T})}\{$$

$$\beta(w^{(1)}(\tau),v(\cdot),u(\cdot),\xi^{(1)}(\cdot))\}=$$

$$=c^{(e)}_{\beta}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot)). \qquad (20)$$

Let $\tilde{v}^{(e)}(\cdot)\in\mathbf{V}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot))\subseteq\mathbf{V}(\overline{\tau,T};u(\cdot))$. Then from relations (13), (9), and (10), and properties of the reachable set $\mathbf{G}^{(1)}(\tau,z(\tau),u(\cdot),v(\cdot),T)$ for the dynamical system (2), (4), (5), and relations (17), (18), and taking into account that $\mathbf{V}(\overline{\tau,T};u(\cdot))$ is finite set, the following equality is true

$$c^{(e)}_{\beta}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot))=$$

$$=\min_{v(\cdot)\in\ \mathbf{V}(\overline{\tau,T};u(\cdot))}\ \max_{\xi^{(1)}(\cdot)\in\ \Xi^{(1)}(\overline{\tau,T})}\{$$

$$\beta(w^{(1)}(\tau),v(\cdot),u(\cdot),\xi^{(1)}(\cdot))\}=$$

$$=\max_{\xi^{(1)}(\cdot)\in\ \Xi^{(1)}(\overline{\tau,T})}\beta(w^{(1)}(\tau),\tilde{v}(\cdot),u(\cdot),\xi^{(1)}(\cdot))=$$

$$=\max_{z(T)\in\mathbf{G}^{(1)}(\tau,z(\tau),u(\cdot),\tilde{v}(\cdot),T)}<e^{(1)},z(T)>_s=$$

$$=\kappa^{(e)}_{\beta}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot),\tilde{v}^{(e)}(\cdot))=$$

$$= \min_{v(\cdot)\in \mathbf{V}(\overline{\tau,T};u(\cdot))} \kappa_\beta^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot), v(\cdot)) =$$

$$=< e^{(1)}, \tilde{z}^{(e)}(T) >_s=$$

$$= \tilde{c}_\beta^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot)). \quad (21)$$

From (20) it follows that if $\tilde{v}^{(e)}(\cdot) \in \tilde{\mathbf{V}}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot))$ then $\tilde{v}^{(e)}(\cdot) \in \mathbf{V}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot))$, and consequently the following inclusion holds

$$\tilde{\mathbf{V}}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot)) \subseteq \mathbf{V}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot)),$$

and from (21) it follows that if $\tilde{v}^{(e)}(\cdot) \in \mathbf{V}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot))$ then $\tilde{v}^{(e)}(\cdot) \in \tilde{\mathbf{V}}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot))$, and consequently the following inclusion holds

$$\mathbf{V}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot)) \subseteq \tilde{\mathbf{V}}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot)).$$

Then from this inclusions it follows that equality (19) is true, and the validity of Theorem 1 is completely proved.

Next, consider the algorithm for solving the problem 2.

Let $\mathbf{G}(\tau, y(\tau), u(\cdot), v(\cdot), T)$ is a reachable set [Krasovskii and Subbotin, 1988] of all admissible phase states of the object I at time moment T corresponding to the fixed collection $(\tau, y(\tau), u(\cdot), v(\cdot)) \in \{\tau\} \times \mathbb{R}^r \times \mathbf{U}(\overline{\tau,T}) \times \mathbf{V}(\overline{\tau,T}; u(\cdot))$.

Then, on the basis of the above definitions and results of the works [Shorikov, 1997], [Shorikov, 2005] the procedure of the construction the solution of the Problem 2 for the discrete-time dynamical system (1)–(5) can be represented as a sequence consisting from solving of the following three sub-problems:

1) constructing of the reachable set $\mathbf{G}(\tau, y(\tau), u(\cdot), v(\cdot), T)$ (note, that this set can be construct by finding a solutions of a finite sequence one-step operations only, and it is convex, closed and bounded set in the space $\mathbb{R}^r$ [Shorikov, 1997], [Shorikov, 2005]) ;

2) maximization of the convex terminal functional $\tilde{\alpha}$ which is defined by the relations (6)–(8) on the sets $\mathbf{G}(\tau, y(\tau), u(\cdot), v(\cdot), T)$ and $\mathbf{G}^{(1)}(\tau, z(\tau), u(\cdot), v(\cdot), T)$, namely, the formation of the following number:

$$\lambda_\alpha^{(e)}(\overline{\tau,T}, w(\tau), u(\cdot), v(\cdot)) =$$

$$= \mu\hat{\gamma}(\tilde{y}^{(e)}(T)) + \mu^{(1)} < e^{(1)}, \tilde{z}(e)(T) >_s=$$

$$= \max_{y(T)\in \mathbf{G}(\tau,y(\tau),u(\cdot),v(\cdot),T)} \mu\hat{\gamma}(y(T)) +$$

$$+ \max_{z(T)\in \mathbf{G}^{(1)}(\tau,z(\tau),u(\cdot),v(\cdot),T)} \mu^{(1)} < e^{(1)}, z(T) >_s \quad (22)$$

