REACHABILITY OF SWITCHED LINEAR SYSTEMS WITH A TREE LIKE STRUCTURE

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Abstract
A reduced form of switched linear systems is introduced as a simplification of switched linear systems. Reachability properties of the original systems are studied by using the reachability of the reduced systems. This will yield a simplification of the involved computations.

Key words
hybrid system, reachability, equivalence

1 Introduction

Hybrid systems have been attracting much attention in the recent past years because of the arising problems are not only academically challenging but also of practical importance in a wide field of applications ranging from manufacturing systems to information processes and modeling ecosystems, among others [Mosterman. 2007].

Switched linear systems belong to a special class of hybrid control systems which comprises a collection of subsystems described by linear dynamics (differential/difference equations) together with a switching rule that specifies the switching between the subsystems (see [Sun and Ge, 2005]-[Yang, 2002]).

The paper is organized as follows: Section 2 is devoted to reviewing switched linear systems by giving some motivating examples and main results about behavior of a switched linear system.

Section 3 study reachability property by giving a new method which reduces the case (A_σ, B_σ) to the case (A_σ, B). The reduced system allows a simplification of the reachability condition and permits the construction of a tree that represents the behaviour of the system. An example is also given.

Thanks to this tree structure, Section 4 provides an algorithm to check reachability of a switched linear system. Section ends with a discussion about the computational improvement given by this new method.

2 Switched Systems

Definition 2.1. *A Switched Linear System is given by an evolution equation on the form*

$$x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \,\Big\} \qquad (1)$$

where $x \in \mathbb{K}^n$ is the internal state of system, $u \in \mathbb{K}^m$ is a external input (or control) of system, together with

$$\sigma(t) = \varphi(t, \sigma(t-1), x(t)) \,\Big\} \qquad (2)$$

which is the next command function. Commands (or switches) are finite sequences (words) on the finite alphabet $\Sigma = \{0, ..., s-1\}$. We call n, m and s the dimensions of the system, of the input and of the switching alphabet respectively.

For simplicity we denote a Switched Linear System by $\Gamma = (A_\sigma, B_\sigma)$.

It is interesting to study the behavior of a given switched linear system for a fixed sequence $\underline{\sigma}$ of commands (or switching signals) and a fixed sequence $\underline{u}$ of external inputs.

First we need a preparatory result.

Lemma 2.2. *The behavior of a switched linear system Γ is given by the equalities*

$$\left.\begin{array}{rcl} \Phi_\Gamma(x_0, \underline{\sigma}\tau, \underline{u}v) &=& A_\tau \Phi_\Gamma(x_0, \underline{\sigma}, \underline{u}) + B_\tau v \\ \Phi_\Gamma(x_0, \tau, v) &=& A_\tau x_0 + B_\tau v \end{array}\right\} \qquad (3)$$

Where $\underline{\sigma} \in \Sigma^*$, $\tau \in \Sigma$, $\underline{u} \in (\mathbb{K}^m)^*$ and $v \in \mathbb{K}^m$.

Proof.- Direct application of the definition of switched linear system ■

Theorem 2.3. *Let $\Gamma : x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)$ be a switched linear system .The behavior of system Γ from initial state x_0, with sequence $\underline{\sigma} = \sigma(0)\sigma(1)\cdots\sigma(s)$ of commands, and sequence $\underline{u} = u(0)u(1),...,u(s)$ of controls is*

$$\left.\begin{array}{l}\Phi_\Gamma(x_0,\underline{\sigma},\underline{u}) = A_{\sigma(s)}A_{\sigma(s-1)}\cdots A_{\sigma(2)}A_{\sigma(1)}x_0+\\ +\sum_{i=0}^{s} A_{\sigma(s)}A_{\sigma(s-1)}\cdots A_{\sigma(i+1)}B_{\sigma(i)}u(i)\end{array}\right\} \quad (4)$$

Proof.-The case $s = 1$ is clear, we prove the result by induction. Assume the result for s; that is,

$$\left.\begin{array}{l}\Phi_\Gamma(x_0,\underline{\sigma},\underline{u}) = A_{\sigma(s)}A_{\sigma(s-1)}\cdots A_{\sigma(2)}A_{\sigma(1)}x_0+\\ +\sum_{i=0}^{s} A_{\sigma(s)}A_{\sigma(s-1)}\cdots A_{\sigma(i+1)}B_{\sigma(i)}u(i)\end{array}\right\} \quad (5)$$

