Calculation of the Periodic Solutions of Two Physical Systems with Time Delay after a Hopf Bifurcation

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***Abstract*— This paper presents an application of the two newly developed methods for the calculation of the periodic solutions, resulted from a Hopf bifurcation, of a first order time-delay system arising from laser dynamics and a single inertial neural model with time delay. The two methods involve easy computation only and yield estimation of the bifurcated periodic solution with high accuracy.**

I. Introduction

Many physical systems can be modelled by delay differential equations of the first order [1]-[5], including the famous Ikeda system [2] and the Lang-Kobayashi system [4]. Due to the existence of a time-delay, such systems may exhibit complicated nonlinear dynamics. They may admit periodic solution resulted from a Hopf bifurcation, quasi-periodic solution resulted from secondary Hopf bifurcation or from double-Hopf bifurcation, and even chaotic solution. In the literature, the problem of Hopf bifurcation of time-delay systems has been investigated intensively [6]-[9]. The existence of a Hopf bifurcation can be determined from linear stability analysis. As for the bifurcation direction, the stability of the bifurcated periodic solution, the contribution of the nonlinear terms must be taken into consideration. Due to the Hopf bifurcation theory [6], at the vicinity of the Hopf bifurcation, the frequency of the periodic solution is close to that of the linearized system at the critical point. A routine for calculating the frequency of high-order approximation with respect to the small parameter is given in [10]. The amplitude of the periodic solution can be estimated usually by using the center manifold reduction plus the normal form theory, or the singular perturbation methods that are widely applied in nonlinear dynamics, including the method of averaging, the Poincaré-Lindstedt method, and the method of multiple scales. Each of the methods has advantages over the others. For example, the center manifold reduction is a tool with rigorous mathematics, but it involves usually a great deal of tedious computation. The singular perturbation methods are simpler than the center manifold reduction from the viewpoint of computation, but they may fail for some systems. The Poincaré-Lindstedt method works not only in determining the amplitude but also the frequency of the emerging periodic solution, but it does not work in studying the stability of the solution. Recently, a method named "pseudo-oscillator analysis" was developed in [11] for the Hopf bifurcation of scalar time-delayed systems. It shows that the main features of a Hopf bifurcation of a scalar delayed system can be determined by an artificially generated oscillator. The method leads to easy computation only, and gives accurate prediction on the periodic solution. But the pseudo-oscillator analysis is not applicable to Hopf bifurcation analysis of coupled equations with time delay(s), to which a new iteration method is proposed in [12]. Analysis shows that both the two methods may yield better results than the center manifold reduction [13]. Other routines for solving similar problems include the work of [14]. This paper aims at introducing the two methods, on the basis of [11][12], to calculate the periodic solution after a Hopf bifurcation of a first order time-delay system arising from laser dynamics and a single inertial neural model with time delay, presented in Sections II and III respectively, and it ends up in Section IV with a few remarks.

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II. Hopf Bifurcation of a First-order System

Consider firstly the following scalar time-delay system arising from laser physics [1]

$$\dot{x}(t) = -(\frac{\pi}{2} + \varepsilon)\sin x(t-1) \tag{1}$$

where $|\varepsilon| \ll 1$ is a small parameter. Eq. (1) undergoes a Hopf bifurcation at $\varepsilon = 0$, because (i). for small $\varepsilon < 0$, the solution $x = 0$ of Eq. (1) is asymptotically stable; (ii). at $\varepsilon = 0$, the characteristic function $p(\lambda) := \lambda + (\frac{\pi}{2} + \varepsilon)\mathrm{e}^{-\lambda}$ has a pair of complex conjugate roots $\lambda = \pm\mathrm{i}\,\frac{\pi}{2}$, and the other roots of $p(\lambda)$ have negative real parts; (iii). $\Re\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\varepsilon}\right]_{\varepsilon=0} \neq 0$, where $\Re(z)$ stands for the complex conjugate of z. The key features in the vicinity of the Hopf bifurcation can be determined from the following truncated equation

$$\dot{x}(t) + (\frac{\pi}{2} + \varepsilon)\left(x(t-1) - \frac{x^3(t-1)}{6} + \frac{x^5(t-1)}{120}\right) = 0 \tag{2}$$

because Hopf bifurcation is a local property. In what follows, three schemes will be used to find out the periodic solution.

