Projective Synchronization of the Nonlinear Bloch Equations

U. E. Vincent, J. A. Laoye and F. Ayedun

***Abstract*— In this paper, the projective synchronization (PS) in a drive-response system of the nonlinear Bloch equations (NBE) is considered. A generalized nonlinear control that is effective in achieveing PS with a constant, as well as different scaling factors is proposed. We prove the stability of the PS state using the Lyapunov stability theory and give numerical evidence of PS in the NBE.**

***Index Terms*— Nonlinear Bloch equations, chaos, projective synchronization**

I. INTRODUCTION

During the begining of the last decade, one of the most fascinating discoveries that transformed research in the field of nonlinear dynamics and chaos theory is the fact that two or more chaotic systems evolving from different initial conditions can be made to synchronize, either by coupling the systems (locally or globally) or by forcing them [2]. This was demostrated by Pecora and Carroll [1]. Synchronization can be understood as a state in which two or more systems (with dynamics that can either be periodic or chaotic) adjust each other giving rise to a common dynamical behaviour [2]. In view of its practical applications in diverse disciplines such as biological, chemical, neurological systems as well as secure communications, cryptography, and so on [2], [3], [4], the phenomenon of synchronization has been widely investigated in several practical systems. In recent years, different kinds of synchronization phenomena have been found in different systems: Complete (or identical) [1], generalized [5], phase [6], lag [7], anticipatory [7], [8], measure [9], [10], reduced-order [11], [12] and projective synchronization [13], [14], [15], [16], [17].

For complete or full synchronization, the trajectories of two chaotic systems converges, i.e. $y(t) = x(t)$. However, an interesting case, namely projective synchronization, first reported in [13] is characterized by a situation when a slave chaotic system synchronizes with the projection of the master chaotic system with a constant scaling factor. The condition for this kind of synchronization to occur in both partially and nonpartially discrete and continuous chaotic systems were considered in [14], [15], [16].

In this paper, we study the projective synchronization in the nonlinear Bloch equations (NBE). The dynamics of ensemble of spins usually described by Bloch equations is very important for the understanding of the underlying physical process of nuclear magnetic resonance. In this direction, Abergel [18] examined the linear set of equations originally proposed by Bloch based on chaos theory; while Ucar et al. [19] extended the work of Abergel [18] and also demonstrated the possibility of realizing chaos synchronization between two nonlinear Bloch equations. Additionally, Moukam Kakmeni et al [20], examined the dynamics of a variant NBE, extended to account for both the bi-axial property of the magnets to which the set of spins belongs and the presence of a back action from the probe. In [20], the synchronization behaviour of the NBE were considered based on adaptive approach, using both linear and nonlinear feedback couplings; while Park [21] studied the synchronization of the NBE with uncertain parameters. More recently, chaos suppression in the NBE were considered in [22], [23].

II. MODEL

The model is derived from magnetization $\mathbf{M}$ precessing in the magnetic induction field B_0 in the presence of a constant radiofrequency field $\mathbf{B_1}$ with intensity $\mathbf{B_1} = \frac{\omega_1}{\gamma}$ and frequency ω_{rf} and consists of the three nonlinear modified Bloch equations (NBE) given in dimensionless units as

$$\begin{aligned}
\dot{x} &= \delta y + \lambda z(x \sin \quad - y \cos \quad) - \frac{x}{\tau_2}, \\
\dot{y} &= -\delta x - z + \lambda z(x \cos \quad + y \sin \quad) - \frac{y}{\tau_2}, \qquad (1) \\
\dot{z} &= y - \lambda \sin \quad (x^2 + y^2) - \frac{z-1}{\tau_1},
\end{aligned}$$

where the dots denotes time derivatives, δ, λ, and are the system parameters; and τ_1 and τ_2 are longitudinal time and transverse relaxation time respectively. The dynamics of system (1) has been extensively studied in Ref. [18], [19] for a fixed subset of the system parameters $(\delta, \lambda, \tau_1, \tau_2)$ and for a space area range of the radiation damping feedback . The regions of that would admit chaotic solutions were obtained. For instance, the NBE exhibits chaotic behaviour for $\delta = -0.4\pi$, $\lambda = 30$, $\; = 0.173, \tau_1 = 5$ and $\tau_2 = 2.5$. Further detailed analysis and the rich dynamics of this system is given in [24]

III. PROJECTIVE SYNCHRNIZATION

Our goal is to obtain a generalized nonlinear control that could, beside full synchronization achieve the projective synchronization between two identical NBE systems with known parameters and evolving from different initial conditions - using the same and different scaling factors. Let us consider

U. E. Vincent (*Corresponding author*) is with the Department of Physics, Lancaster University, Lancaster LA1 4YB, UK; and Department of Physics, Olabisi Onabanjo University, P.M.B. 2002 Ago-Iwoye, Nigeria. u.vincent@tu-clausthal.de.

