

The Role of Unbound Wavefunctions in Energy Quantization and Quantum Bifurcation

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Abstract

The energy eigenvalues of a confined quantum system are traditionally determined by solving the time-independent Schrödinger equation for square-integrable solutions. The resulting bound solutions give rise to the well-known phenomenon of energy quantization, but the role of unbound solutions, which are not square integrable, is still unknown. In this paper, we release the square-integrable condition and consider a general solution to the Schrödinger equation, which contains both bound and unbound wavefunctions. With the participation of unbound wavefunctions, we derive universal quantization laws from the discrete change of the number of zero of the general wavefunction, and meanwhile we propose a quantum dynamic description of energy quantization, in terms of which a new phenomenon regarding the synchronization between energy quantization and quantum bifurcation is revealed.

Keywords: Unbound Wavefunctions, Quantum Hamilton-Jacobi Formalism, Energy quantization, Quantum bifurcation, Complex Quantum Trajectory.

1. Introduction

Energy levels of a confined system are traditionally determined by solving the Schrödinger equation for square-integrable solutions. However, unbound solutions are usually ruled out under the statistical interpretation of wavefunctions. In this article, we point out that both bound and unbound wavefunctions are found to participate in the formation of energy quantization, which should be equivalently important to the wave function. Besides participating in the formation of energy quantization, unbound wavefunctions appear to be closely related to the spin motion.

The independence of energy quantization with the square-integrable condition was first revealed by Leacock and Padgett [1,2] based on the quantum

Hamilton-Jacobi (H-J) formalism, which has been developed since the inception of quantum mechanics along the line of Dirac [3] and Schwinger [4]. The main advantage of classical H-J formalism is to give the frequencies of a periodic motion directly without solving the equations of motion. Analogous to its classical counterpart, the advantage of quantum H-J formalism is recognized as a method of finding energy eigenvalues directly without solving the related Schrödinger equation. This novel approach to determining eigen energies can be conceived of as an extension of the Wilson-Sommerfeld quantization rule [5] of the action variable J :

$$J = J(E) = \frac{1}{2\pi} \oint_c p(x) dx = n\hbar, \quad (1)$$

where $p(x)$ is called quantum momentum function (QMF) and is related to the quantum action function S as

$$p = \partial S / \partial x \quad (2)$$

with S satisfying the quantum H-J equation

$$\frac{\partial S}{\partial t} + \left[\frac{p^2}{2m} + V - \frac{i\hbar}{2m} \frac{\partial x}{\partial p} \right]_{p=\partial S/\partial x} = 0 \quad (3)$$

The quantum H-J equation becomes the Schrödinger equation by the substitution $S = -i\hbar \ln \Psi$. Leacock and Padgett [1,2] proposed an ingenious method to evaluate $J(E)$ without actually solving $p(x)$ from the quantum H-J equation (3). They showed that for a given potential $V(x)$, $J(E)$ can be computed simply by a suitable deformation of the complex contour c and change of variables in Eq. (1). Once $J(E)$ has been found, the eigen energy E_n can be determined by solving E in terms of the integer n via the relation $J(E) = n\hbar$.

The quantum H-J approach to determining eigen energies has two significant implications. Firstly, this approach suggests that the eigen energy E_n stems from the quantization of the action variable J , rather than from the quantization of the total energy E itself. Precisely speaking, the eigen energy E_n is the

specific energy E at which the action variable $J(E)$ jumps between the quantized level $n\hbar$, i.e., $J(E_n) = n\hbar$. Inspired by this implication, the first goal of this paper is to reveal the internal mechanism causing the quantization of the action variable J and find out its relation to the energy quantization.

Secondly, the quantum H-J approach implies that the square-integrable condition is not required throughout the process of determining energy eigenvalues, which means that whether wavefunctions are bound or not is unconcerned upon evaluating eigen energies. Based on this observation, our second goal here is to expose how bound and unbound wavefunctions cooperate to form the observed energy levels within a confining potential.

