Peculiarities of synchronization of quasiperiodic oscillations

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Quasiperiodic oscillations with two and more independent frequencies are widely encountered in the contemporary natural sciences. They appear when electro-magnetic oscillations are modulated by information signals (radio-engineering), accompany the transition to turbulence in fluid flows (theoretical physics), describe the motion of planets in the cosmic space (theoretical mechanics), etc. Quasiperiodic oscillations describe biophysical, ecological and even social evolutionary processes. In phase space they are associated with limit sets in the form of n-dimensional tori. The analysis of stability, bifurcations and synchronization of quasiperiodic oscillations are determined by resonances and bifurcations of n-dimensional tori. This is a rather complex and in many ways unsolved problem. In our research we deal with the case of quasiperiodic oscillations whose image in the phase space represents an attracting limit set in the form of a two-dimensional ($2D$) torus.

In our case we study quasiperiodic oscillations in an autonomous dissipative dynamical system that realizes the regime of stable self-sustained oscillations with two independent frequencies. Non-autonomous two-frequency oscillations that are observed when a limit cycle system is periodically driven cannot be a subject of our research. In such systems only one of two frequencies (the oscillation frequency on a limit cycle) is independent. The external force frequency is given and is not defined by the system properties.

In the present talk we propose the autonomous four-dimensional dissipative dynamical system that demonstrates the regime of stable two-frequency oscillations. The interaction of two generators is analyzed. The phenomenon of external and mutual synchronization of two-frequency oscillations is observed, for which winding number locking on a two-dimensional torus takes place [1]. The peculiarities of external synchronization of a resonant limit cycle on a torus are studied. It has been established numerically and experimentally that in the resonant conditions the synchronization effect takes place at one of the two basic frequencies of the system. Oscillations at the second basic frequency remain unsynchronized. Our results convincingly indicate a principal difference between synchronization of the resonant limit cycle on the torus and of a typical limit cycle [2].

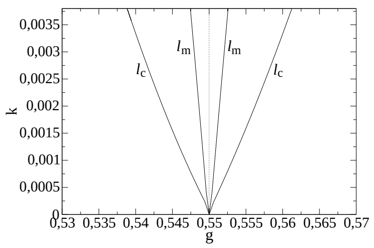
Equations of the autonomous dissipative dynamical system in R^4 that demonstrates stable two-frequency motions have the following form:

$$\begin{aligned} \dot{x} &= mx + y - x\varphi - dx^3, \\ \dot{y} &= -x, \\ \dot{z} &= \varphi, \\ \dot{\varphi} &= -\gamma\varphi + \gamma\Phi(x) - gz. \end{aligned} \tag{1}$$

System (1) is characterized by four controlling parameters: m is the excitation parameter, d is the parameter of nonlinear dissipation, γ is the attenuation parameter, and g is the sluggishness parameter of the filter. Two of those parameters are most important: the parameter of excitation m of the generator and the parameter of sluggishness g that characterizes the resonance frequency of the filter. Nonlinear function $\Phi(x)$ is asymmetrical with respect to the variable x and has the form: $\Phi(x) = H(x)x^2$, $H(x)$ – Heviside function.

By means of the generator of quasi-periodic oscillations (1) one can treat a new problem in the theory of synchronization: the synchronization of a two-dimensional torus. That problem is a natural generalization from the problem of synchronization of a limit cycle to the case of interaction of two generators with quasi-periodic motions.

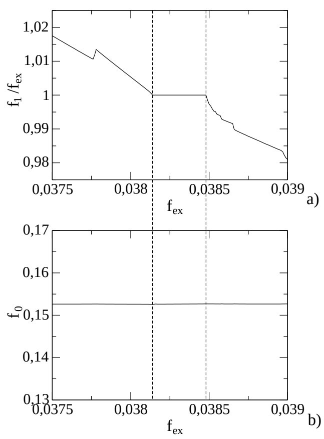
Fig. 1



In Fig. 1 on the plane of the coupling parameter k and the sluggishness parameter g of one of the interacting generators the structure of synchronization regions is uncovered. In the large "tongue" of synchronization, which is bounded by the lines l_c, the basic frequencies of quasi-periodic motions appear to be locked. With this, the modulation frequencies remain different. Hence, the phenomenon of partial synchronization of quasi-periodic motions takes place. In the narrower region bounded by the bifurcational lines l_m, locking of the modulation frequencies and, correspondingly, locking of the winding numbers is observed.

In cases of rational frequencies relation the limit cycle is observed in the phase space. That limit cycle is a resonant structure on two-dimensional torus. Let's consider for example the case of resonance $f_1 : f_0 = 1 : 4$.

Fig. 2



We calculate the oscillation spectrum of $x(t)$ variable as the external signal frequency f_{ex} is varied. The numerical results are pictured in Fig. 2. As seen from Fig. 2,a, the frequency f_1 locking, i.e., synchronization, is observed within the region $f_e \simeq 0.0381 \div 0.0385$. In the synchronization region (Fig. 2,a) the modulation frequency f_1 is locked by the external force and the condition $f_1/f_{\text{ex}} = 1$ is fulfilled. At the same time, the data plotted in Fig. 2,b indicate that the frequency f_0 is not synchronized by the external force. If we would deal with a typical limit cycle, then the spectral line at the frequency $f_0 = 4f_1$, as well as at any harmonic nf_1, would demonstrate the synchronization effect.

In conclusion we note that the obtained result can be used to diagnose the presence of a resonant torus in a system. If the torus exists, then its basic frequencies will demonstrate the effect of synchronization independently.

[1] V. Anishchenko, S. Nikolaev, and J. Kurths, Phys. Rev. E **73**, 056202 (2006).
[2] V. Anishchenko, S. Nikolaev, and J. Kurths, Phys. Rev. Letters (2007) (in press).