THE HERTZ CONTACT PROBLEM AND ITS VOLUMETRIC REDUCTION WITH COMPUTATIONAL APPLICATIONS

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Abstract

A method of computational reduction of an elastic contact model for rigid bodies in frame of the Hertz contact model is considered. An algorithm to transform the outer surfaces geometric properties to the local contact coordinates system is described. It tracks permanently in time the surfaces of the bodies which are able to contact.

An approach to compute the normal elastic force is represented. That one deals with the reduction to one transcendental scalar equation that includes the complete elliptic integrals of the fist and second kinds. Simulation of the Hertz model was accelerated essentially due to use of the differential technique to compute the complete elliptic integrals and due to the replacement of the implicit transcendental equation by the differential one.

Based on the Hertz contact problem classic solution an invariant form for the force function which depends on the geometric properties of an intersection for the undeformed rigid bodies volumes, so-called volumetric model, is proposed then. The resulting reduced expression for the force function supposed to be in use in cases of the classic contact theory hypotheses are broken. The expression derived has been applied to several cases of the elastic bodies contacting, and in particular back to the source Hertz model itself.

The volumetric model showed a reliable behavior and an acceptable accuracy. Finally an implementation of the ball bearing dynamics computer model as an example of the contact models application is under consideration.

Key words

Hertz contact model, theorem of existence and uniqueness, volumetric contact model, ball bearing model.

1 Introduction

It is known [Gonthier, Lange, McPhee, 2007] to compute a force of the elastic bodies interaction at a contact several different approaches are applied: (a) the classical Hertz model [Hertz, 1882], (b) the model based on the polygonal approximation of the contacting surfaces [Hippmann, 2004] applied to cases of the surfaces of a complex shape, (c) the volumetric model [Gonthier, Lange, McPhee, 2007]. In our model we follow the classical Hertz approach, and the normal force computation method is a main topic of our analysis. To handle with the surfaces at the contact we apply an approach defining the surfaces using an equations of constraints. For definiteness and simplicity to simulate the tangent contact force one uses a regularized model of the Coulomb friction [Kosenko, 2005]. This is sufficient enough to simulate the dynamics over time of the machine under simulation lifecycle. May be some additional complications for the friction model, e. g. an account of the lubrication of any type, will be needed.

2 Reduction in Vicinity of Contact

Keeping a frame of the formalism applied previously to simulate a unilateral constraint [Kosenko, 2005] consider its particular case corresponding to mechanics of elastic contact interaction for two rigid bodies, identified hereafter as A and B. Their outer surfaces, see Figure 1, being at contact supposed sufficiently regular.

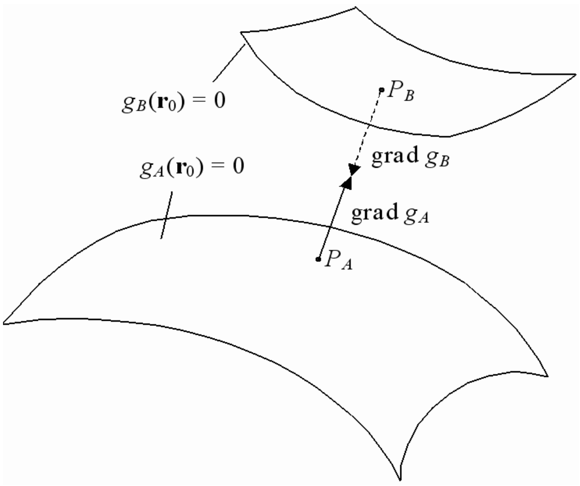

Figure 1. Vicinity of the Contact Area

How to compute the two opposing nearest points P_A and P_B of the surfaces to be tracked all over the simulation process? Applying the notations obvious enough we start here by reproducing the system of eight scalar equations

