# Control of Mechanical Systems under Uncertainty by Small Forces

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***Abstract*— We consider a mechanical system governed by Lagrange's equations under the assumption that its matrix of inertia is not known exactly and that the system is subjected to unknown bounded disturbances. We assume also that the disturbances are smaller than the control forces which in turn do not exceed the potential forces. A bounded control is proposed which, under certain conditions, steers the system from an arbitrary initial state to the prescribed terminal state in finite time. To construct the control law and to justify it, the Lyapunov function given implicitly is used. The algorithm employs a linear feedback control with the gains which are functions of the phase variables. The gains increase and tend to infinity as the phase variables tend to zero; nevertheless, the control forces are bounded and meet the imposed constraint.**

**The proposed approach is illustrated by the results of the computer simulation of steering a plain two-link pendulum.**

## I. Introduction

A lot of approaches to designing control for dynamical systems with uncertain parameters are based on the stability theory and consist in constructing regimes ensuring the asymptotic stability of the desired motion (in particular, the terminal state) of the system. In contrast to these approaches, we are searching for the control laws bringing the system to a prescribed terminal state in finite time. In recent years, new approaches to constructing constrained controls for steering perturbed mechanical systems with many degrees of freedom into a prescribed terminal state in finite time have been developed [1], [2], [3]. In [4], [5] a method was elaborated which enables one to construct feedback control for a generic Lagrangian mechanical system under the assumption that the kinetic energy matrix is unknown and the system is subjected to uncertain force disturbances. The resulting control meets an imposed constraint and steers the system to an arbitrary, but given in advance, terminal state in finite time (the duration of the process is not fixed in advance). In addition, the control function is continuously differentiable everywhere, except for the terminal state. The proposed method can be applied if the control force prevail over all other forces acting on the system including the potential forces. In the present investigation the method is extended to the case when the control is small in comparison with the potential forces.

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## II. Statement of the Problem

Consider a mechanical system governed by Lagrange's equations of the second kind

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q} - \frac{\partial T}{\partial q} = u + s - \frac{\partial P}{\partial q}. \qquad (1)$$

Here $q \in R^n$ is the vector of the generalized coordinates, $\dot q \in R^n$ is the vector of the generalized velocities, $T$ is the kinetic energy, $P$ is the potential energy. The kinetic energy of the system has the form

$$T(q,\dot q) = \frac{1}{2}\langle A(q)\dot q, \dot q\rangle$$

where $\langle \cdot,\cdot\rangle$ stands for the scalar product.

The potential energy $P(q) \in C^1$ is a bounded from below function which has a point of global minimum. Without loss of the generality we assume that this point coincides with the phase space origin $q = 0$.

The vector of generalized control forces $u \in R^n$ is bounded

$$\|u\| \le U,\ U > 0 \qquad (2)$$

and so is the vector of unknown generalized forces $S$

$$\|s(t,q,\dot q)\| \le S_0,\ S_0 > 0$$

which will be called disturbance.

Let us stress that the system under consideration is fully actuated because the number of degrees of freedom of the system coincides with the dimension of the control force vector.

We assume that the disturbance is smaller than the control force which in turn does not exceed the potential force.

Let the positive definite symmetric matrix of the kinetic energy $A(q) \in C^1$ have the form

$$A(q) = A_0(q) + A_1(q) \qquad (3)$$

where the matrix $A_0(q)$ is known and positive-definite but the matrix $A_1(q)$ is unknown.

The potential energy is also presented as a sum of the given function $P_0(q)$ and the unknown function $P_1(q)$

$$P(q) = P_0(q) + P_1(q)$$

The matrix $A_1$ and the vector-function $\partial P_1/\partial q$ are assumed to be small in comparison with $A_0$ and the control vector-function $u$ respectively.

We assume also that the eigenvalues of the matrices $A(q)$ and $A_0(q)$ for any $q$ belong to the interval $[m, M],\ 0 < m \le M$. This implies the inequalities

$$\forall q, z \in R^n \quad mz^2 \le \langle A(q)z, z\rangle \le Mz^2$$

$$mz^2 \leq \langle A_0(q)z, z\rangle \leq Mz^2$$

The matrix $A_1(q)$ and the partial derivatives of the matrices $A(q), A_0(q)$, and $A_1(q)$ are bounded uniformly in $q$, that is

$$\|\frac{\partial A(q)}{\partial q_i}\| \leq C_1,\ \|\frac{\partial A_0(q)}{\partial q_i}\| \leq C_1,\ \|A_1(q)\| \leq D_1$$

$$\|\frac{\partial^2 A}{\partial q_i \partial q_j}(q)\| \leq C_2,\ \|\frac{\partial A_1(q)}{\partial q_i}\| \leq D_2,\ \|\frac{\partial P_1}{\partial q}\| \leq p_0$$

$$C_1, C_2, D_1, D_2, p_0 > 0, \quad i = 1, \ldots, n$$

($\|\cdot\|$ means the Euclidean norm of a vector or a matrix).