Consequently, by 2.2

$$\begin{array}{l}\Phi_\Gamma(x_0,\underline{\sigma}\sigma(s+1),\underline{u}u(s+1)) =\\ = A_{\sigma(s+1)}A_{\sigma(s)}\cdots A_{\sigma(2)}A_{\sigma(1)}x_0 +\\ +A_{\sigma(s+1)}\left(\sum_{i=0}^{s} A_{\sigma(s)}A_{\sigma(s-1)}\cdots A_{\sigma(i+1)}B_{\sigma(i)}u(i)\right)\\ +B_{\sigma(s+1)}u(s+1) =\\ = A_{\sigma(s+1)}A_{\sigma(s)}\cdots A_{\sigma(1)}A_{\sigma(0)}x_0\\ +\sum_{i=0}^{s} A_{\sigma(s)}A_{\sigma(s-1)}\cdots A_{\sigma(i+1)}B_{\sigma(i)}u(i)\end{array}$$

■

Reachability is a central property of (dynamical) systems. In the switched linear case we research the set of internal states that can be reached by a given switched linear system Γ for any sequence of commands $\underline{\sigma}$ and any sequence of external inputs $\underline{u}$.

Definition 2.4. *Let $\Gamma : x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)$ be a switched linear system. Let x_0 and ω be two internal states. We say that ω is switched reachable from initial state x_0 if there exists a chain of commands $\underline{\sigma}$ and a chain of inputs $\underline{u}$ such that*

$$\Phi_\Gamma(x_0,\underline{\sigma},\underline{u}) = \omega \,\} \quad (6)$$

We will denote this fact by $x_0 \rightsquigarrow_{(\underline{\sigma},\underline{u})} \omega$ or simply by $x_0 \rightsquigarrow \omega$.

We say that Γ is reachable if for every pair of states x_1, x_2, one has that $x_1 \rightsquigarrow x_2$.

Note that for switched systems reachability is also equivalent to controllability (see [Xie and Wang, 2003]). Interested reader is referred to [Xie and Wang, 2003] and [Sun and Ge, 2005] for the study of reachability and other related properties.

3 Reduced form of a system

Next we introduce a way to study a given switched linear system Γ by using a new switched linear system $\widetilde{\Gamma}$ directly obtained from Γ. Main advantage is that $\widetilde{\Gamma} = \left(\widetilde{A}_\sigma, \widetilde{B}\right)$ and all subsystems have same control matrix $\widetilde{B}$. This will yield a simplification of reachability calculations.

Let $\Gamma = (A_\sigma, B_\sigma)$ be a switched linear system, we define a new switched linear system $\widetilde{\Gamma} = \left(\widetilde{A}_\sigma, \widetilde{B}\right)$, where

$$\left.\widetilde{A}_\sigma = \begin{pmatrix} 0 & 0 \\ B_\sigma & A_\sigma \end{pmatrix} \widetilde{B} = \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix}\right\} \quad (7)$$

During the rest of the paper, a system $\widetilde{\Gamma} = \left(\widetilde{A}_\sigma, \widetilde{B}\right)$ will denote a system obtained from a system $\Gamma = (A_\sigma, B_\sigma)$ as explained. If system Γ is of n dimension we will say that system $\widetilde{\Gamma}$ is also of dimension n, not $n+m$ (the dimension of the new matrix $\widetilde{A}$) as we may suppose.

Behavior of Γ and of $\widetilde{\Gamma}$ are closely related. In fact we have that reachability from zero-state is an equivalent notion in Γ and in $\widetilde{\Gamma}$. First we need to note an easy previous result.