J. A. Laoye is with Department of Physics, Olabisi Onabanjo University, P.M.B. 2002 Ago-Iwoye, Nigeria.

F. Ayedun is with the Department of Physics and Mathematics, Tai Solarin University of Education, Ijebu-Ode, Nigeria.

a driver NBE system given by

$$
\begin{aligned}
\dot{x}_1 &= \delta y_1 + \lambda z(x_1 \sin \quad - y \cos \quad) - \frac{x_1}{\tau_2}, \\
\dot{y}_1 &= -\delta x_1 - z_1 + \lambda z_1(x_1 \cos \quad + y_1 \sin \quad) - \frac{y_1}{\tau_2}, \quad (2) \\
\dot{z}_1 &= y - \lambda \sin \quad (x_1^2 + y_1^2) - \frac{z_1 - 1}{\tau_1},
\end{aligned}
$$

and a response NBE system given by

$$
\begin{aligned}
\dot{x}_2 &= \delta y_2 + \lambda z_2(x_2 \sin \quad - y_2 \cos \quad) - \frac{x_2}{\tau_2} + u_1, \\
\dot{y}_2 &= -\delta x_2 - z_2 + \lambda z_2(x_2 \cos \quad + y_2 \sin \quad) - \frac{y_2}{\tau_2} + u_2 \quad (3) \\
\dot{z}_2 &= y - \lambda \sin \quad (x_2^2 + y_2^2) - \frac{z_2 - 1}{\tau_1} + u_2,
\end{aligned}
$$

where the $u_i (i = 1, 2, 3)$ are control inputs to be determined. In order to find suitable control inputs $u_i (i = 1, 2, 3)$, such that the system (2) synchronizes with a projection, $\alpha_i (i = 1, 2, 3)$ of system (3), we define the synchronization error dynamics between the driver and the response systems as $e_1 = x_2 - \alpha_1 x_1$, $e_2 = y_2 - \alpha_2 y_1$ and $e_3 = z_2 - \alpha_3 z_1$. This implies that there exist a constant matrix $\alpha = diag(\alpha_1, \alpha_2, \alpha_3)$ such that the $\lim_{t\to\infty} ||e_i||(i = 1, 2, 3) = 0$.

With the above definition of the error states, we obtain the following error dynamics system:

$$
\begin{aligned}
\dot{e}_1 &= \delta e_2 + \lambda[z_2(x_2 \sin \quad - y_2 \cos \quad) \\
&\quad -\alpha_1 z_1(x_1 \sin \quad - y_1 \cos \quad)] - \frac{e_1}{\tau_2} + u_1, \\
\dot{e}_2 &= -\delta e_1 - e_3 + \lambda[z_2(x_2 \cos \quad + y_2 \sin \quad) \\
&\quad -\alpha_2 z_1(x_1 \cos \quad + y_1 \sin \quad)] - \frac{e_2}{\tau_2} + u_2, \quad (4) \\
\dot{e}_3 &= e_2 - \lambda[\sin \quad (x_2^2 + y_2^2) \\
&\quad -\alpha_3 \sin \quad (x_1^2 + y_1^2)] - \frac{e_z - 1}{\tau_1} + u_3.
\end{aligned}
$$

Notice that in the absence of the control inputs $u_i (i = 1, 2, 3)$, the systems (4) would have equilibrium at $(0, 0, 0)$. Thus, if suitable control inputs are chosen such that the equilibrium is not changed, then the synchronization problem would reduced to achieving the asymptotic solution of the error system (4). In this regards, we employ the Lyapunov stability theory to ascertain the asymptotic stability of system (4) at the origin (0,0,0) for suitable choice of control inputs.