The phase variables $q, \dot{q}$ are supposed to be available for measuring at every time instant.

*Problem.* For a given initial state $(q_0, \dot{q}_0)$ construct a control law meeting constraint (2) and steering system (1) to the prescribed terminal state $(q_*, \dot{q}_*)$ in a finite time, whatever the matrix $A_1(q)$, the function $\partial P_1/\partial q$, and the disturbance $s$, satisfying the above conditions, be.

To illustrate the statement of the problem, let us consider a problem of steering a plain two-link pendulum to a prescribed terminal state by means of joint torques (Fig. 1). Suppose that we do not know exactly the mass-inertia parameters of the pendulum. Then the kinetic energy matrix $A$ and the force of gravity $\partial P/\partial q$ are not known exactly also, and we come to the control problem described above. It will be shown below that the pendulum can be brought to the terminal state by the control which may be small in comparison with the force of gravity.

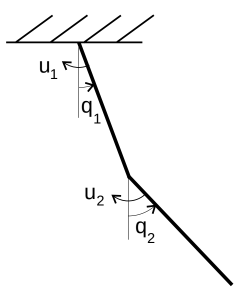


Fig. 1. Controlled double pendulum

## III. Control Algorithm

The system will be brought to the terminal state in two steps. First, we steer system (1) into the neighbourhood of the phase space origin $q = 0$ (which is the point of minimum for the potential energy $P(q)$, in agreement with our assumption). The control algorithm for such steering is not presented in this paper. To make the system approach the minimal point of the potential energy one can use, for example, the control law

$$u = -\frac{U}{\|\dot{q}\|}\dot{q}$$

which is applied below for transferring the double pendulum into the neighbourhood of the phase space origin.

Denote by $(q_*^1, \dot{q}_*^1)$ the ending point of the trajectory of system (1) at the first stage of process.

At the second stage of steering for some "reference" system we design a trajectory starting in the neighbourhood of the phase space origin and ending in the prescribed terminal state $(q_*, \dot{q}_*)$. Then, using the trajectory tracking procedure we steer the original system under consideration to the prescribed terminal state.

### A. The Reference System

Denote

$$T_0(q, \dot{q}) = \frac{1}{2}\langle A_0(q)\dot{q}, \dot{q}\rangle$$

and consider the reference system

$$\frac{d}{dt}\frac{\partial T_0}{\partial \dot{q}} - \frac{\partial T_0}{\partial q} = u' - \frac{\partial P_0}{\partial q} \tag{4}$$

System (4) does not contain uncertainties and has the following symmetry property: if a feedback control $u'(q, \dot{q})$ steers system (4) from the state $(q^1, \dot{q}^1)$ to the state $(q^2, \dot{q}^2)$ then the control $u'(q, -\dot{q})$ steers it from the state $(q^2, \dot{q}^2)$ to the state $(q^1, \dot{q}^1)$. The same is true also for open-loop controls $u'(t)$ and $u'(\bar{t} - t),\ t \in [t, \bar{t}]$, where $\bar{t}$ is the total time of motion. To justify this property it is sufficient to note that the only difference between the equations for the forward and backward motion is the sign of the variables the control functions depend on.

Let the control $u'$ transfer the reference system from the neighbourhood of the phase space origin to the state $(q_*, \dot{q}_*)$ and satisfy the constraint $\|u'\| \leq U/2$. To design such control one can use, for example, the function

$$u'(q, \dot{q}) = -\frac{U}{2\|\dot{q}\|}\dot{q}$$

which meets the imposed constraint and brings system (4) to some point $(q_*^0, \dot{q}_*^0)$ lying in the neighbourhood of the phase space origin. Then, according to the symmetry property the control function

$$u'(q, \dot{q}) = \frac{U}{2\|\dot{q}\|}\dot{q}$$

transfers system (4) from the state $(q_*^0, \dot{q}_*^0)$ to the terminal state $(q_*, \dot{q}_*)$. Denote by $\tilde{q}(t), \dot{\tilde{q}}(t)$ the vector-functions describing the trajectory of such transferring. We shall call this trajectory the reference trajectory.

Thus, the point $(q_*^1, \dot{q}_*^1)$ is the initial state of system (1) at the beginning of the second stage of the motion, and the reference trajectory starts at the point $(q_*^0, \dot{q}_*^0)$. Both points lie in the neighbourhood of the phase space origin.