A. The Pseudo-Oscillator Analysis [11]

The pseudo-oscillator analysis constructs firstly a pseudo-oscillator associated with Eq. (2) as follows

$$\ddot{x} + \frac{\pi^2}{4}\,x(t) + \phi(\dot{x}(t), x(t-1)) = 0 \tag{3}$$

where $\phi(\dot{x}(t), x(t-1))$ stands for the left hand side of Eq. (2). Near the Hopf bifurcation, the *stationary* solution of Eq. (1) has a form

$$x(t) = r(\varepsilon t)\cos(\omega(\varepsilon)t + \theta(\varepsilon t)) = r\cos(\omega_0 t + \theta) + O(\varepsilon) \tag{4}$$

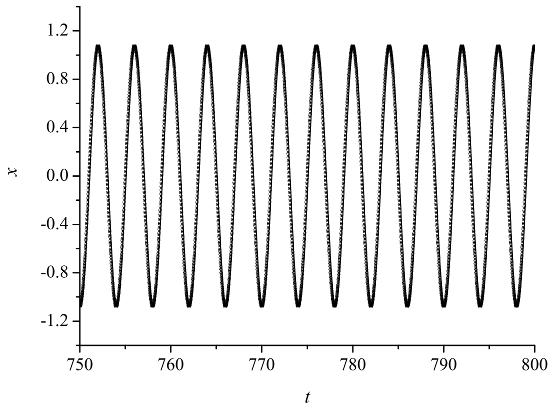

Fig. 1. The time history of $x(t)$ of the stable periodic solution for Eq. (1) with $\varepsilon = 0.25$, the two curves obtained by using numerical simulation and by using $x(t) = 1.0742\cos(\frac{\pi}{2}\,t)$ respectively are almost the same, where the stepsize of the integration via Runge-Kutta method is 0.005.

as done in applications of the method of multiple scales, where $r := r(0),\ \theta := \theta(0)$ for short. It means that Eq. (4) is slightly perturbed from $\ddot{x}(t) + \frac{\pi^2}{4}\,x(t) = 0$. So one can define an energy function

$$E = \frac{1}{2}\dot{x}^2 + \frac{\pi^2}{8}x^2$$

and computes the power function

$$\frac{\mathrm{d}E}{\mathrm{d}t}\Big|_{Eq.(3)} = -\phi(\dot{x}(t), x(t-1))\cdot\dot{x}(t) \approx -\phi(\dot{x}_*(t), x_*(t-1))\cdot\dot{x}_*(t)$$

where $x_*(t) = r\cos(\frac{\pi}{2}\,t + \theta)$ is the main part of the bifurcated periodic solution. It follows that approximately, the power function is periodic in t with period $T = \frac{2\pi}{\pi/2} = 4$, and thus it can be replaced with the following averaged *pseudo-power* function

$$\frac{\mathrm{d}E}{\mathrm{d}t}\Big|_{Eq.(3)} \approx h(r) := -\frac{1}{T}\int_0^T \phi(\dot{x}_*(t), x_*(t-1))\cdot\dot{x}_*(t)\mathrm{d}t$$

Usually, the averaging technique requires

$$\frac{\mathrm{d}E}{\mathrm{d}t}\Big|_{Eq.(3)} \approx 0 \tag{5}$$

This condition always holds if the scaling $x \to \varepsilon x$ is made, considering that Hopf bifurcation is a local property. In applications, however, such a scaling is not necessary. Straightforward computation gives

$$h(r) = (\frac{1}{1536}\pi^2 + \frac{1}{768}\varepsilon\pi)\,r^6 - (\frac{1}{64}\pi^2 + \frac{1}{32}\varepsilon\pi)\,r^4 + \frac{1}{4}\varepsilon\pi\,r^2 \tag{6}$$