Proposition: The drive-response system of NBE (3) and (4) can approach projective synchronization asymptotically with the same scaling factor $\alpha_1 = \alpha_2 = \alpha_3$ or different scaling factors $\alpha_1 \neq \alpha_2 \neq \alpha_3$, if the controllers $u_i (i = 1, 2, 3)$ are chosen such:

$$
\begin{aligned}
u_1 &= \left(\frac{1}{\tau_2} - 1 + k_1\right) e_1 - \delta e_2 - \lambda[z_2(x_2 \cos \quad + y_2 \sin \quad) \\
&\quad -\alpha_2 z_1(x_1 \cos \quad + y_1 \sin \quad)], \\
u_2 &= \delta e_1 + \left(\frac{1}{\tau_2} - 1 + k_2\right) e_2 + e_3 - \lambda[z_2(x_2 \cos \quad + y_2 \sin \quad) \\
&\quad -\alpha_2 z_1(x_1 \cos \quad + y_1 \sin \quad)], \quad (5) \\
u_3 &= -e_2 + \left(\frac{1}{\tau_1} - 1 + k_3\right) e_3 + \lambda[\sin \quad (x_2^2 + y_2^2) \\
&\quad -\alpha_3 \sin \quad (x_1^2 + y_1^2)].
\end{aligned}
$$

and feedback gains $k_i (i = 1, 2, 3) \leq 0$.

Proof: Consider the following Lyapunov function:

$$
V = \frac{1}{2} \sum_{i=1}^{3} (1 - k_i) e_i^2, \tag{6}
$$

whose time derivative along the trajectories is

$$
\dot{V} = \sum_{i=1}^{3} (1 - k_i) e_i \dot{e}_i. \tag{7}
$$

If $\dot{e}_i = -e_i$ and $k_i \leq 0$ then,

$$
\dot{V} = -\sum_{i=1}^{3} (1 - k_i) e_i^2 < 0. \tag{8}
$$

Based on the Lyapunov stability theory, the error dynamical system (4) is asymptotically stable at the origin $(0, 0, 0)$; therefore projective synchronization is achieved between the drive-response NBE (2) and (3) regardless of the value of the scaling factor.

IV. Numerical Results

Previous studies on chaos synchronization of the nonlinear Bloch equations considered the simple case where $\alpha = diag(\alpha_1, \alpha_2, \alpha_3) = diag(1, 1, 1)$ which corresponds to complete or full synchronization. In the general case where $\alpha = diag(\alpha_1, \alpha_2, \alpha_3) \neq diag(1, 1, 1)$ and also the more stringent case where $\alpha_1 \neq \alpha_2 \neq \alpha_3$, we have shown that synchronization could be also achieved as long as the inequality (8) is satisfied. Note also that the case $\alpha = -diag(1, 1, 1)$ corresponds to the so-called complete anti-synchronization. Thus, with the controller (5), one can achieve four kinds of synchroniation, namely: complete synchronization, complete anti-synchronization, projective synchronization and projective anti-synchronization. In what follows, we give some numerical simulations to confirm our theoretical analysis and we fix the parameters as follows: $\delta = -0.4\pi$, $\lambda = 30$, $\quad = 0.173, \tau_1 = 5$ and $\tau_2 = 2.5$ so that the stable chaotic behaviour is obtained.

First we consider the case $\alpha = diag(\alpha_1, \alpha_2, \alpha_3) \neq diag(1, 1, 1)$. In Fig. 1(a), we give numerical results for $diag(\alpha_1, \alpha_2, \alpha_3) = diag(2, 2, 2)$ and $k_i = -200$. Here, we activate the control at $t \geq 20$. It is clear that PS has been achieved.

In Fig. 1(b), we also plot the projection of the chaotic attractors for the driver NBE (red) and the response NBE