(note, that the solving of this problem is reduced to solving a linear and a convex mathematical programming problems [Shorikov, 1997], [Shorikov, 2005]);

3) constructing of the set $\tilde{\mathbf{U}}^{(e)}(\overline{\tau,T}, w(\tau))$ and the number $\tilde{c}_\alpha^{(e)}(\overline{\tau,T}, w(\tau))$ from solving the following optimization problem:

$$\tilde{\mathbf{U}}^{(e)}(\overline{\tau,T}, w(\tau)) = \{\tilde{u}^{(e)}(\cdot) : \ \tilde{u}^{(e)}(\cdot) \in \mathbf{U}(\overline{\tau,T}),$$

$$\tilde{c}_\alpha^{(e)}(\overline{\tau,T}, w(\tau)) = \lambda_\alpha^{(e)}(\overline{\tau,T}, w(\tau), \tilde{u}^{(e)}(\cdot), \tilde{v}^{(e)}(\cdot)) =$$

$$= \min_{u(\cdot)\in \mathbf{U}(\overline{\tau,T})} \min_{\tilde{v}^{(e)}(\cdot)\in \tilde{\mathbf{V}}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot))} \{$$

$$\lambda_\alpha^{(e)}(\overline{\tau,T}, w(\tau), u(\cdot), \tilde{v}^{(e)}(\cdot))\}\} \quad (23)$$

(note, that the set $\mathbf{U}(\overline{\tau,T})$ is a finite set at the space $\mathbb{R}^p$, and the finite set $\tilde{\mathbf{V}}^{(e)}(\overline{\tau,T}, w^{(1)}(\tau), u(\cdot))$ is constructed from (18), and then the solving of this problem is reduced to solving a finite discrete optimization problem).

Taking into consideration (6)–(8), (14), (18)–(23), and the conditions stipulated for the system (1)–(5), one can prove (analogy as in works [Shorikov, 1997], [Shorikov, 2005]), that the following assertion is valid.

Theorem 2. For fixed time interval $\overline{\tau,T} \subseteq \overline{0,T}$ ($\tau < T$), admissible on the levels I and II in the two level hierarchical control system for the discrete-time dynamical system (1)–(5) realizations τ-positions $w(\tau) = \{\tau, y(\tau)\} \in \mathbf{W}(\tau)$ ($w(0) = w_0 \in \mathbf{W}_0$) and $w^{(1)}(\tau) = \{\tau, z(\tau)\} \in \mathbf{W}^{(1)}(\tau)$ ($w^{(1)}(0) = w_0^{(1)} \in \mathbf{W}_0^{(1)}$) of the players P and E, respectively, the set $\tilde{\mathbf{U}}^{(e)}(\overline{\tau,T}, w(\tau))$ of the admissible program controls $\tilde{u}^{(e)}(\cdot) \in \mathbf{U}(\overline{\tau,T})$ of the player P on the level I of the control system and the number $\tilde{c}_\alpha^{(e)}(\overline{\tau,T}, w(\tau))$ are constructed from a finite number procedures of solving the linear and convex mathematical programming problems, and the finite discrete optimization problem, and the following equalities are true:

$$\tilde{\mathbf{U}}^{(e)}(\overline{\tau,T}, w(\tau)) = \mathbf{U}^{(e)}(\overline{\tau,T}, w(\tau));$$

$$\tilde{c}_\alpha^{(e)}(\overline{\tau,T}, w(\tau)) = c_\alpha^{(e)}(\overline{\tau,T}, w(\tau)), \quad (24)$$

where the set $\mathbf{U}^{(e)}(\overline{\tau,T},w(\tau))$ and the number $c_\alpha^{(e)}(\overline{\tau,T},w(\tau)))$ determined by the relation (14).

We note that the proof of Theorem 2 is completely analogous to the proof of Theorem 1.

Then, on the basis of the above algorithms of solving the Problems 1 and 2 the procedure of constructing the solution of the Problem 3 for the discrete-time dynamical system (1)–(5) can be represented as a sequence consisting from solving of the following two sub-problems:

1) for any control $\tilde{u}^{(e)}(\cdot) \in \tilde{\mathbf{U}}^{(e)}(\overline{\tau,T})$ of the player P the constructing of the set $\bar{\mathbf{V}}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),\tilde{u}^{(e)}(\cdot))$ and the number $\bar{c}_\alpha^{(e)}(\overline{\tau,T},w(\tau))$ from solving the following optimization problem:

$$\bar{\mathbf{V}}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),\tilde{u}^{(e)}(\cdot)) = \{\bar{v}^{(e)}(\cdot):$$

$$\bar{v}^{(e)}(\cdot) \in \tilde{\mathbf{V}}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),\tilde{u}^{(e)}(\cdot)),$$

$$= \bar{c}_\alpha^{(e)}(\overline{\tau,T},w(\tau)) = \lambda_\alpha^{(e)}(\overline{\tau,T},w(\tau),\tilde{u}^{(e)}(\cdot),\bar{v}^{(e)}(\cdot)) =$$

$$= \min_{u(\cdot)\in\mathbf{U}(\overline{\tau,T})} \min_{\tilde{v}^{(e)}(\cdot)\in\tilde{\mathbf{V}}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),u(\cdot))} \{$$

$$\lambda_\alpha^{(e)}(\overline{\tau,T},w(\tau),u(\cdot),\tilde{v}^{(e)}(\cdot))\}\} \qquad (25)$$

