Virtual outputs with uniformly asymptotically stable zero dynamics and feedback design

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***Abstract*— Conceptions of relative degree and minimum phase are connected to many control problems. To apply these conceptions to nonautonomous nonlinear systems one needs to know an output map which renders an nonautonomous affine system to be uniformly minimum phase. Necessary and sufficient conditions of the existence of such outputs are presented in the multy-output case. Obtained conditions result in new setting of the stabilization problem and in a new set of time-varying stabilizing feedbacks. An example of a hovercraft trajectory stabilization is considered.**

I. Introduction

One of stabilizing control design methods for a nonlinear system is based on changes of variables in the state space and in the control space. The goal of these changes of variables usually consists in obtaining an equivalent system for which we know how to solve the control problem. For example, one can often try to transform the original nonlinear system into a linear one or a partially linear one. These methods can be applied both to autonomous systems [2] and to nonautonomous systems [8].

It is well-known that the stabilization problem can be solved by the zero value output stabilization if the autonomous system under investigation is a minimum phase one [1], [2], [9]. It is worth to note that even in case of the complete knowledge of the state vector the use of properties of minimum phase systems often allows essentially enlarge a set of stabilizing feedbacks. For nonautonomous systems same ideas can be applied.

From physical point of view asymptotically stable zero dynamics are an internal property of the nonlinear dynamical system which garantees that the system is asymptotically stable by the part of variables. This property may be helpful for feedback design because one needs to control only a set of "unstable" variables. So one can hope that controls will be smaller and designed feedback will be more favourable for a stabilized system.

In this paper any smooth time-varying output map of an affine system will be called a virtual output similarly to the use of term "virtual control" in the backstepping method [3]. This output may not be a real output of an affine system.

Lyapunov converse theorems are used hereafter, so uniform asymptotic stability of zero dynamics is considered.

Uniform asymptotic stability of zero dynamics depends on choice of a virtual output. So one needs to know where an affine system can be provided by virtual outputs with uniform asymptotically stable zero dynamics and how to find such outputs if any exists. For autonomous affine systems with scalar control one can find solutions of these problems for virtual outputs with relative degree 1 and 2 (in the general case) and for virtual outputs with relative degree greater then 2 (in the special case) in [4], [5], [6]. For SISO nonautonomous systems this problem is partially solved in [7].

The main contribution of this paper consists in finding a virtual time-varying multy-output for an nonautonomous affine system with multiple control whose associated zero dynamics is uniformly asymptotically stable.

The structure of this paper is as follows. In Section 2 necessary definitions are introduced and related results are recalled. Section 3 contains necessary and sufficient conditions of the existence of virtual outputs of relative degree 1 and 2 for nonautonomous affine systems with the corresponding uniformly asymptotically stable zero dynamics. A method of finding of such virtual outputs is proposed as well. In Section 4 an illustrative example is considered.

II. Preliminaries

Consider an nonautonomous affine control system of the form

$$\begin{array}{l}\dot{x} = A(x,t) + B_1(x,t)u_1 + \ldots + B_m(x,t)u_m,\\ x \in \mathbb{R}^n,\ u = (u_1,\ldots,u_m)^{\mathrm{T}} \in \mathbb{R}^m.\end{array} \qquad (1)$$

Here $A(0,t) = 0$, $A(x,t) = (a_1(x,t),\ldots,a_n(x,t))^{\mathrm{T}}$, $B(x,t) = (B_1(x,t),\ldots,B_m(x,t))^{\mathrm{T}}$, $B_j(x,t) = (b_j^1(x,t),\ldots,b_j^n(x,t))^{\mathrm{T}}$, $j = 1,\ldots,m$, $\mathrm{rank}B(0,t) = m$, $a_i(x,t)$, $b_j^i(x,t) \in C^\infty(\Omega \times [0,+\infty))$, $\Omega \subseteq \mathbb{R}^n$ is an open set which contains equilibrium point $x = 0$.

System (1) is one-to-one corresponded on $\Omega \times [0,+\infty)$ to vector fields

$$\begin{array}{l}A = \sum_{i=1}^n a_i(x,t)\frac{\partial}{\partial x_i} + \frac{\partial}{\partial t},\ B_j = \sum_{i=1}^n b_j^i(x,t)\frac{\partial}{\partial x_i},\\ j = 1,\ldots,m.\end{array} \qquad (2)$$

Denote Lie derivative of a function $\varphi(x,t)$ along a vector field X as $L_X\psi(x,t)$.

Let a vector function $\varphi(x,t) = (\varphi_1(x,t),\ldots,\varphi_m(x,t))^T$, $(\varphi_i(x,t) \in C^\infty(\Omega) \times [0,+\infty)$, $\varphi_i(0,t) = 0$, $i = 1,\ldots,m)$ be a m–dimensional virtual output of system (1).