2. Time Average along a Complex Quantum Trajectory

The necessity of considering time average along a complex quantum trajectory comes from the fact that the action variable J introduced in Eq. (1) is equal to the time-averaged kinetic energy, as will be shown below. For a particle confined by a time-independent potential $V(x)$, we have wavefunction $\Psi(t, x) = e^{-iEt/\hbar}\psi(x)$ and quantum action function $S(t, x) = -i\hbar \ln \Psi(t, x)$, with which the Quantum H-J equation (3) can be recast into the following form

$$H(x, p) = \left[\frac{p^2}{2m} + V(x) - \frac{i\hbar}{2m} \frac{\partial p}{\partial x} \right]_{p=-i\hbar d \ln \psi / dx} = -\partial S / \partial t = E \quad (4)$$

This is the energy conservation law in the quantum H-J formalism, indicating that the conserved total energy E comprises three terms: the kinetic energy $E_k = p^2/2m$, the applied potential $V(x)$, and the quantum potential $Q = -(i\hbar/2m)\partial p/\partial x$. When expressed in terms of the wavefunction $\psi_E(x)$, Eq. (4) becomes the time-independent Schrödinger equation:

$$\frac{\hbar^2}{2m} \frac{d^2 \psi_E}{dx^2} + (E - V(x))\psi_E = 0, \quad (5)$$

The energy conservation law (4) is valid for any solution $\psi_E(x)$ to the Schrödinger equation (5), either bound or unbound. However, under the statistical interpretation of $\psi_E(x)$, only the square-integrable eigenfunctions $\psi_n(x)$ are allowed in Eq. (5), which in turn limit the total energy E to the discrete eigenvalues E_n . If the square-integrable condition is released, the total energy E will be still conserved, but no longer quantized, because the participation of unbound wavefunctions in Eq. (4) can result in an arbitrary total energy E other than

E_n .

It is evident that once wavefunctions are allowed to be unbound, the total energy E can take any continuous value and consequently lose its feature of quantization. However, even if the total energy E is allowed to be varied continuously, there exist intrinsic quantization laws from which the eigen energy E_n can be recovered. This issue was first addressed by Leacock and Padgett [1,2]. As stated in the introductory section, they expressed the action variable $J(E)$ as a continuous function of the total energy E and then applied the quantization law $J(E_n) = n\hbar$ to solve for the energy eigenvalue E_n [6,7]. In other words, they found that contrary to the common understanding, the eigen energy E_n labels the n^{th} quantization level of the action variable J , rather than the quantization level of the total energy E .

To investigate the role of unbound wavefunctions in the formation of energy quantization of E_k and Q , we need an alternative operation to replace the expectation (assemble average) $\langle f(x, p) \rangle_\psi = \langle \psi | f(\hat{x}, \hat{p}) | \psi \rangle$ of a quantum observable $f(x, p)$, because unbound wavefunctions ψ fail to serve as probability density functions. In the face of unbound wavefunctions, the complex quantum trajectory method [8-10] developed from quantum H-J formalism can provide the time average of $f(x, p)$ along a complex trajectory to replace the usual assemble average $\langle f(x, p) \rangle_\psi$. The complex quantum trajectory describing the particle's motion in a confined system can be solved from the dynamic equation

$$\frac{dx}{dt} = \frac{p(x)}{m} = \frac{1}{m} \frac{\partial S(t, x)}{\partial x} = -\frac{i\hbar}{m} \frac{d \ln \psi_E(x)}{dx} \quad (6)$$

where $\psi_E(x)$ is a general solution to Eq. (5) with $E \in \mathbb{R}^+$. The resulting complex trajectory $x(t)$ serves as a physical realization of the complex contour c appearing in Eq. (1), which allows the contour integral to be evaluated along the particle's path of motion.

The time average of $f(x, p)$ along the particle's trajectory $x(t)$ is defined as

$$\langle f(x, p) \rangle_T = \frac{1}{T} \int_0^T f(x(t), p(t)) dt \quad (7)$$

where T is the period of the trajectory surrounding the equilibrium point. The action variable J defined in Eq. (1) is a ready example of taking time average along a complex contour. Letting c be a closed trajectory solved from Eq. (6) with period T , we can rewrite the contour integral (1) as a time integral

$$J = \frac{1}{2\pi} \oint_c p(x) dx = \frac{1}{\pi} \int_0^T E_k dt = \frac{2}{\omega} \langle E_k \rangle_T \quad (8)$$

where $\omega = 2\pi/T$ is the angular frequency of the periodic motion. Therefore, the Wilson-Sommerfeld quantization law $J = n\hbar$ is simply an alternative expression of the energy quantization law $\langle E_k \rangle_T = n(\hbar\omega/2)$.

By using the dynamic equation (6), the time average of an arbitrary function $f(x, p)$ given by Eq. (7) can be computed by a contour integral

$$\langle f(x, p) \rangle_T = i \frac{m\omega}{2\pi\hbar} \oint_c f(x) \frac{\psi_E(x)}{\psi_E'(x)} dx \quad (9)$$

where c is the closed contour traced by $x(t)$ on the complex plane, and the symbol “prime” denotes the differentiation with respect to x . Since QMF $p(x)$ can be expressed as a function of x , we simply write $f(x, p)$ as $f(x)$ in the integrand. According to the residue theorem, the value of $\langle f(x, p) \rangle_T$ is determined only by the poles of the integrand enclosed by the contour c and is independent of the actual form of c . We will see below that the discrete change of the number of poles in the integrand leads to the quantization of $\langle f(x, p) \rangle_T$.