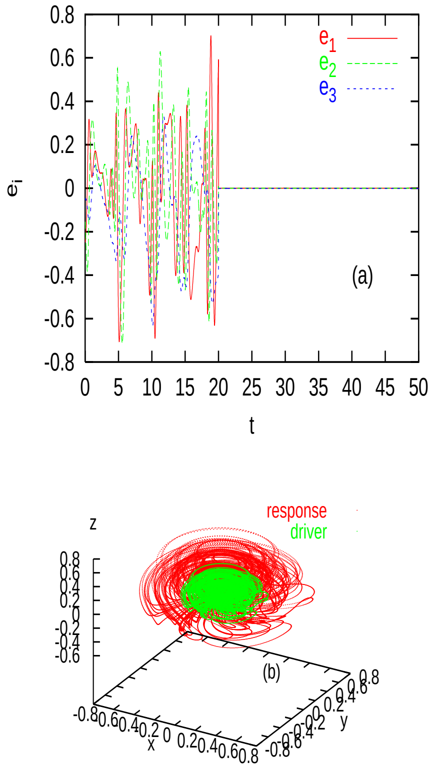

Fig. 1. (a) Time history of the error dynamical systems $e_i (i = 1, 2, 3)$ showing the projective synchronization and (b) projection of the chaotic attractors of the drive-response NBE system in the projective synchronized state corresponding to (a) for $k_i = -200$ and $diag(\alpha_1, \alpha_2, \alpha_3) = (2, 2, 2)$. Parameters are: $\delta = -0.4\pi$, $\lambda = 30$, $= 0.173, \tau_1 = 5$ and $\tau_2 = 2.5$

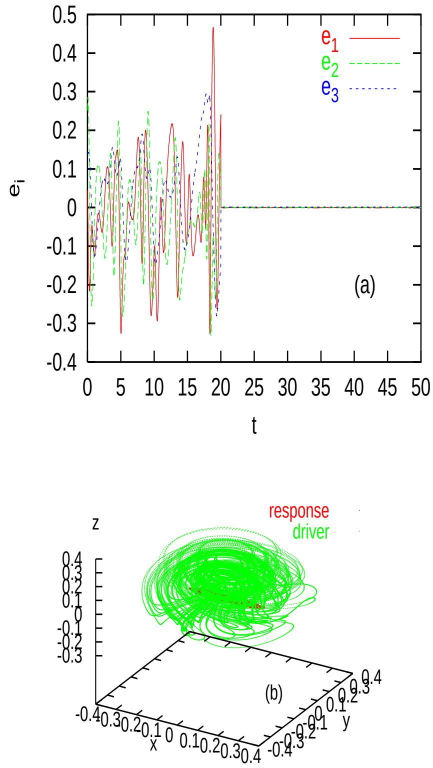

Fig. 2. (a) Time history of the error dynamical systems $e_i (i = 1, 2, 3)$ showing the projective synchronization and (b) projection of the chaotic attractors of the drive-response NBE system in the projective synchronized state corresponding to (a) for $k_i = -200$ and $diag(\alpha_1, \alpha_2, \alpha_3) = (0.5, 0.01, 0.1)$. Parameters are: $\delta = -0.4\pi$, $\lambda = 30$, $= 0.173, \tau_1 = 5$ and $\tau_2 = 2.5$

(blue) corresponding to the Fig. 1. In Fig. 2(a), we set $\alpha_1 \neq \alpha_2 \neq \alpha_3$, i.e $(\alpha_1, \alpha_2, \alpha_3) = (0.5, 0.01, 0.1)$ and the feedback gain $k_i (i = 1, 2, 3) = -200$. Again, we find that the PS is achieved. Fig. 2(b) shows the projection of the attractors in the (x_1, y_1, z_1)(blue) and (x_2, y_2, z_2) (red) phase space. Notice the difference between Fig. 1(b) and 2(b). In both Figs. 1(b) and 2(b), we find attractor scaling, in which case, the size of the chaotic attractor (in the x_2, y_2, z_2) phase space shown in Fig. 1(b) is enlarged; whereas its size is reduced in Fig. 2(b) - suggesting that the system dynamics as well as its complexity could be control by means of appropriate scaling factors.

V. Conclusion

In this paper, we have obtained a generalized nonlinear controls that is effective in realizing the projective synchronization for the nonlinear Bloch equations using the same and different values of the scaling factors. The control inputs is generalized in the sense that one can achieve the complete synchronization by setting the scaling factors, $\alpha_i = 1$; anti-synchronization for $\alpha_i = -1$; projective synchronization for $\alpha_i \neq 1$, projective anti-synchronization for $\alpha_i = -\alpha_j$ ($\alpha_j \neq 1$ or $\alpha_j \neq 0$) and modified projective synchronization $\alpha_1 \neq \alpha_2 \neq \alpha_3$. The condition for the projective synchronization to occur was obtained using the Lyapunov stability theory. Numerical simulations were carried out to verify the effectiveness of the control inputs obtained for the NBE.

Acknowledgment

The work of UEV is supported by the Alexander von Humboldt Foundation in Bonn, Germany and the Royal Society of London through the Newton International Fellowship scheme. He is grateful to Olabisi Onabanjo University for granting research leave. The authors thank D. Mayer for carefully reading through the manuscript and his valuable comments.