Next we utilize the trajectory tracking procedure. We design a control $u''$ that meets the constraint

$$|u''| \leq \frac{U}{2} \tag{5}$$

brings system (1) from the state $(q_*^1, \dot{q}_*^1)$ to the reference trajectory $\tilde{q}(t), \dot{\tilde{q}}(t)$ in finite time, and steers the system along this trajectory to the terminal state $(q_*, \dot{q}_*)$.

To this end let us consider the auxiliary control problem for the equations in deviations.


Fig. 2. Trajectory tracking

### B. Auxiliary Control Problem

Denote by $x, \dot{x}$ the deviations of the state of the original system from the reference trajectory

$$x(t) = q(t) - \tilde{q}(t), \quad \dot{x}(t) = \dot{q}(t) - \dot{\tilde{q}}(t)$$

The equations in deviations can be written as follows

$$\tilde{A}(t, x)\ddot{x} = S + u'' \tag{6}$$

Here

$$\tilde{A}(t, x) = A(\tilde{q}(t) + x)$$

and the function $S$ satisfies the inequality [6]

$$\|S\| \leq \left(\sqrt{n}C_1Q_2 + \frac{3n}{2}C_2Q_1^2\right)\|x\| +$$

$$+3\sqrt{n}C_1Q_1\|\dot{x}\| + \frac{3}{2}\sqrt{n}C_1\|\dot{x}\|^2 + s_0 + p_0$$

with the assumption that

$$|\dot{\tilde{q}}| \leq Q_1, \quad \|\ddot{\tilde{q}}\| \leq Q_2$$

Since the trajectory of the original system starts from the point $(q_*^1, \dot{q}*^1)$ at the time instant $t_0$ at the beginning of the second stage, the initial deviations are

$$x_0 = x(t_0) = q_*^1 - q_*^0, \quad \dot{x}_0 = \dot{x}(t_0) = \dot{q}_*^1 - \dot{q}_*^0$$

*Auxiliary problem.* Design a feedback control $u''(x, \dot{x})$ meeting constraint (5) and steering system (6) to the terminal state $x = \dot{x} = 0$ in finite (unfixed) time.

Denote

$$\tilde{A}_0(t, x) = A_0(\tilde{q}(t) + x)$$

$$\tilde{T}_0(t, x, \dot{x}) = \frac{1}{2}\langle\tilde{A}_0(t, x)\dot{x}, \dot{x}\rangle$$

The desired control is chosen as follows

$$u''(t, x, \dot{x}) = -a(t, x, \dot{x})\tilde{A}_0(t, x)\dot{x} - b(t, x, \dot{x})x \tag{7}$$

where

$$a^2(t, x, \dot{x}) = \frac{b(t, x, \dot{x})}{M}, \; b(t, x, \dot{x}) = \frac{3U^2}{32V(t, x, \dot{x})} \tag{8}$$

$$V(t, x, \dot{x}) = \tilde{T}_0 + \frac{1}{2}b(t, x, \dot{x})x^2 + \tag{9}$$

$$+\frac{1}{2}a(t, x, \dot{x})\langle\tilde{A}_0(t, x)\dot{x}, x\rangle, \quad x^2 + \dot{x}^2 > 0$$

Relations (8) and (9) define the functions $a$, $b$, and $V$ in an implicit form.

The justification of the proposed control law is based on Lyapunov direct method. The function $V$ plays a principal role in the present investigation. Given his function one can find the feedback factors $a$ and $b$ through the above relations, and, consequently, the control $u$ according to formula (7). In addition, the function $V$ has the dimension of energy and serves as a Lyapunov function for the system under consideration. It tends to zero as the system approaches the terminal state. Since the function $V$ appears in the denominators in relations (8) and (9), the feedback factors tend to infinity as the trajectory approaches the origin. Nevertheless, the proposed control does not go beyond the admissible boundaries.

The justification of the control is based on the following propositions.

*Theorem 1.* In the domain $x^2 + \dot{x}^2 > 0$ there exist continuously differentiable positive functions $a, b$ and $V$ satisfying (8) and (9).

*Theorem 2.* The function $V$ satisfies the inequalities

$$V_- \leq V(t, x, \dot{x}) \leq V_+$$

where

$$V_- = \frac{1}{8}\left(m\dot{x}^2 + \left[m^2\dot{x}^4 + \frac{3U^2x^2}{16}\right]^{1/2}\right)$$

$$V_+ = \frac{3}{8}\left(M\dot{x}^2 + \left[M^2\dot{x}^4 + \frac{3U^2x^2}{16}\right]^{1/2}\right)$$

*Theorem 3.* The derivative of the function $V(t) = V(t, x, \dot{x})$ satisfies the inequality

$$\dot{V}(t) \leq -\frac{\delta}{3}V^{1/2}(t), \quad t \geq t_0 \tag{10}$$

along the trajectory of system (6). Here the constant $\delta$ depends on the given parameters $m, M, C_1, C_2, D_1, D_2, S_0$, and $p_0$.