$$\begin{aligned} \operatorname{grad} g_A\left(\mathbf{r}_{P_A}\right) &= \lambda \cdot \operatorname{grad} g_B\left(\mathbf{r}_{P_B}\right), \\ \mathbf{r}_{P_A} - \mathbf{r}_{P_B} &= \mu \cdot \operatorname{grad} g_B\left(\mathbf{r}_{P_B}\right), \\ g_A\left(\mathbf{r}_{P_A}\right) &= 0, \\ g_B\left(\mathbf{r}_{P_B}\right) &= 0. \end{aligned} \tag{1}$$

defining the coordinates x_{P_A}, y_{P_A}, z_{P_A}, x_{P_B}, y_{P_B}, z_{P_B} of the outer surfaces opposing points P_A, P_B, see Figure 1. Here the coordinate vectors $\mathbf{r}_{P_A} = \left(x_{P_A}, y_{P_A}, z_{P_A}\right)^T$, $\mathbf{r}_{P_B} = \left(x_{P_B}, y_{P_B}, z_{P_B}\right)^T$ are defined with respect to (w. r. t.) the absolute coordinate frame $O_0x_0y_0z_0$ of reference (AF) usually connected to the multibody system base body B_0. Note the functions $g_A(\mathbf{r}_0) = g_A(\mathbf{r}_0, t)$, $g_B(\mathbf{r}_0) = g_B(\mathbf{r}_0, t)$ are really a time dependent ones, and define the outer surfaces current spatial position of the bodies at a contact w. r. t. AF. The values λ, μ are an auxiliary variables.

It turned out by the computational practice the most suitable approach to implement a system of algebraic equations like (1) is to replace it by the system of DAEs properly derived from (1). It can be done by introducing an additional variables being the time derivatives and composing the differential subsystem of the form

$$\dot{\mathbf{r}}_{P_A} = \mathbf{u}_{P_A}, \dot{\mathbf{r}}_{P_B} = \mathbf{u}_{P_B}, \dot{\lambda} = \xi, \dot{\mu} = \eta, \tag{2}$$

completed by the algebraic one

$$\begin{aligned} \left[\boldsymbol{\omega}_A, \operatorname{grad} g_A\right] + {} & \\ T_A \operatorname{Hess} f_A T_A^T \left(\mathbf{u}_{P_A} - \mathbf{v}_{P_A}\right) - {} & \\ \xi \operatorname{grad} g_B - \lambda\left(\left[\boldsymbol{\omega}_B, \operatorname{grad} g_B\right] + \right. & \\ \left. T_B \operatorname{Hess} f_B T_B^T \left(\mathbf{u}_{P_B} - \mathbf{v}_{P_B}\right)\right) &= \mathbf{0}, \\ \mathbf{u}_{P_A} - \mathbf{u}_{P_B} - \eta \operatorname{grad} g_B - {} & \\ \mu\left(\left[\boldsymbol{\omega}_B, \operatorname{grad} g_B\right] + \right. & \\ \left. T_B \operatorname{Hess} f_B T_B^T \left(\mathbf{u}_{P_B} - \mathbf{v}_{P_B}\right)\right) &= \mathbf{0}, \\ \left(\operatorname{grad} g_A, \mathbf{u}_{P_A}\right) - \left(\operatorname{grad} f_A, T_A^T \mathbf{v}_{P_A}\right) &= 0, \\ \left(\operatorname{grad} g_B, \mathbf{u}_{P_B}\right) - \left(\operatorname{grad} f_B, T_B^T \mathbf{v}_{P_B}\right) &= 0. \end{aligned} \tag{3}$$

where the vectors $\mathbf{v}_{P_A}$, $\mathbf{v}_{P_B}$ are a velocities of the bodies physical points currently located at the geometric points P_A, P_B, $\boldsymbol{\omega}_A$, $\boldsymbol{\omega}_B$ are the angular velocities of the bodies. Matrices $\operatorname{Hess} f_A$, $\operatorname{Hess} f_B$ are the Hesse ones of the functions f_A, f_B defining the bodies outer surfaces w. r. t. the bodies central principal coordinate systems. The the functions f_A, f_B relate to the ones g_A, g_B according to the equations

$$g_\alpha(\mathbf{r}_0) = f_\alpha\left[T_\alpha^T\left(\mathbf{r}_0 - \mathbf{r}_{O_\alpha}\right)\right] \; (\alpha = A, B),$$

where T_A, T_B are the orthogonal matrices defining current orientation of the bodies.