Assume that there exist numbers $\rho_i \geq 1$, $i = 1,\ldots,m$, such that following two conditions are fulfiled: if $k < \rho_i - 1$ then $L_{B_j}L_A^k\varphi_i(x,t)$, $1 \leq j \leq m$, are equal to zero in some

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neiborhood of the point $x = 0$, $t \geq 0$; the matrix

$$A_\rho(x,t) = \begin{pmatrix} L_{B_1}L_A^{\rho_1-1}\varphi_1(x,t) & \dots & L_{B_m}L_A^{\rho_1-1}\varphi_1(x,t) \\ \dots & \dots & \dots \\ L_{B_1}L_A^{\rho_m-1}\varphi_m(x,t) & \dots & L_{B_m}L_A^{\rho_m-1}\varphi_m(x,t) \end{pmatrix} \quad (3)$$

is nonsingular at the point $x = 0$, $t \geq 0$. In this case $\rho = (\rho_1, \dots, \rho_m)$ is said to be the vector relative degree of system (1) with the virtual multi-output $y = \varphi(x,t)$ at the point $x = 0$.

If the vector relative degree ρ is equal to $(1, \dots, 1)$ at point $x = 0$ then the matrix

$$\begin{pmatrix} L_{B_1}\varphi_1(x,t) & \dots & L_{B_m}\varphi_1(x,t) \\ \dots & \dots & \dots \\ L_{B_1}\varphi_m(x,t) & \dots & L_{B_m}\varphi_m(x,t) \end{pmatrix} \quad (4)$$

is nonsingular at the point $x = 0$ with $t \geq 0$. If $\rho_i > 1$ then from first condition it follows that a function $\varphi_i(x,t)$ is a solution of the system of PDE equations

$$L_{B_j}L_A^k\varphi_i = 0, \quad k = \overline{0,\, \rho_i - 2}, j = \overline{1,\, m}, \quad (5)$$

in some neighborhood of the point $x = 0$ with $t \geq 0$. So if $|\rho| = \rho_1 + \dots + \rho_m = n$ then necessary and sufficient conditions of existence of such virtual multy-output $\varphi(x)$ are conditions of equivalence of affine system (1) to a linear control system in some neighborhood of the point $x = 0$ (see [8]).

If these conditions are fulfilled then a control

$$u = A_\rho^{-1}(x,t) \cdot \begin{pmatrix} -L_A^{\rho_1}\varphi_1(x,t) - \sum_{k=0}^{\rho_1-1} c_{1k}L_A^k\varphi_1(x,t) \\ \dots \\ -L_A^{\rho_m}\varphi_m(x,t) - \sum_{k=0}^{\rho_m-1} c_{mk}L_A^k\varphi_m(x,t) \end{pmatrix} \quad (6)$$

stabilizes the equilibrium point $x = 0$ with respect to $\varphi(x,t)$ and its Lie derivatives. Here one need to choose the matrix of coefficients $\{c_{ij}\}$ in (6) so that all roots of equations $\lambda_i^{\rho_i} + \sum_{j=0}^{\rho_i-1} c_{ij}\lambda_i^j = 0$, $1 \leq i \leq m$ have negative real parts.

If above conditions are not fulfilled then there is no a virtual multy-output with $|\rho| = \rho_1 + \dots + \rho_m = n$. In this case for any virtual multy-output a vector relative degree at the point $x = 0$ is not defined or $\rho_1 + \dots + \rho_m = |\rho| < n$. If a relative degree is defined, $|\rho| < n$ and the distribution $G = \mathrm{span}\{B_1, \dots, B_m\}$ is involutive, where vector fields B_j, $j = 1, \dots, m$, corresponds to system (1), then in some neighborhood of the point $x = 0$ there exists a change of variables

$$\begin{array}{l} z^i = \Phi^i(x,t),\ \eta = \Psi(x,t), t = t, \\ z^i \in \mathbb{R}^{\rho_i},\ \eta \in \mathbb{R}^{n-|\rho|},\ i = \overline{1, m}, \end{array} \quad (7)$$

where $\Phi^i(0,t) = 0$, $\Psi(0,t) = 0$, $\Phi^i(x,t) = (\varphi_i(x,t), L_A\varphi_i(x,t), \dots, L_A^{\rho_i-1}\varphi_i(x,t))^{\mathrm{T}}$, $z^i = (z_1^i, \dots, z_{\rho_i}^i)^{\mathrm{T}}$, $1 \leq i \leq m$, $\eta = (\eta_1, \dots, \eta_{n-|\rho|})^{\mathrm{T}}$, that transforms affine system (1) with the virtual multy-output $y = \varphi(x,t)$ to the normal form