Lemma 3.1. *Given a system $\Gamma = (A_\sigma, B_\sigma)$ and its reduced system $\widetilde{\Gamma} = \left(\widetilde{A}_\sigma, \widetilde{B}\right)$ we have:*

$$\left.\begin{array}{l}\Phi_{\widetilde{\Gamma}}\left(\begin{pmatrix}\underline{0}\\ \underline{0}\end{pmatrix}, \sigma(0)\cdots\sigma(s), \begin{pmatrix}u(1)\\0\end{pmatrix}\cdots\begin{pmatrix}u(s)\\0\end{pmatrix}\begin{pmatrix}z\\0\end{pmatrix}\right)\\ = \begin{pmatrix} z \\ \Phi_\Gamma(\underline{0},\sigma(1)\cdots\sigma(s),u(1)\cdots u(s))\end{pmatrix}\end{array}\right\} \quad (8)$$

Proof.- It is easily checked by induction on s ■

Theorem 3.2. *Switched Linear System Γ is reachable from $\underline{0}$ if and only if system $\widetilde{\Gamma}$ is reachable from $\begin{pmatrix}\underline{0}\\ \underline{0}\end{pmatrix}$.*

Proof.- Suppose that system Γ is reachable from zero and let us prove that every internal state $\begin{pmatrix}\underline{\omega_1}\\ \underline{\omega_2}\end{pmatrix}$ of $\widetilde{\Gamma}$ can be reached from zero. Since Γ is reachable from zero it follows that

$$\underline{\omega_2} = \Phi_\Gamma(\underline{0},\sigma(1)\cdots\sigma(s),u(1)\cdots u(s)) \,\} \quad (9)$$

Consequently

$$\Phi_{\widetilde{\Gamma}}\left(\begin{pmatrix}\underline{0}\\ \underline{0}\end{pmatrix}, \sigma(0)\cdots\sigma(s), \begin{pmatrix}u(1)\\0\end{pmatrix}\cdots\begin{pmatrix}u(s)\\0\end{pmatrix}\begin{pmatrix}\omega_1\\0\end{pmatrix}\right)$$

$= \begin{pmatrix} \underline{\omega}_1 \\ \underline{\omega}_2 \end{pmatrix}$

And we are done. The converse result is proved in a similar way ■

Definition 3.3. *Let* $\Gamma : x(t+1) = A_{\sigma(t)}x(t) + Bu(t)$ *be a switched linear system. Denote by* $\mathrm{Reach}^s(\Gamma)$ *the linear subspace of all reachable states from* $\underline{0}$ *with at most* s *commands; that is to say*

$$\mathrm{Reach}^s(\Gamma) = \{\underline{x} : \Phi_\Gamma(\underline{0}, \underline{\sigma}, \underline{u}) = \underline{x}; |\underline{\sigma}|, |\underline{u}| \le s\} \Big\} \tag{10}$$

In the classical case of linear systems without switch it is well known that Γ is reachable if and only if $\mathrm{Reach}^n(\Gamma) = \mathbb{K}^n$. We will state the same result for the case of switched linear systems when $\mathbb{K}$ is an infinite field.

Obviously we have that $\mathrm{Reach}^s(\Gamma)$ is a subset $\mathrm{Reach}^{s+1}(\Gamma)$ for all s. But we can say something more:

Lemma 3.4. *In a system* $\Gamma = (A_\sigma, B)$*:*

$$\left.\begin{array}{l} \mathrm{Reach}^s(\Gamma) = \mathrm{Reach}^{s+1}(\Gamma) \Rightarrow \\ \Rightarrow \mathrm{Reach}^{s+1}(\Gamma) = \mathrm{Reach}^{s+2}(\Gamma) \end{array}\right\} \tag{11}$$

Proof.- It is sufficient to prove that the following statement yields a contradiction:

$$\mathrm{Reach}^s(\Gamma) = \mathrm{Reach}^{s+1}(\Gamma) \subsetneq \mathrm{Reach}^{s+2}(\Gamma) \Big\} \tag{12}$$

Let $\underline{x} \in \mathrm{Reach}^{s+2}(\Gamma) - \mathrm{Reach}^{s+1}(\Gamma)$ and assume that

$$\underline{x} = \Phi_\Gamma(\underline{0}, \sigma(0)\sigma(1)\cdots\sigma(s+1), u(0)u(1)\cdots u(s+1))$$

Then it follows that

$$\begin{aligned} \underline{x}' &= \Phi_\Gamma(\underline{0}, \sigma(0)\sigma(1)\cdots\sigma(s), u(0)u(1)\cdots u(s)) \\ &\in \mathrm{Reach}^{s+1} = \mathrm{Reach}^s(\Gamma) \end{aligned}$$

Consequently

$$\underline{x}' = \Phi_\Gamma(\underline{0}, \tau(0)\tau(1)\cdots\tau(s-1), v(0)v(1)\cdots v(s-1))$$

for some $\underline{\tau}, \underline{v}$.