(note, that the sets $\tilde{\mathbf{V}}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),\tilde{u}^{(e)}(\cdot))$ and $\tilde{\mathbf{U}}^{(e)}(\overline{\tau,T};w(\tau))$ are constructed from relations (23) and (18), respectively, and then the solving of this problem is reduced to solving a finite discrete optimization problem);

2) for any control $\tilde{u}^{(e)}(\cdot) \in \tilde{\mathbf{U}}^{(e)}(\overline{\tau,T})$ of the player P and any control $\bar{v}^{(e)}(\cdot) \in \bar{\mathbf{V}}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),\tilde{u}^{(e)}(\cdot))$ of the player E the constructing of the number $\bar{c}_\beta^{(e)}(\overline{\tau,T},w^{(1)}(\tau),\tilde{u}^{(e)}(\cdot))$ from solving the finite discrete optimization problem described by relation (17) and satisfies the following relation:

$$\bar{c}_\beta^{(e)}(\overline{\tau,T},w^{(1)}(\tau),\tilde{u}^{(e)}(\cdot)) =$$

$$= \kappa_\beta^{(e)}(\overline{\tau,T},w^{(1)}(\tau),\tilde{u}^{(e)}(\cdot),\bar{v}^{(e)}(\cdot)). \qquad (26)$$

Taking into consideration (18)–(26), and the conditions stipulated for the system (1)–(5), one can prove that the following assertion is valid.

Theorem 3. For fixed time interval $\overline{\tau,T} \subseteq \overline{0,T}$ ($\tau < T$) and admissible on the level I of the control in two-level hierarchical dynamical system (1)–(5) realization the τ-position $w(\tau) = \{\tau,y(\tau),z(\tau)\} \in \mathbf{W}(\tau)$ ($w(0) = \{0,y_0,z_0\} = w_0 \in \mathbf{W}_0$) of the player P and admissible on the level II of the control process for this dynamical system the realization τ-position $w^{(1)}(\tau) \in \mathbf{W}^{(1)}(\tau)$ ($w^{(1)}(0) = w_0^{(1)} \in \mathbf{W}_0^{(1)}$) of the player E which formed due from the τ-position $w(\tau)$, and admissible realization of the program minimax control $u^{(e)}(\cdot) \in \tilde{\mathbf{U}}^{(e)}(\overline{\tau,T},w(\tau)) = \mathbf{U}^{(e)}(\overline{\tau,T},w(\tau))$ of the player P on the level I of the control process, which formed due from the solution of the Problems 1 and 2, the set $\bar{\mathbf{V}}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),u^{(e)}(\cdot))$ and the number $\bar{c}_\beta^{(e)}(\overline{\tau,T},w^{(1)}(\tau),u^{(e)}(\cdot))$ which form due from (25) and (26), respectively, are constructed from a finite number procedures of solving the linear and the convex mathematical programming problems and the finite discrete optimization problem, and the following equalities are true:

$$\bar{\mathbf{V}}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),u^{(e)}(\cdot)) =$$

$$= \hat{\mathbf{V}}^{(e)}(\overline{\tau,T},w^{(1)}(\tau),u^{(e)}(\cdot));$$

$$\bar{c}_\alpha^{(e)}(\overline{\tau,T},w(\tau)) = c_\alpha^{(e)}(\overline{\tau,T},w(\tau));$$

$$\bar{c}_\beta^{(e)}(\overline{\tau,T},w^{(1)}(\tau),u^{(e)}(\cdot)) =$$

$$= c_\beta^{(e)}(\overline{\tau,T},w^{(1)}(\tau),u^{(e)}(\cdot)). \qquad (27)$$

We note that the proof of Theorem 3 is completely analogous to the proof of Theorem 1.

Note, that on the basis of the above algorithm of solving the Problems 1–3 the procedure of the construction a solution of the main problem of two-level hierarchical minimax program control by the final states of the objects I and II for the discrete-time dynamical system (1)–(5) in the presence of perturbations can be formed from realization of a finite number procedures of solving the linear and the convex mathematical programming problems, and the finite discrete optimization problems.

6 Conclusion

Thus, in this paper we have presented the mathematical formalization of the main problem of two-level hierarchical minimax program control by the final states of the objects I and II for the discrete-time dynamical system (1)–(5) with incomplete information. This paper proposes an algorithm for solving this problem, which is a realization of a finite sequence procedures of solving the linear and the convex mathematical programming problems, and the finite discrete optimization problems.

Results obtained in this paper are based on the studies [Krasovskii and Subbotin, 1988]–[Filippova, 2016] and can be used for computer simulation, design and construction of multilevel control systems for actual technical an economic dynamical processes operating under deficit of information and uncertainty. Mathematical models of such systems are presented, for example, in [Fradkov, 2003]–[Tarbouriech and Garcia, 1997].

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