$$\begin{array}{l} \dot{z}_1^i = z_2^i, \dots, \dot{z}_{\rho_i-1}^i = z_{\rho_i}^i, \\ \dot{z}_{\rho_i}^i = f_i(z,\eta,t) + \\ \qquad g_{i1}(z,\eta,t)u_1 + \dots + g_{im}(z,\eta,t)u_m, \\ \dot{\eta} = q(z,\eta,t),\ y = (z_1^1, z_1^2, \dots, z_1^m)^{\mathrm{T}}, i = \overline{1, m}, \end{array} \quad (8)$$

where $f(0,0,t) = 0$, $q(0,0,t) = 0$, $z = ({z^1}^{\mathrm{T}}, {z^2}^{\mathrm{T}}, \dots, {z^m}^{\mathrm{T}})^{\mathrm{T}}$, and the matrix $\{g_{ij}(0,0,t)\}_{i,j=\overline{1,m}}$ is nonsingular with $t \geq 0$.

The system

$$\dot{\eta} = q(0,\eta,t), \quad (9)$$

is corresponded to system (8) and is called zero dynamics. If the equilibrium point $\eta = 0$ of (9) is uniformly assimptotically stable then affine system (1) with the mulpitle output $y = \varphi(x,t)$ is called uniformly minimum phase (at the point $x = 0$).

Suppose that $y = \varphi(x,t)$, $\varphi(0,t) = 0$ is a virtual multy-output of system (1) and a vector relative degree $\rho_1 + \dots + \rho_m = |\rho|$ is defined at the equilibrium point $x = 0$. If the system is uniformly minimum phase at this point then the input of type (6) locally stabilizes the equilibrium point $x = 0$ (with respect to $\varphi(x,t)$ and its Lie derivatives) of the system.

III. Stabilization of An Affine System with $\rho = (1, \dots, 1)$

Suppose that the function

$$\begin{array}{l} y = h(x,t) = (h_1(x,t), \dots, h_m(x,t))^{\mathrm{T}}, \\ h_i(x,t) \in C^\infty(\Omega),\ h_i(0,t) = 0, i = 1, \dots, m, \end{array} \quad (10)$$

is the virtual multy-output of system (1) and the vector relative degree of system (1), (10) at the point $x = 0$ is equal to $\rho = (1, \dots, 1)$, $|\rho| = m$. Assume that the distribution $G = \mathrm{span}\{B_1, \dots, B_m\}$ is involutive. Rewrite system (1), (10) in the normal form

$$\begin{array}{l} \dot{z} = f(z,\eta,t) + g_1(z,\eta,t)u_1 + \\ \qquad \dots + g_m(z,\eta,t)u_m, \end{array} \quad (11)$$

$$\dot{\eta} = q(z,\eta,t). \quad (12)$$

Here $z = (z_1, \dots, z_m)^{\mathrm{T}} \in \mathbb{R}^m$, $u = (u_1, \dots, u_m)^{\mathrm{T}} \in \mathbb{R}^m$, $\eta = (\eta_1, \dots, \eta_{n-m}) \in \mathbb{R}^{n-m}$, $f(z,\eta,t) = (f_1(z,\eta,t), \dots, f_m(z,\eta,t))^T$, $f(0,0,t) = 0$, $g_j(z,\eta,t) = (g_j^1(z,\eta,t), \dots, g_j^m(z,\eta,t))^T$, $j = 1, \dots, m$,

$$G(z,\eta,t) = \begin{pmatrix} g_1^1(z,\eta,t) & \dots & g_m^1(z,\eta,t) \\ \dots & \dots & \dots \\ g_1^m(z,\eta,t) & \dots & g_m^m(z,\eta,t) \end{pmatrix},$$

$\det G(0,0,t) \neq 0$, $q(0,0,t) = 0$, $z = h(x,t)$, $\eta = \Psi(x,t)$, $\Psi(0,t) = 0$.

Theorem 1: Assume that the distribution $G = \mathrm{span}\{B_1, \dots, B_m\}$ generated by vector fields of system (1) is involutive. Let system (11)-(12) be the normal form of system (1). System (1) has a virtual multy-output with the vector relative degree $\rho = (1, \dots, 1)$ at the point $x = 0$ and with uniforly assimptotically stable zero dynamics iff the equilibrium point $\eta = 0$ of the nonlinear system

$$\dot{\eta} = q(v,\eta,t) \quad (13)$$

with the control $v = (v_1, \dots, v_m)^{\mathrm{T}}$ is uniformly stabilizable by a smooth feedback $v = v(\eta,t) = (v_1(\eta,t), \dots, v_m(\eta,t))^{\mathrm{T}}$, $v(0,t) = 0$. Any smooth stabilizing feedback for system (13) corresponds the virtual output

$y = \varphi(x,t) = h(x,t) - v(\Psi(x,t),t)$ *for system (1) with the vector relative degree* $\rho = (1,\ldots,1)$ *at the point* $x = 0$ *and uniformly asymptotically stable zero dynamics.*

Let system (11)-(12) be the normal form of system (1). If affine system (1) with output $y = \varphi(x,t)$ (from theorem 1) is uniformly minimum-phase then control (6) locally uniformly stabilizes the point $x = 0$ with respect to $\varphi(x,t)$ and its Lie derivatives.