On the other hand $\underline{x} = \Phi_\Gamma(\underline{x}', \sigma(s+1), u(s+1))$. Therefore

$$\begin{aligned} \underline{x} = \Phi_\Gamma \;\; & (\underline{0}, \tau(0)\tau(1)\cdots\tau(s-1)\sigma(s+1), \\ & , v(0)v(1)\cdots v(s-1)u(s+1)) \in \\ \in \;\; & \mathrm{Reach}^{s+1}(\Gamma) \end{aligned}$$

which is a contradiction ■

First note that $\mathrm{Reach}^n(\Gamma)$ is not a linear subspace of the state space $\mathbb{K}^n$ but it is finite union of linear subspaces of $\mathbb{K}^n$ (see [Sun and Zheng, 2001]). To be concise, if we denote by $\mathrm{Reach}_{\sigma(0)\cdots\sigma(s-1)}(\Gamma)$ the set of reachable states from zero by using the sequence of commands $\underline{\sigma}$ then

$$\mathrm{Reach}_{\underline{\sigma}}(\Gamma) = \mathrm{Im}(B, A_{\sigma(1)}B, ..., A_{\sigma(s-1)}\cdots A_{\sigma(1)}B) \Big\} \tag{13}$$

and consequently

$$\mathrm{Reach}^n(\Gamma) = \bigcup_{|\underline{\sigma}|=n} \mathrm{Reach}_{\underline{\sigma}}(\Gamma) \Big\} \tag{14}$$

Since we are working on infinite fields, a union of subspaces is the whole vector space if and only if one of involved subspaces is. Thus:

Lemma 3.5. *Given a system* $\Gamma = (A_\sigma, B)$ $\mathrm{Reach}^n(\Gamma) = \mathbb{K}^n$ *if and only if* $\mathrm{Reach}_{\underline{\sigma}}(\Gamma) = \mathbb{K}^n$ *for some* $\underline{\sigma}$ *with* $|\underline{\sigma}| = n$.

That is to say, if a system is reachable, all the states of the system can be reached with just one chain $\underline{\sigma}$ of switching signals (and the adequate inputs).

On the other hand it is not difficult to check that $\mathrm{Reach}_{\underline{\sigma}}(\Gamma)$ is a linear subspace of $\mathrm{Reach}_{\underline{\sigma\tau}}(\Gamma)$ for all $\underline{\tau} \in \Sigma^*$. Therefore dimensions can only increase n times (all of them are subspaces of $\mathbb{K}^n$). Consequently the chain

$$\cdots \subseteq \mathrm{Reach}^s(\Gamma) \subseteq \mathrm{Reach}^{s+1}(\Gamma) \subseteq \cdots \Big\} \tag{15}$$

stabilizes at index n. If an internal state cannot be reached using n commands then it can never be reached.

Above discussion is the proof of the following result:

Theorem 3.6. *Let* $\mathbb{K} = \mathbb{R}, \mathbb{C}$. *Let* $\Gamma : x(t+1) = A_{\sigma(t)}x(t) + Bu(t)$ *be a switched linear system. Then* Γ *is reachable from* $\underline{0}$ *if and only if* $\mathrm{Reach}^n(\Gamma) = \mathbb{K}^n$

As main consequence we have the criterion of reachability of switched linear systems with common input matrix B in terms of reachability from zero.

Theorem 3.7. *Let* $\mathbb{K} = \mathbb{R}, \mathbb{C}$. *Let* $\Gamma : x(t+1) = A_{\sigma(t)}x(t) + Bu(t)$ *be a switched linear system. Then* Γ *is reachable if and only if* Γ *is reachable from* $\underline{0}$

Proof.- $\Rightarrow$ is straightforward. To prove the converse note that, by previous result we have that Γ is reachable from zero if and only if

$$\mathrm{Reach}^n(\Gamma) = \bigcup_{|\underline{\sigma}|=n} \mathrm{Reach}_{\underline{\sigma}}(\Gamma) = \mathbb{K}^n \Big\} \tag{16}$$