References

[1] L. M. Pecora and T. L. Carroll, *Synchronization in chaotic systems*, Phys. Rev. Lett. **64** (1990) 821-824.

[2] A. Pikovsky, M. Rosemblum and J. Kurths, *Synchronization: A universal concept in nonlinear science*, (Cambridge University Press, New York, 2001).

[3] U. E. Vincent, A. N. Njah and J. A. Laoye, "Controlling chaos and deterministic directed transport in inertia ratchets using backstepping control," Physica D **231** (2007) 130-136.

[4] U. E. Vincent and J. A. Laoye, "Synchronization and Control of directed transport in chaotic ratchets via active control," Phys. Lett. A **363** (2007) 91-95.

[5] L. Kocarev and U. Parlitz, "Generalized synchronization, predictability and equivalence of unidirectional coupled dynamical systems," Phys. Rev. Lett. **76** (1996) 1816-1819.
[6] J. L. Mateos and F. R. Alatriste, "Phase synchronization in tilted inertial ratchets as chaotic rotators," Chaos **18** (2008) 043125.
[7] D. V. Senthikumar and M. Lakshmanan, "Transition from anticipatory to lag synchronization via complete synchronization in time-delay systems," Phys. Rev. E **71** (2005) 016211.
[8] H. U. Voss, "Dynamic Long term anticipation of chaotic states," Phys. Rev. Lett. **87** (2001) 014102.
[9] A. Hampton and H. D. Zanette, "Measure synchronization in coupled hamiltonian systems," Phys Rev Lett **83** (1999) 2179-2182.
[10] U. E. Vincent, A. N. Njah and O. Akinlade, "Measure synchronization in a coupled hamiltonian system associated with Nonlinear Schrodinger equation," Mod. Phys. Lett. B **19** (2005) 737-742.
[11] R. Femat and G. Solís-Perales, "Synchronization of chaotic systems with different order," Phys. Rev. E **65** (2002) 036226.
[12] U. E. Vincent and R. Guo, "A Simple Adaptive Control for full and reduced-order synchronization of uncertain time-varying chaotic systems," Comm. Nonl. Sc. Num. Sim. **14** (2008) 3925-3932.
[13] R. Mainieri and J. Rehacek, "Projective synchronization in three dimensional chaotic systems," Phys. Rev. Lett. **82** (1999) 3042-3045.
[14] D. Xu. W. L. Ong and Z. Li, "Criteria for the occurrence of projective synchronization in chaotic systems of arbitrary dimension," Phys. Lett. A **305** (2002) 167-172.
[15] D. Xu, C. Y. Chee and C. Li, "A necessary condition of projective synchronization in discrete-time systems of arbitrary dimensions," Chaos, Solitons & Fractals **22** (2004) 175-180.
[16] G. When and D. Xu, 'Nonlinear observer control for full-state projective synchronization in chaotic continuous-time systems," Chaos, Solitons & Fractals **26** (2005) 71-77.
[17] Z. Li and D. Xu, "A secure communication scheme using projective synchronization," Chaos, Solitons & Fractals **22** (2004) 477-481.
[18] D. Abergel, "Chaotic solutions of the feedback driven Bloch equations," Phys. Lett. A **302** (2002) 17-22.
[19] A. Ucar A, K. E. Lonngren and E. W. Bai, "Synchronization of chaotic behaviour in nonlinear Bloch equations," Phys. Lett. A **314** (2003) 96-101.
[20] F. M. Moukam Kakmeni, J. P. Nguenang and T. C. Kofane, "Chaos synchronization in bi-axial magnets modeled by Bloch equations," Chaos, Solitons & Fractals **30** (2006) 690-699.
[21] J. H. Park, "Chaos synchronization of nonlinar Bloch equations," Chaos, Solitons & Fractals **27** (2006) 357-361.
[22] U. E. Vincent, "Recursive active control for controlling chaos in Nonlinear Bloch equations," In: International Physics and Control Society Conference 2007 (PhysCon 2007). 1176-1179
[23] L. Peng, S. Cai and Z. Chen, "Chaos suppression by feedback control in nuclear magnetic resonance," Physica B **396** (2007) 57-61.
[24] D. Ghosh, A. R. Chowdhury and P. Saha,"Bifurcation continuation, chaos and chaos control in nonlinear Bloch system," Comm. Nonl. Sc. Num. Sim. **13** (2008) 1461-1471.