As usual for the Hertz approach we suppose the bodies A and B don't create any obstacles for their relative motion. If 3D-regions bounded by the bodies outer surfaces don't intersect then the contact computer model has to generate a zero wrench in the direction of each body. Simultaneously it has to generate the radius vectors $\mathbf{r}_{P_A}$, $\mathbf{r}_{P_B}$ of opposing with each other points P_A, P_B.

Based on (1) note the variable μ indicates the contact of the bodies A and B. Indeed, for definiteness suppose the outer surfaces in vicinities of the points P_A, P_B are such that vectors of gradients $\operatorname{grad} g_A(\mathbf{r})$, $\operatorname{grad} g_B(\mathbf{r})$ are directed outside the each body. Then we have the following cases at hand: (a) $\mu > 0$ means the contact absent; (b) $\mu \leq 0$: the contact takes place. If $\mu < 0$ then the bodies supposed to penetrate each other, though really begin to deform in a region of the contact. In the sequel we follow the simplest elastic contact model originating from Hertz [Hertz, 1882]. Computational analysis will be performed for the case of contacting only, see Figure 2. For simplicity and definiteness the surfaces are showed convex in Figure 2 though it is not necessary at all in general for our implementation.

To represent the Hertz contact model in its classical form first of all we have to construct an auxiliary base in vicinity of the contact. First base is composed by three unit vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ such that $\boldsymbol{\gamma} = \mathbf{n}_A$, where $\mathbf{n}_A$ is the unit vector along the gradient $\operatorname{grad} g_A(\mathbf{r})$ collinear to the z-axis in Figure 2. As it was for the derivation of the opposing points the most appropriate move to compute the proper base $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$ is to construct a relevant subsystem of DAEs. First of all start

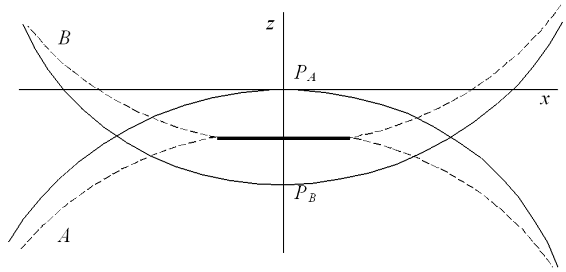

Figure 2. Local Coordinate System

with differential equation for $\boldsymbol{\gamma}$. It has the form

$$\dot{\boldsymbol{\gamma}} = |\mathrm{grad}\, g_A|^{-1} \left[(\mathrm{grad}\, g_A)^{\cdot} - (\mathbf{n}_A, (\mathrm{grad}\, g_A)^{\cdot})\, \mathbf{n}_A \right].$$

After that we can right down the chain of equations

$$\boldsymbol{\Omega} = [\boldsymbol{\gamma}, \dot{\boldsymbol{\gamma}}]\,,\ \dot{\boldsymbol{\alpha}} = [\boldsymbol{\Omega}, \boldsymbol{\alpha}]\,,\ \boldsymbol{\beta} = [\boldsymbol{\gamma}, \boldsymbol{\alpha}]\,,$$

defining successively the angular velocity $\boldsymbol{\Omega}$ of the unit vector $\boldsymbol{\gamma}(t)$ rotation, the differential equation for the unit vector $\boldsymbol{\alpha}$, and the unit vector $\boldsymbol{\beta}$ completing the local base under construction. Actually the vector $\boldsymbol{\Omega}$ is an angular velocity of the base triple $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$ w. r. t. AF.