And from 3.5 we have that $\mathbb{K}^n = \mathrm{Reach}_{\underline{\sigma}}(\Gamma)$ for some $\underline{\sigma}$ such that $|\underline{\sigma}| = n$. In particular,

$$x_2 - A_{\sigma(s)} \cdots A_{\sigma(0)} x_1 \in \mathrm{Reach}_{\underline{\sigma}}(\Gamma)$$

Hence one has the equality

$$x_2 - A_{\sigma(s)} \cdots A_{\sigma(0)} x_1 = \Phi_\Gamma(\underline{0}, \underline{\sigma}, \underline{u})$$

which is equivalent to the equality

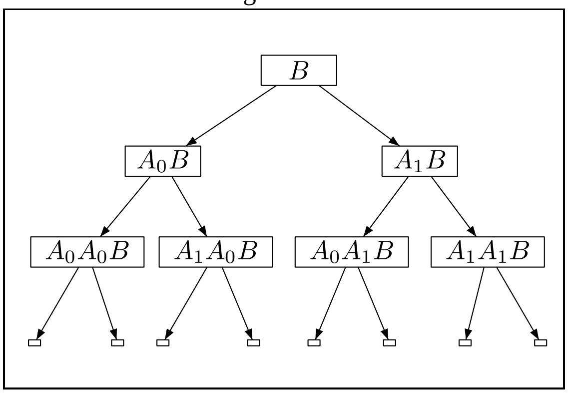
$$x_2 = \Phi_\Gamma(x_1, \underline{\sigma}, \underline{u})$$

Therefore $x_1 \rightsquigarrow x_2$ for all x_1, x_2 and Γ is reachable ■

As a consequence of 3.2, 3.6 and 3.7 we obtain the main result of the paper:

Corollary 3.8. *Let $\Gamma = (A_\sigma, B_\sigma)$ be a switched linear system and $\widetilde{\Gamma} = \left(\widetilde{A}_\sigma, \widetilde{B}\right)$ its reduced system. Then Γ is reachable if and only if $\mathrm{Reach}^n(\widetilde{\Gamma}) = \mathbb{K}^n$*

Thus to obtain the reachable states of a switched linear systems it is sufficient to obtain $\sum_{k=0}^{n-1} (\#\Sigma)^k$ blocks that need to be adequately arranged. In the case of a switched linear system $\Gamma : x(t+1) = A_{\sigma(t)} x(t) + Bu(t)$ where $\sigma \in \{0,1\}$ (i.e. two subsystems) we need to evaluate the following tree of block matrices:



We write down an explicit example for a switched linear system proposed in [Xie and Wang, 2003]:

Example 3.9. *Consider the three-dimensional ($n = 3$) single-input ($m = 1$) switched linear system ($\mathbb{K} = \mathbb{R}$) given by $\Gamma = (A_\sigma, B_\sigma, \Sigma = \{0,1\})$ where:*

$$A_0 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, B_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix}, B_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Now, system $\widetilde{\Gamma}$ is given by

$$\widetilde{A_0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \widetilde{A_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & -2 \end{pmatrix}, \widetilde{B} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now system is reachable if and only if $\widetilde{\Gamma}$ is reachable using a chain of at most $n + m = 4$ commands. Note the reverse indices from the sequence of commands and the indices of matrices. It is not difficult to complete the table and obtain that $\underline{\sigma} = x010$, with $x = 0|1$, is a sequence of commands that reaches every state (using adequate input sequence $\underline{u}$) because

$$\mathrm{span}\{\widetilde{B}, \widetilde{A}_0\widetilde{B}, \widetilde{A}_0\widetilde{A}_1\widetilde{B}, \widetilde{A}_0\widetilde{A}_1\widetilde{A}_0\widetilde{B}\} =$$
$$= \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\} =$$
$$= \mathbb{R}^{n+m}.$$

So, the original system $\Gamma = (A_\sigma, B_\sigma, \Sigma = \{0,1\})$, is reachable using the switching signal $\underline{\sigma} = 010$ (the same chain without the first symbol) and the adequate input sequence.

4 Computational considerations

Using the previous results we can build and algorithm to easily check the reachability of a switched linear system. If we start with a switched system in the form $\Gamma = (A_\sigma, B_\sigma)$, first we need to transform it to the reduced form $\widetilde{\Gamma} = \left(\widetilde{A}_\sigma, \widetilde{B}\right)$.