If $\varepsilon < 0$, then $h(r) < 0$ holds locally, so the trivial solution $x = 0$ of Eq. (1) is asymptotically stable. If $\varepsilon > 0$, then $x = 0$ is unstable, and a bifurcated periodic solution $x(t) \approx r_0\cos(\frac{\pi}{2}\,t + \theta)$ emerges, with the amplitude r_0 determined from $h(r_0) = 0$, namely from

$$(\frac{\pi}{384} + \frac{\varepsilon}{192})\,r_0^4 - (\frac{\pi}{16} + \frac{\varepsilon}{8})\,r_0^2 + \varepsilon = 0 \tag{7}$$

A positive root r_0 exists only if $\varepsilon > 0$, thus the bifurcation is supercritical. With such a r_0 (the smaller one if the fourth order equation has two positive roots, because Hopf bifurcation is a local property), the emerging periodic solution is given by

$$x(t) \approx x_0(t) = r_0\,\cos(\frac{\pi}{2}\,t)$$

up to a shift of the phase angle. The periodic solution is stable because $h'(r_0) < 0$.

As shown in Figure 1, the pseudo-oscillator analysis yields very accurate prediction on the emerging periodic solution, and as shown in [11], it provides a better result than the multiple scaling method that has been applied in [1], where $r_0 = 4\sqrt{\varepsilon/\pi}$ was obtained.

B. The First Iteration Scheme [12]

The bifurcated periodic solution can also be obtained simply by using the following iteration scheme

$$\ddot{x}_{n+1}(t) = \ddot{x}_n(t) - \phi(\dot{x}_n(t), x_n(t-1)) \tag{8}$$

for $n = 0, 1, 2, \cdots$. Let $x_0(t) = r\cos(\frac{\pi}{2}\,t + \theta)$ be the initial iteration, then one has

$$\begin{aligned} x_1(t) &= x_0(t) + \tfrac{4}{25\pi^2}\left(\tfrac{\varepsilon}{1920} + \tfrac{\pi}{3840}\right) r^5 \sin\left(\tfrac{5\pi}{2}\,t + 5\theta\right) \\ &+ \tfrac{4}{\pi^2}\left(\left(\tfrac{\pi}{384} + \tfrac{\varepsilon}{192}\right) r^5 - \left(\tfrac{\pi}{16} + \tfrac{\varepsilon}{8}\right) r^3 + r\,\varepsilon\right) \sin\left(\tfrac{\pi}{2}\,t + \theta\right) \\ &- \tfrac{4}{9\pi^2}\left(\left(\tfrac{\varepsilon}{384} + \tfrac{\pi}{768}\right) r^5 - \left(\tfrac{\pi}{48} + \tfrac{\varepsilon}{24}\right) r^3\right) \sin\left(\tfrac{3\pi}{2}\,t + 3\theta\right) \end{aligned}$$

If the initial guess is good enough, then the term involving $\sin\left(\frac{\pi}{2}\,t + \theta\right)$ in $x_1(t)$ should disappear, which results in Eq. (7). Because the terms of high-order frequency in $x_1(t)$ contributes little to the periodic solution, the approximation $x(t) \approx x_0(t)$ can be kept unchanged.

C. The Second Iteration Scheme [12]

The iteration method works for general time-delay systems. For simplicity, let us revisit Eq. (1) or Eq. (2). To this end, we firstly define a linear operator (corresponding to the bifurcation point $\varepsilon = 0$) $L:\ \mathcal{C} := C([-1, 0], \mathbb{R}) \to \mathcal{C}$ and its adjoint operator $L^*:\ \mathcal{C}^* := C([0, 1], \mathbb{R}) \to \mathcal{C}^*$ as follows

$$L(\phi) = \begin{cases} \frac{\mathrm{d}\phi}{\mathrm{d}\theta}, & \theta \in [-1, 0) \\ -\frac{\pi}{2}\phi(-1), & \theta = 0 \end{cases}$$

$$L^*(\psi) = \begin{cases} -\frac{\mathrm{d}\psi}{\mathrm{d}s}, & s \in (0, 1] \\ -\frac{\pi}{2}\psi(1), & s = 0 \end{cases}$$

in the sense that

$$(\psi, L\phi) = (L^*\psi, \phi), \quad (\forall\phi \in \mathcal{C},\ \ \forall\psi \in \mathcal{C}^*)$$

with respect to the bilinear form

$$(\psi, \phi) = \psi(0)\phi(0) - \frac{\pi}{2}\int_{-1}^0 \psi(\xi + 1)\phi(\xi)\mathrm{d}\xi$$