3. The Origin of Energy Quantization

Let $\psi_E(x)$ be a general solution to the Schrödinger equation (5) with a given energy $E \in \mathbb{R}^+$. We can treat the time average $\langle f(x, p) \rangle_T$ as a function of the total energy E by noting that $\langle f(x, p) \rangle_T$ is computed by Eq. (9) with given wavefunction $\psi_E(x)$, which in turn depends on the energy E . The quantization of $\langle f(x, p) \rangle_T$ is a phenomenon that $\langle f(x, p) \rangle_T$ manifests a stair-like distribution as the total energy E increases monotonically. We will derive several quantization laws originating from such a stair-like behavior of $\langle f(x, p) \rangle_T$, which are universal for all confined quantum systems.

Firstly, we consider the quantization of the time-averaged kinetic energy $E_k = p^2/2m$. By substituting $f(x, p) = p^2/2m$ into Eq. (9) with $p = -i\hbar\psi_E'(x)/\psi_E(x)$, we obtain

$$\langle E_k \rangle_T = \frac{1}{T} \int_0^T \frac{1}{2m} p^2 dt = \frac{\hbar\omega}{4\pi i} \oint_c \frac{\psi_E'(x)}{\psi_E(x)} dx \quad (10)$$

To evaluate the above contour integral, we recall a formula from the residue theorem:

$$\oint_c \frac{f'(x)}{f(x)} dx = 2\pi i (Z_f - P_f) \quad (11)$$

where Z_f and P_f are, respectively, the numbers of

zero and pole of $f(x)$ enclosed by the contour c . Using this formula in Eq. (10) yields

$$\langle E_k \rangle_T = (\hbar\omega/2)(Z_{\psi} - P_{\psi}) = (\hbar\omega/2)n_{\psi} \quad (12)$$

where the integer $n_{\psi} = Z_{\psi} - P_{\psi}$ is the difference between the numbers of zero and pole of $\psi_E(x)$. It appears that the time average of the particle's kinetic energy in a confined potential is an integer multiple of $\hbar\omega/2$. This is an universal quantization law independent of the confining potential $V(x)$. Using Eq. (12) in Eq. (8), we recover the Wilson-Sommerfeld quantization law $J = n_{\psi}\hbar$.

The other quantized energy is the quantum potential Q . The evaluation of Eq. (9) with $f(x, p) = Q = -(i\hbar/2m)dp/dx$ gives

$$\langle Q \rangle_T = \frac{\hbar}{2miT} \int_0^T \frac{dp}{dx} dt = \frac{\hbar\omega}{4\pi i} \oint_c \frac{p'(x)}{p(x)} dx \quad (13)$$

Applying formula (11) once again, we arrive at the second energy quantization law

$$\langle Q \rangle_T = (\hbar\omega/2)(Z_p - P_p) = (\hbar\omega/2)n_p \quad (14)$$

where integer $n_p = Z_p - P_p$ is the difference between the numbers of zero and pole of $p(x) = -i\hbar\psi_E'(x)/\psi_E(x)$. Like the quantization of $\langle E_k \rangle_T$, Eq. (14) reveals that the value of $\langle Q \rangle_T$ is an integer multiple of $\hbar\omega/2$, irrespective of the confining potential $V(x)$.

The Kinetic energy E_k and the quantum potential energy Q , individually, are quantized quantities, and their combination leads to another quantization law. This can be verified from the combination of Eq. (10) and Eq. (13):

$$\langle E_k + Q \rangle_T = \frac{\hbar\omega}{4\pi i} \oint_c \left[\frac{\psi_E'(x)}{\psi_E(x)} + \frac{p'(x)}{p(x)} \right] dx \quad (15)$$

$$= \frac{\hbar\omega}{4\pi i} \oint_c \frac{\psi_E''(x)}{\psi_E'(x)} dx = \frac{\hbar\omega}{2} n_{\psi'} \quad (16)$$

where integer $n_{\psi'} = Z_{\psi'} - P_{\psi'}$ is the difference between the numbers of zero and pole of $\psi_E'(x)$.