Using the base $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$ built up above it is easy enough to compose the matrix $T = [\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}]$ consisting of the columns composed themselves by the coordinates of the unit vectors. Actually T is the transfer matrix between coordinates of AF and the current local base $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$. This makes it possible to express the outer surfaces equations in coordinates of the local system (LF) having an origin at the point P_A, see Figure 2. Under the general assumptions of the regularity for the bodies outer surfaces we can construct easily the procedure transforming the surfaces equations permanently in time to the LF such that they can be resolved w. r. t. the variable z in the explicit form

$$z = a'_\alpha x^2 + 2c'_\alpha xy + b'_\alpha y^2. \tag{4}$$

The further reduction comes to a transformation to canonical representation of the quadratic form

$$q(x, y) = ax^2 + 2cxy + by^2, \tag{5}$$

derived as a difference between the forms (4) such that $a = a'_B - a'_A$, $b = b'_B - b'_A$, $c = c'_B - c'_A$.

The transformation is implemented simply as a rotation about the z-axis of the system $P_A xy$ to achieve the coefficient c vanishes. Finally the function (5) becomes having the form

$$q(x, y) = Px^2 + Qy^2 \tag{6}$$

with the additional condition $0 < P \leq Q$.

3 The Hertz Model

According to the known technique [Landau, Lifshitz, 1999] to compute the total normal force at the contact we have to solve the system

$$\begin{aligned} \frac{FD}{\pi} \int_0^\infty \frac{d\xi}{\sqrt{(\alpha+\xi)(\beta+\xi)\xi}} &= h, \\ \frac{FD}{\pi} \int_0^\infty \frac{d\xi}{(\alpha+\xi)\sqrt{(\alpha+\xi)(\beta+\xi)\xi}} &= P, \\ \frac{FD}{\pi} \int_0^\infty \frac{d\xi}{(\beta+\xi)\sqrt{(\alpha+\xi)(\beta+\xi)\xi}} &= Q, \end{aligned} \tag{7}$$

of three transcendental equations provided the coefficients P, Q from the representation (6) and depth of mutual penetration, so-called mutual approach, $h = |\mathbf{r}_{P_B} - \mathbf{r}_{P_A}|$ are already have been computed. The system (7) has three unknown variables: α, β, F, where the values α, β are the semi-major axes squared of the contact spot ellipse, and F is the total normal elastic force really distributed over the contact area. The parameter D summarizing elastic properties of the contacting bodies depends on: Poisson's ratios ν_A, ν_B and Young's moduli E_A, E_B.

Using the substitution $\xi \mapsto \eta$ ($\xi = \lambda\eta$) in elliptic integrals of (7) we can separate the last two equations of (7). Indeed, introducing new scaled unknown variables α', β' according to formulae $\alpha' = \alpha/\lambda$, $\beta' = \beta/\lambda$ we can deduce the two mentioned equations to the closed system of ones w. r. t. α', β' if the scaling factor λ satisfies the norming condition $FD\pi^{-1}\lambda^{-3/2} = 1$.

Furthermore, we can reduce this system of equations to the one-dimensional transcendental equation

$$\frac{1}{2}\mathsf{K}(c) \left(\frac{d\mathsf{K}(c)}{dc} \right)^{-1} - (1 - c) = \frac{P}{Q} \tag{8}$$

w. r. t. the unknown value $c = k^2 = 1 - \beta'/\alpha'$, the elliptic integral modulus square. Here $\mathsf{K}(c)$ is the complete elliptic integral of the first kind. As one can clearly see we interpret the complete elliptic integrals as a functions of c using the work [Whittaker, Watson, 2002] as a pattern. Note that the inequality $\alpha' \geq \beta'$, which is equivalent to the condition $P \leq Q$, satisfied above. As one can see here the value k actually has a geometric sense exactly of the contact spot ellipse eccentricity.

Once the solution of the equation (8) had been found we can obtain immediately the values

$$\alpha' = \left(\frac{4}{Q} \frac{d\mathsf{K}(c)}{dc} \right)^{2/3},\ \beta' = \alpha'(1 - c).$$

Using the first equation of (7) and normalizing condition we then find the value of the scaling factor λ

thus arriving to the Hertz problem solution: the normal force and the contact ellipse semi-major axes values $F = \pi D^{-1}\lambda\sqrt{\lambda}$, $a = \sqrt{\lambda\alpha'}$, $b = \sqrt{\lambda\beta'}$.