Then solving two eigenvalue problems $L\phi = \mathrm{i}\,\frac{\pi}{2}\phi$, $(\phi \in \mathcal{C})$, and $L^*\psi = -\mathrm{i}\,\frac{\pi}{2}\psi$, $(\psi \in \mathcal{C}^*)$, give the basis matrices $\Phi(\theta)$ and $\Psi(s)$ as follows

$$\Phi(\theta) = [\sin(\frac{\pi}{2}\theta),\ \cos(\frac{\pi}{2}\theta)]$$

$$\Psi(s)=\frac{4}{\pi^2+4}\left[\begin{array}{c}\pi\cos(\frac{\pi}{2}s)+2\sin(\frac{\pi}{2}s)\\ 2\cos(\frac{\pi}{2}s)-\pi\sin(\frac{\pi}{2}s)\end{array}\right]$$

satisfying $(\Psi,\Phi)=I_2$ (the identical matrix).

Let $x(t)$ be the solution of Eq. (2), $x_t(\theta):=x(t+\theta)\in\mathcal{C}$, then it has a decomposition

$$x_t(\theta)=\Phi(\theta)z(t)+v(\theta),\quad z=(\Psi,x_t),\ v\in Q \tag{9}$$

Because $x_t(\theta)$ depends on t, so does $v(\theta)$. From the definition, z can be found to satisfy the following differential equation

$$\dot{z}=\Omega z+\Psi(0)F(\Phi(0)z+v(0),\Phi(-\tau)z+v(-\tau),p) \tag{10}$$

where

$$\Omega=\left[\begin{array}{cc}0 & -\omega_0\\ \omega_0 & 0\end{array}\right]$$

In addition, v is governed by a differential equation with respect to t. Such a procedure above is also required for the center manifold reduction.

Now, because v is in the stable manifold Q, any bounded solution of v must have $v=O(|\varepsilon|)$ as $t\to+\infty$. Consequently, in the vicinity of the Hopf bifurcation, z is governed by an ODE

$$\dot{z}\approx\Omega z+\Psi(0)F(\Phi(0)z,\Phi(-\tau)z,p) \tag{11}$$

Namely, $z=[z_1,z_2]^T$ is governed by the following ordinary differential equation

$$\left[\begin{array}{c}\dot{z}_1\\ \dot{z}_2\end{array}\right]\approx\left[\begin{array}{cc}0 & -\frac{\pi}{2}\\ \frac{\pi}{2} & 0\end{array}\right]\left[\begin{array}{c}z_1\\ z_2\end{array}\right]+\eta\left[\begin{array}{c}\pi\\ 2\end{array}\right] \tag{12}$$

where

$$\eta=\frac{4(\varepsilon z_1-\frac{1}{12}(\pi+2\varepsilon)z_1^3+\frac{1}{240}(\pi+2\varepsilon)z_1^5)}{\pi^2+4}$$

Taking

$$[z_{1,0},\ z_{2,0}]^T=[-r\sin(\frac{\pi}{2}\,t+\theta),\ r\cos(\frac{\pi}{2}\,t+\theta)]^T$$

as the initial guess of the iteration, and define

$$\left[\begin{array}{c}\dot{z}_{1,k+1}\\ \dot{z}_{2,k+1}\end{array}\right]=\left[\begin{array}{cc}0 & -\frac{\pi}{2}\\ \frac{\pi}{2} & 0\end{array}\right]\left[\begin{array}{c}z_{1,k}\\ z_{2,k}\end{array}\right]+\eta_k\left[\begin{array}{c}\pi\\ 2\end{array}\right] \tag{13}$$

for $k=0,1,2\cdots$, where

$$\eta_k=\frac{4(\varepsilon z_{1,k}-\frac{1}{12}(\pi+2\varepsilon)z_{1,k}^3+\frac{1}{240}(\pi+2\varepsilon)z_{1,k}^5)}{\pi^2+4}$$