The above theorems have been proved in [5] for the rheonomic mechanical systems. For the system in deviations the proof employs similar ideas and is not presented in this paper.

By integrating inequality (10) we obtain the following estimate from above for the time $\tau$ it takes for system (6) to come to the terminal state $x = \dot{x} = 0$:

$$\tau \leq \frac{6}{\delta}V^{1/2}(t_0, x_0, \dot{x}_0)$$

Thus, the control algorithm described above ensures vanishing of the deviations $x(t), \dot{x}(t)$ in finite time. This implies that control (7) steers system (1) to the reference trajectory $\tilde{q}(t), \dot{\tilde{q}}(t)$ and confines the system to this trajectory (Fig. 2). Therefore, system (1) comes along the reference trajectory to the terminal state $(q_*, \dot{q}_*)$ in finite time.

## IV. Computer simulation results

To demonstrate the efficiency of the proposed algorithm we present the results of the computer simulation of steering a plain two-link pendulum by means of joint torques. In all figures the dashed lines correspond to the first link and the solid lines correspond to the second link of the pendulum.

At the first stage the pendulum was transferred from the initial state

$$q_1 = \frac{\pi}{2},\ q_2 = \frac{\pi}{2}\ \text{rad},\ \dot{q}_1 = \dot{q}_2 = 0\ \text{rad/s}$$

into the neighbourhood of the phase space origin. The graphs in Fig. 3 describes the behavior of the angular coordinates of the links during such steering.

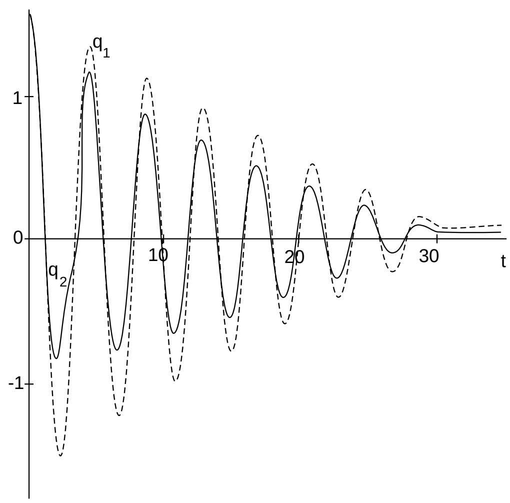


Fig. 3. The first stage of steering

At the second stage the pendulum was brought to the terminal state

$$q_1^* = \frac{3\pi}{4},\ q_2^* = \frac{3\pi}{4}\ \text{rad},\ \dot{q}_1^* = \dot{q}_2^* = 0\ \text{rad/s}.$$

The graphs in Fig. 4 depict the time history of the angular coordinates at the second stage of the motion.

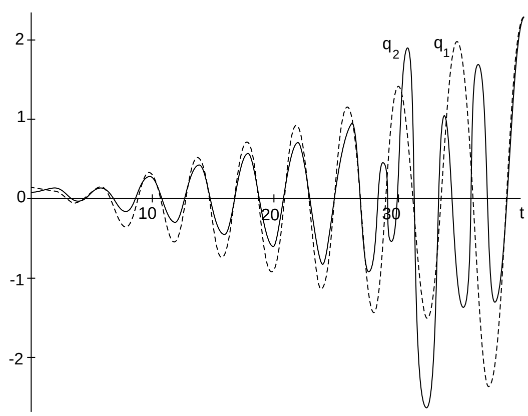


Fig. 4. The second stage of steering

Fig. 5 shows the graphs of the deviations of the current state of the system from the reference trajectory at the second stage.

## V. Conclusion

The control laws constructed based upon the approach proposed bring generic nonlinear mechanical systems to a given state in a finite time. These laws are robust with respect

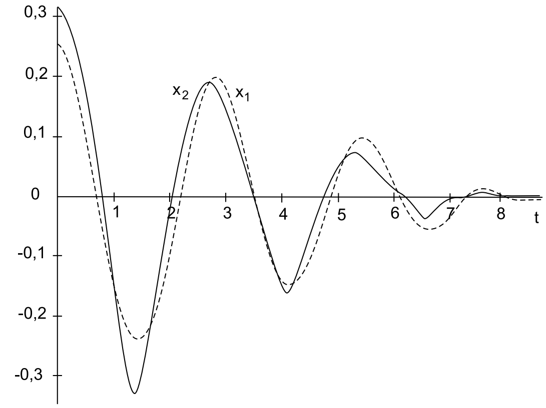


Fig. 5. Deviation from the reference trajectory

to change in the mass-inertia parameters of the system, since they do not require knowing the kinetic energy matrix of the system, and they happened to be effective even under the action of uncontrolled perturbing forces. The approach is also applicable for the systems subjected to action of large potential forces.

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