Theorem 2: Let system (11)-(12)be the normal form of system (1). If the equilibrium point $\eta = 0$ *of the nonlinear system*

$$\dot{\eta} = q(v,\eta,t) \tag{14}$$

with the control $v = (v_1,\ldots,v_m)^{\mathrm{T}}$ *is uniformly stabilizable by the smooth dynamical feedback*

$$\begin{array}{l} v = v(\tau,\eta,t) = (v_1(\tau,\eta,t),\ldots,v_m(\tau,\eta,t))^{\mathrm{T}},\\ v(0,0,t) = 0,\\ \dot{\tau} = \Theta(\tau,\eta,t), \quad \tau \in \mathbb{R}^p,\\ \Theta = (\Theta_1,\ldots,\Theta_p)^{\mathrm{T}}, \quad \Theta(0,0,t) = 0, \end{array} \tag{15}$$

then the equilibrium point $x = 0$ *of system (1) is uniformly stabilizable by some dynamical feedback (with respect to* $\varphi(x,t)$ *and its Lie derivatives).*

IV. Stabilization of an affine system with $|\rho| > m$.

For system (1) let us choose some virtual multy-output (10) with the vector relative degree of system (1),(10) at the point $x = 0$ being equal to $\rho = (\rho_1,\ldots,\rho_m)$, where $|\rho| = \rho_1 + \ldots + \rho_m > m$, $\rho_i \geq 1$, $i = \overline{1,m}$. Assume that the distribution $G = \mathrm{span}\{B_1,\ldots,B_m\}$ is involutive.

Theorem 3: Assume that (8) is the normal form of affine system (1) with multy-output (10) in some neighborhood of the point $x = 0$*, and* $q(z,\eta,t) = p(y,\eta,t) = p(z_1^1, z_1^2,\ldots,z_1^m,\eta,t)$.

Affine system (1) has virtual multy-output with the relative degree $\rho = (\rho_1,\ldots,\rho_m)$ *at the point* $x = 0$ *and uniformly asymptotically stable zero dynamics iff the equilibrium point* $\eta = 0$ *of the nonlinear system*

$$\dot{\eta} = p(v,\eta,t) \tag{16}$$

with the control $v = (v_1,\ldots,v_m)^{\mathrm{T}}$ *is uniformly stabilizable by the smooth feedback* $v = v(\eta,t) = (v_1(\eta,t),\ldots,v_m(\eta,t))^{\mathrm{T}}$, $v(0,t) = 0$*. Every uniformly stabilizing feedback of above type in (16) corresponds the virtual multy-output* $y = h(x,t) - v(\Psi(x,t))$ *of system (1) with relative degree* $\rho = (\rho_1,\ldots,\rho_m)$ *at the point* $x = 0$*, and the zero dynamics corresponding to this output is uniformly asymptotically stable.*

Theorem 4: Assume that (8) is the normal form of affine system (1) with virtual multy-output (10) in some neighborhood of the point $x = 0$, $t \geq 0$*, and* $q(z,\eta,t) = p(y,\eta,t) = p(z_1^1, z_1^2,\ldots,z_1^m,\eta,t)$*. If the equilibrium point* $\eta = 0$ *of the nonlinear system*

$$\dot{\eta} = p(v,\eta,t) \tag{17}$$

with the control $v = (v_1,\ldots,v_m)^{\mathrm{T}}$ *is uniformly stabilizable by the smooth dynamical feedback*

$$\begin{array}{c} v = v(\tau,\eta,t) = (v_1(\tau,\eta,t),\ldots,v_m(\tau,\eta,t))^{\mathrm{T}},\\ \dot{\tau} = \Theta(\tau,\eta,t), \tau \in \mathbb{R}^p, \quad \Theta = (\Theta_1,\ldots,\Theta_p)^{\mathrm{T}},\\ v(0,0,t) = 0, \quad \Theta(0,0,t) = 0, \end{array} \tag{18}$$

then the equilibrium point $x = 0$ *of system (1) is uniformly stabilizable by the dynamical feedback (with respect to* $h(x,t)$ *and its Lie derivatives).*

V. Example

The example of trajectory stabilization of a hovercraft will be considered.

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