then the first iteration gives

$$z_{1,1}=-r\sin(\frac{\pi}{2}t+\theta)+c_1(r)\,\cos(\frac{\pi}{2}t+\theta)+\text{h.f.t}$$
$$z_{2,1}=c_2(r)\,\cos(\frac{\pi}{2}t+\theta)+\text{h.f.t}$$

where h.f.t stands for high frequency terms, and

$$c_1(r)=\frac{r((2\varepsilon+\pi)r^4-24(2\varepsilon+\pi)r^2+384\varepsilon)}{48(\pi^2+4)}$$
$$c_2(r)=\frac{r((2\varepsilon+\pi)r^4-24(2\varepsilon+\pi)r^2+384\varepsilon+24\pi^3+96\pi)}{24\pi(\pi^2+4)}$$

Here again Eq. (7) is obtained because the coefficient of $\cos(\frac{\pi}{2}\,t+\theta)$ in $z_{1,1}$ should be zero. In this case, the approximation of the (stationary) periodic solution is found to be (up to a shift of the phase angle)

$$x(t)\approx\Phi(0)z_0(t)=r_0\cos(\frac{\pi}{2}\,t) \tag{14}$$

III. Hopf Bifurcation of a Second-order System

Now we consider the following time-delay system after a Hopf bifurcation [15]

$$\ddot{x}(t)+a\dot{x}(t)+bx(t)-c\left(f(x(t))-hf(x(t-\tau))\right)=0 \tag{15}$$

where $f(x)=\tanh(x)$. The characteristic quasi-polynomial for the trivial solution $x=0$ is

$$p(\lambda)=\lambda^2+a\lambda+b-c(1-h\mathrm{e}^{-\lambda\tau})$$

Let h be the bifurcation parameter, then a Hopf bifurcation occurs at $h=h_0$ only if for fixed a,b,c,τ, there is a $\omega_0>0$ such that $p(\mathrm{i}\omega_0)=0$, namely

$$\left\{\begin{array}{l}-\omega_0^2+b-c(1-h_0\cos(\omega_0\tau))=0\\ a\omega_0-ch\sin(\omega_0\tau)=0\end{array}\right. \tag{16}$$

We assume that the system admits a Hopf bifurcation at $h=h_0$. Because Hopf bifurcation is a local property, the main features of the system near the Hopf bifurcation can be determined from the approximate system

$$\ddot{x}(t)+a\dot{x}(t)+bx(t)-c\left(f_0(x(t))-hf_0(x(t-\tau))\right)=0 \tag{17}$$

where $f_0(x)=x-\frac{1}{3}x^3$ is the third-order truncated function of $f(x)=\tanh x$.

The problem of Hopf bifurcation of Eq. (15) has been investigated in [15] by means of the center manifold reduction and normal form theory. It has also been studied in [16] by using the pseudo-oscillator analysis and found that the pseudo-oscillator analysis yields better prediction on the emerging periodic solution than the center manifold reduction. From the viewpoint of computation, the pseudo-oscillator analysis is really simple. In what follows, the iteration method is applied to calculate the periodic solution.

To this end, an iterative sequence is firstly constructed:

$$\ddot{x}_{n+1}(t)=-a\dot{x}_n(t)-bx_n(t)+c\left(f_0(x_n(t))-hf_0(x_n(t-\tau))\right) \tag{18}$$

for $n=0,\ 1,\ 2,\ \cdots$. Let h be a value close to h_0, and let

$$x_0(t)=r\cos(\omega_0 t)$$

be the initial guess of the emerging periodic solution (up to a shift of the phase angle), then the first iteration gives

$$\begin{aligned}\ddot{x}_1(t)=&a\,r\omega\sin(\omega t)+(-\frac{c}{4}r^3+r(c-b))\cos(\omega t)\\ &+\frac{c\,h}{4}(-4r+r^3)\cos(\omega t-\omega\tau)-\frac{c}{12}r^3\cos(3\omega t)\\ &+\frac{c\,h}{12}r^3\cos(3\omega t-3\omega\tau)\end{aligned}$$

or in short

$$\ddot{x}_1(t)=\alpha_1(r)\sin(\omega_0 t)+\alpha_2(r)\cos(\omega t)+\text{h.f.t}$$

where the coefficient α_1 reads

$$\alpha_1(r)=r\,a\,\omega_0+\left(\frac{1}{4}r^3-r\right)c\,h\,\sin(\omega_0\tau)$$