Nevertheless numeric implementation usually requires a further reduction of the model in a manner we already mentioned above: use preferably the differential equations (evidently to overcome the potential problems on the DAE system index reduction stage with a software at hand). To this end we have to remind the known ODEs concerned the complete elliptic integrals of the first $\mathsf{K}(c)$ and the second $\mathsf{E}(c)$ kind between one another [Whittaker, Watson, 2002]

$$\frac{d\mathsf{K}}{dc} = \frac{\mathsf{E} - (1-c)\mathsf{K}}{2c(1-c)},\ \frac{d\mathsf{E}}{dc} = \frac{\mathsf{E} - \mathsf{K}}{2c}.$$

Furthermore, instead of (8) then we should use its differential version

$$\left[3\left(\frac{d\mathsf{K}}{dc}\right)^2 - \mathsf{K}\frac{d^2\mathsf{K}}{dc^2}\right]\dot{c} = 2\left(\frac{d\mathsf{K}}{dc}\right)^2\dot{C},$$

where $C = P/Q$. In this way the complete integrals become additional state variables, and simultaneously we have yet another way to compute elliptic integrals in dynamics, note: exclusively fast and sufficiently accurate way.

One else formal issue remains unresolved yet: whether the equation (8) has a unique solution c? It turns out the following analytic result takes place:

Theorem 1. *For the value $C = P/Q \in (0,1]$ the equation (8) has exactly one solution on the set $c \in [0,1)$.*

4 The Volumetric Model

Staying in frame of the traditional Hertz model and taking into account that the expression for the normal force has the form

$$F_{\mathrm{elast}} = -e(P,Q)h^{3/2},$$

where while changing the value h the values P, Q don't change, we conclude the potential energy of elastic deformations is represented by the expression

$$U_{\mathrm{elast}} = \frac{2}{5}e(P,Q)h^{5/2}.$$

On the other hand using the volumetric approach one can try to represent the same potential energy as follows

$$U_{\mathrm{elast}} = f\left(\frac{b}{a}\right)V^{\nu}S^{\sigma}p^{\delta},$$

where V is the volume of the bodies undeformed material intersected, S is the area of the intersection projection onto the xy-plane of the LF, p is the perimeter of that projection, a, b ($0 < a \le b$) are the semi-major axes of the contact ellipse. It turned out if $\nu = 2$, $\sigma = -7/4$, $\delta = 1/2$ then the function

$$V_{\mathrm{elast}} = 0.357469\frac{8}{15\pi^{1/4}\left(\theta_A + \theta_B\right)}\frac{V^2p^{1/2}}{S^{7/4}},$$

differs from the exact Hertz U_{elast} by 0.5% of its value in wide range of the contact ellipse shapes: surely for $b/a \in [0.1, 1]$. Here

$$\theta_\alpha = \frac{1-\nu_\alpha^2}{\pi E_\alpha},\ (\alpha = A, B).$$

Since in the case of the Hertz model the contact spot is the ellipse then the values V, S, p are to be computed by the expressions

$$V = \frac{\pi h^2}{2\sqrt{PQ}},\ S = \frac{\pi h}{2\sqrt{PQ}},\ p = \frac{4\sqrt{h}\mathsf{E}(c_1)}{\sqrt{P}},$$

where the elliptic integral modulus squared this time has the expression $c_1 = 1 - P/Q$. Then taking into account that

$$F_{\mathrm{elast}} = -\frac{\partial U_{\mathrm{elast}}}{\partial h},$$

we get the formula for the approximate value of the normal force at the contact

$$F_{\mathrm{elast}} = -0.357469\frac{2}{3\left(\theta_A + \theta_B\right)}\frac{\sqrt{\mathsf{E}(c_1)}}{P^{3/8}Q^{1/8}}h^{3/2}. \tag{9}$$

Numeric experimental verification showed an application of the above expression for the normal force indeed causes the relative error near the value 0.5% for the contacting bodies configuration coordinates in compare with "exact" Hertz model over long time of simulation. Anyway to estimate with the proper quality the fatigue processes in machines while the lifecycle simulation it is sufficient enough to have an acceptable approximation for the contact forces because the Hertz model itself is surely the quasistatic approximate one for the real processes of an elastic interaction.