Then, we can use the next recursive function to check the reachability of the reduced system. This function returns ”true” (system is reachable) if it finds a base of the vector space $\mathbb{K}^n$ (stored in the variable newVectors) and ”false” (system is not reachable) in other case. This function can be seen as a preorder traversal of the tree described in the previous section. The function is first called with Reachable(1, 0, B, null).

```
Global variables: n, s, A_0 ... A_{S-1}, B.

Reachable(current_n,
          ,current_s,
          ,currentMatrix,
          ,currentVectors)
  newMatrix = A_{current_s} * currentMatrix
  newVectors = completeBase(currentVectors
                            ,newMatrix)
  if dim(newVectors) = n then
    result := true
  else
    if current_n < n then
      for i:=0 to s − 1
```

```
    result := Reachable(current_n+1
                        ,i
                        ,newMatrix
                        ,newVectors)
  else
    result := false
 return result
```

The first two instructions evaluate the current node, building a base of the subspace of currently reached states. If this subspace is not equal to $\mathbb{K}^n$ then the function is called recursively for each s to compute the next level of the tree, until we reach the n level.

Note that we are looking for a base of $\mathbb{K}^n$ in spite of $\mathbb{K}^{n+m}$ in n iterations (not $n+1$). The missing iteration is the one corresponding to the processing of the root of the tree (matrix B). As matrix B always has the same structure, its columns always complete the calculated base to a base of $\mathbb{K}^{n+m}$, so this operations are omitted.

The function completeBase() searches for new vectors for the base of the space. If any of the columns of the matrix ”newMatrix * B” is linearly independent with the vectors of ”currentVectors”, this column is added as a new vector of the base.

The complexity of this function is given by the recursive equation:

$$\left.\begin{array}{l} reachable(n,s) = g\,(n) + s * reachable(n-1,s) \\ reachable(1,s) = g\,(n) \end{array}\right\} \tag{17}$$

Where $g\,(n)$ is the complexity of the first two instructions of the algorithm (the processing of each node). One can verify that the solution of this equation is:

$$\left. reachable(n,s) = g\,(n) * \frac{s^n-1}{s-1} \right\} \tag{18}$$

The complexity of $g\,(n)$ is given by a matrix multiplication and the evaluation of the linear dependency of vectors. The matrix multiplication has complexity n^3. The test of the linear dependency we can be done using a Gauss-Jordan elimination which also has a complexity of n^3.

So, the order of complexity of the algorithm is:

$$\left. reachable\,(n,s) \in O\,(s^n) \right\} \tag{19}$$

This is quite obvious bearing in mind that the algorithm is a preorder traversal of a tree. And the complexity of a preorder traversal is the number of nodes multiplied by the processing of each node (see [Knuth, 1998]).

In the previous known result (from [Sun and Zheng, 2001]), the reachability of a system is checked by calculating n consecutive subspaces ($\mathcal{V}_i$); system is reachable if the last subspace $\mathcal{V}_n$ equals the whole space. The first subspace $\mathcal{V}_1$ is build using $s*n$ matrices. Next subspaces are obtained using $s*n$ matrices multiplications for each one of the matrices calculated for the preceding subspace. So we can see that the complexity of this algorithm is of $O\,((s*n)^n) = O\,(s^n * n!)$.

If we compare both results we can see that there is an important jump in the complexity order of calculating the reachability of a system.

If we suppose the order of the system constant, both methos result in an exponential complexity k^s. But, on the other hand, in the most common case, the number of subsystems will be a small number, and we want to know how the algorithm behaves when the dimension of the system n grows arbitrary. In this case, the previous algorithm has factorial complexity $n!$, the worst possible; while our algorithm has exponential complexity k^n. This is also the complexity order when both dimensions can be freely increased.

Besides this computational improvement, our algorithm can be easily modified to calculate, if the system is reachable, the chain referenced in Lemma 3.5 (the one that reaches all the states of the system). This is because the tree structure under this system represents the behaviour of the system: each path to a leaf node computes the reachable states with each possible chain, and the level n of the tree computes $\mathrm{Reach}^n(\Gamma)$.

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