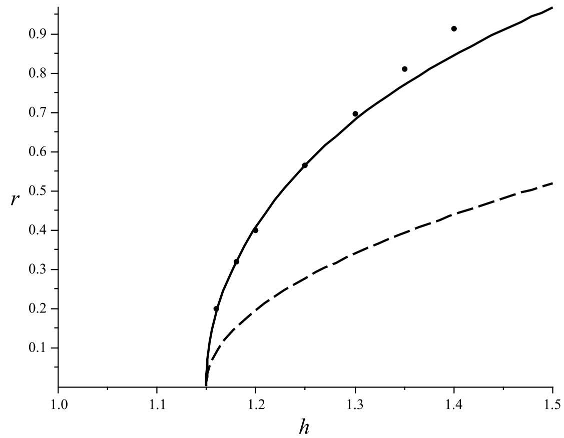

Fig. 2. The amplitude of the stable bifurcated periodic solution for Eq. (15) with $a = 1,\ b = 1.1,\ c = 1$ and $\tau = 1$. The solid line is the plot of $r_i(h)$, obtained by using the iteration method, the dashed line is the plot of $r_c(h)$, obtained by using center manifold reduction, and the dots are obtained by using numerical integration via Runge-Kutta method with fixed step-size 0.005.

If $x_0(t)$ is accurate enough, the term involving $\sin(\omega_0 t)$ in $\ddot{x}_1(t)$ should disappear, thus

$$r\, a\, \omega_0 + \left(\frac{1}{4} r^3 - r\right) c\, h\, \sin(\omega_0 \tau) = 0 \tag{19}$$

which is exactly the same as that obtained by means of the pseudo-oscillator approach [16]. Eq. (19) gives the approximate amplitude as

$$r_i = 2\sqrt{\frac{c\, h \sin(\omega_0 \tau) - a\, \omega_0}{c\, h \sin(\omega_0 \tau)}}$$

Therefore, the bifurcated periodic solution is found to be

$$x(t) \approx x_0(t) = r_i \cos(\omega_0 t) \tag{20}$$

or refined as $x(t) \approx x_1(t)$, up to a shift of the phase angle.

For demonstration, let $a = 1,\ b = 1.1,\ c = 1$ and $\tau = 1$, then the system undergoes a Hopf bifurcation at $h_0 = 1.1496$ for which $p(\lambda)$ has exactly one pair of simple roots $\lambda = \pm\, \mathrm{i}\omega_0$ with $\omega_0 = 0.9017$. On the basis of center manifold reduction and normal form theory, the governing equation for the amplitude r is obtained as following

$$\dot{r} = (h - h_0) K_1 r + K_2 r^3 + \text{h.o.t}$$

where h.o.t stands for higher order terms, $K_1 = 0.2627$ and $K_2 = -0.3408$. Hence the amplitude of the emerging periodic solution is found approximately to be

$$r_c = \sqrt{\frac{K_1(h_0 - h)}{K_2}} \approx \sqrt{0.7708(h - h_0)}$$

As shown in Figure 2, r_c gives much poor prediction on the amplitude of the periodic solution than r_i. The latter is in very good agreement with numerical simulation at the vicinity of the Hopf bifurcation.

If τ is taken as the bifurcation parameter, then similar results can be obtained. Here again, the iteration method gives better estimation of the periodic solution than the center manifold reduction.

IV. Conclusions

The problem of Hopf bifurcation is an 'Old Problem' in nonlinear dynamics and has been investigated intensively in the literature. This paper is an application of two newly developed methods, the pseudo-oscillator analysis and the iteration method, for calculating the periodic solutions resulted from a Hopf bifurcation. The two new methods are easy computational tractable and can produce better estimation of the emerging periodic solutions than the available methods such as the multiple scaling method and the center manifold reduction. Though the two methods are originally developed for delay differential equations, they work actually for ordinary differential equations [12]. On the other hand, as pointed out in [11][12], both the two methods may fail for some systems with or without time delays, while the center manifold reduction works for general systems near a Hopf bifurcation with or without time delays.

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