The formula (9) is essentially simpler than computations in the Hertz model requiring the solution of the transcendental equation. The volumetric algorithm presented here is more reliable than the Hertz one though sometimes due to the differential techniques arranged for the elliptic integrals the Hertz algorithm works even faster than above one.

5 Examples

The procedures described above to compute the normal force of an elastic interaction were implemented on Modelica language in frame of general approach to construct the objects of mechanical constraint [Kosenko et al, 2006]. Strictly speaking in case of the compliant connection the constraint itself is absent. Instead we have an elastic compliance implementing the Hertz contact model. Though the general architecture of the objects interaction conserves completely. Thus for future purpose retain the term “constraint”.

Note the implementation under consideration computes the normal force having besides the elastic Hertzian term the term of viscosity of the form

$$F_{\text{visc}} = -d(h)\dot{h},$$

where h is the mutual approach. This latter term supposed to arise due to the plasticity properties of the material the bodies made of. It is fair natural to consider the coefficient at $\dot{h}$ to depend upon h [Wensing, 1998] since as the mutual approach increases from zero then the contact spot area also increases from zero. Therefore it is quite natural for the plastic resistance to increase continuously from zero.

A tangent force at the contact in our case for the simplicity is implemented as a regularized model of the Coulomb friction [Kosenko, 2005]. Obviously, one can create here even far more complicated models for the tangent force at the contact.

The first example is one of the simplest ones to test an implementations under examination: the contact of the ellipsoid and the plane. The Hertz model and the volumetric one were compared thoroughly in frame of wide range for different regimes of the ellipsoid motion. In particular, the comparison has been done for the case when bouncing over the horizontal surface, one of the stiffest types of motion to simulate including the elastic impacts. The two algorithms showed a high degree of coincidence for the motions under simulation. As an example one can observe the time dependence of the ellipsoid altitude, when bouncing, with the Hertz case plot, red curve, covering exactly the volumetric case plot, blue curve, see Figure 3. In fact in this particular instance of the simulation run the simplest case of the ball having the radius 0.1 unit of length has been under examination.

If we zoom in one of the impact instants, the second from right, at the end of the simulation process then we can observe the slight delay for the Hertz model, see Figure 4.

The next example, the ball bearing model, is built up using the architectural principle announced in [Kosenko, 2005]. For definiteness the bearing was equipped by eight balls. Each ball has two elastic contacts: one with the inner ring, and one with the outer one. In both cases when contacting the ball simultaneously rolls over the surfaces of the toroidal tubes corresponding to the raceways of the inner and outer rings.

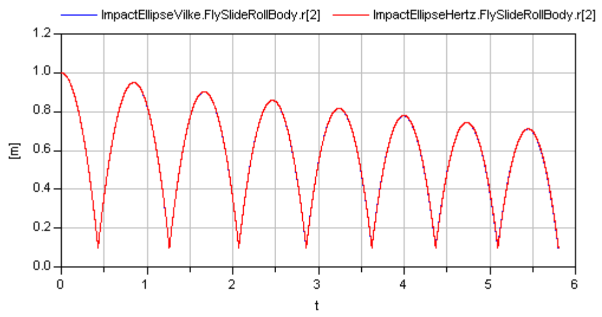

Figure 3. Bouncing Mode: The Ball Center Mass Altitude

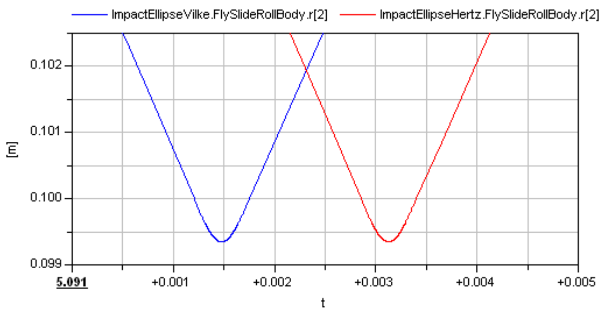

Figure 4. The Impact Instants Delay

Describe in brief the specifications of the contact between the ball and one of the toroidal raceways. The ring always supposed to be denoted as a body A in the contact object of the ball bearing model, while the ball always denoted as B. All we need to complete the constraint specifications is to define the functions f_A, f_B. In our case we have

$$\begin{aligned} f_A(x,y,z) &= 4R_A^2\left(x^2+y^2\right) - \\ &\qquad \left(x^2+y^2+z^2+R_A^2-r_A^2\right)^2, \\ f_B(x,y,z) &= x^2+y^2+z^2-R_B^2, \end{aligned}$$

where r_A is the toroidal pipe radius, R_A is the radius of the circle being an axis of that toroidal pipe, R_B is the ball radius.

The outer and the inner rings supposed to be connected rigidly with outer and inner shafts in our case, attached one with another by the bearing. In the example under consideration the body connected with the outer ring rests w. r. t. AF while the body connected to the inner ring rotates uniformly about z-axis of AF both thus performing the prescribed motion, see the animation image in Figure 5.

To verify the quality of the Hertz and the volumetric models implementation we compared the vectors γ and $\mathbf{n}_A$ as functions of time. The computational experiments showed that their coordinates coincide with a very high accuracy. At last yet another remark: to make the simulation even more faster, at least twice, one can apply the simplified expression of the form

$$F_{\text{elast}} = -eh^{3/2},$$

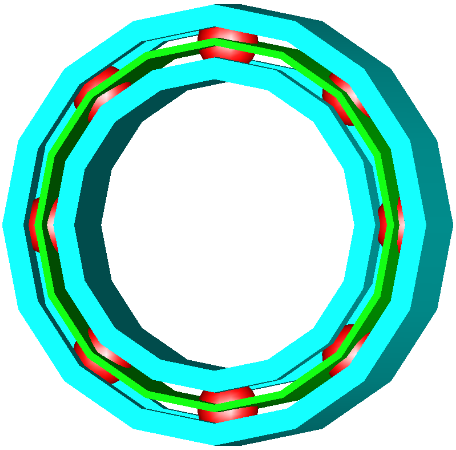

Figure 5. Animation of the Ball Bearing Model

with the constant coefficient e for the normal elastic force at the contact [Lee et al, 2007]. But it is possible only if the geometric properties (curvatures etc.) don't change while simulating the model. Moreover, for different cases of contacting the coefficient e would have different values. Then its value can be computed using the numerical experiment, or even better using the natural physical experiment. If the motion under simulation is perturbed from its pure case with the constant e then immediately its value begins change in time.

6 Conclusions

One can complete the results presented above by the issues splitting to the several main remarks influencing the potential directions of future work:

1. According to an experience accumulated while developing the models simulating the multibody dynamics one can resume the usefulness of an approach when the differential formulations proper applied are more preferable than usual algebraic or even transcendental ones. Mostly it is because the DAE solvers work better, frequently it is the DAE index reduction process, if the source system of the model equations is prepared in the best degree to be differentiated to reduce the index.
2. In particular, it turned out an introduction of the ODEs system components for the elastic bodies outer surfaces tracking for the contact problem conserves an accuracy and simultaneously improves the reliability of the models.
3. Implementation of the complete elliptic integrals using ODEs subsystem also was useful: the models became more reliable and fast. For instance, the Hertz algorithm improved as described above turned out to be even faster than the volumetric one in case of the contact area not very far from the circular case.
4. The volumetric algorithm is more reliable and suitable for wide range of the contact area eccentricities simultaneously providing an accuracy of 0.5% with respect to the Hertz-point algorithm.
5. In addition, the volumetric algorithm makes it possible to implement the contact force computation in a variety of cases far from the Hetrz model for non-elliptic contact spots.

One can outline a broad line of the future work directions such as the development of tangent forces models more complicated, account of the lubrication at the contact, applications to different types of appliances with the rotary motions, or to the problems essentially including the effects of friction when contacting.

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