The three integers, n_{ψ} , n_p and $n_{\psi'}$, are determined by the wavefunction ψ_E , which in turn is solved from Eq. (5) with a prescribed energy $E \in \mathbb{R}^+$. As E increases, the three integers can only change discretely in response to the continuous change of E . For example, let $0 < E_0 < \dots < E_{n-1} < E_n < \dots$ be the sequence of specific energies at which the integer $n_{\psi'}$ experiences a step jump $n-1 \rightarrow n$ as E increases across E_{n-1} . Hence, by treating $\langle E_k + Q \rangle_T$ as a function of E , the value of $\langle E_k + Q \rangle_T$ assumes a stair-like distribution described by

$$\langle E_k + Q \rangle_T = (\hbar\omega/2)n, \quad E_{n-1} < E \leq E_n \quad (17)$$

The values of $\langle E_k \rangle_T$ and $\langle Q \rangle_T$ have a similar

distribution. It is noted that the wavefunction $\psi_E(x)$ solved from Eq. (5) with an energy E in the interval $E_{n-1} < E \leq E_n$ is generally unbound. Our next task is to clarify the roles of these unbound wavefunctions in the quantization process of $\langle E_k \rangle_T$ and $\langle Q \rangle_T$, and to relate the specific energies E_n to the energy eigenvalues.

4. The Role of Unbound Wavefunctions in Energy Quantization

Both bound and unbound wavefunctions contribute to the quantization of $\langle E_k \rangle_T$ and $\langle E_k + Q \rangle_T$. To elucidate how they cooperate to form the observed quantization levels, we consider the typical quantum motion in a harmonic oscillator. The related Schrödinger equation in this case in dimensionless form is

$$d^2\psi/dx^2 + (2E - x^2)\psi = 0 \quad (18)$$

where the total energy E is allowed to be any positive real number. General solution has to be considered to take into account unbound wavefunctions. A general solution to the Schrödinger equation (18) can be expressed in terms of the Whittaker function $W(k, m, z)$ as

$$\psi_E(x) = C_1 x^{-1/2} W(E/2, 1/4, x^2) \quad (19)$$

For a given energy E , the obtained solution $\psi_E(x)$ is generally unbound, except for the eigen energies $E_n = n + 1/2$, $n = 0, 1, 2, \dots$, at which Eq. (19) becomes the eigenfunctions $\psi_n(x) = C_1 e^{-x^2/2} H_n(x)$ for the harmonic oscillator.

All the existing discussions on energy quantization for the harmonic oscillator focus on the bound eigenfunctions and their combinations described by the Hermite polynomial $H_n(x)$. Detailed discussions on the eigen trajectories of the harmonic oscillator were reported in the literature [11,12]. Here we are interested in the energy quantization related to the unbound $\psi_E(x)$ described by Eq. (19) with $E \neq n + 1/2$. According to Eq. (12) and Eq. (16), the quantization of $\langle E_k \rangle_T$ and $\langle E_k + Q \rangle_T$ is determined by the numbers of zero and pole of $\psi_E(x)$ and $\psi'_E(x)$. Examining the expression for Whittaker function $W(k, m, z)$, we find that $\psi_E(x)$ and $\psi'_E(x)$ do not have any pole over the entire complex plane. Using $P_\psi = P_{\psi'} = 0$, we have

$$\begin{aligned} n_\psi &= Z_\psi - P_\psi = Z_\psi, & n_{\psi'} &= Z_{\psi'} - P_{\psi'} = Z_{\psi'}, \\ n_p &= Z_p - P_p = Z_{\psi'} - Z_\psi \end{aligned} \quad (20)$$

Hence the three quantum numbers, n_ψ , n_p and $n_{\psi'}$, can be determined by the two independent integers: Z_ψ and $Z_{\psi'}$, the numbers of zero of $\psi_E(x)$ and

$\psi'_E(x)$, respectively. Regarding the computation of Z_ψ , we find the zero of ψ_E by solving the roots of the Whittaker function according to Eq. (19)

$$W(E/2, 1/4, x^2) = 0, \quad x \in \mathbb{C}, \quad (21)$$

For a given energy E , the resulting root is denoted by $x_s(E)$, and Z_ψ is the number of $x_s(E)$ satisfying Eq. (21). The blue line in Fig. 1 illustrates the variation of Z_ψ with respect to the energy E . Similarly, $Z_{\psi'}$ can be found by solving the roots of $\psi'_E(x) = 0$:

$$\begin{aligned} (2E + 1 - 2x^2) \cdot W(E/2, 1/4, x^2) \\ + 4 \cdot W(E/2 + 1, 1/4, x^2) = 0, \end{aligned} \quad (22)$$

The resulting root is denoted by $x_{eq}(E)$ and the number of $x_{eq}(E)$ satisfying Eq. (22) for a given energy E gives the value of $Z_{\psi'}$. The red line in Fig. 1 illustrates the variation of $Z_{\psi'}$ with respect to the energy E .

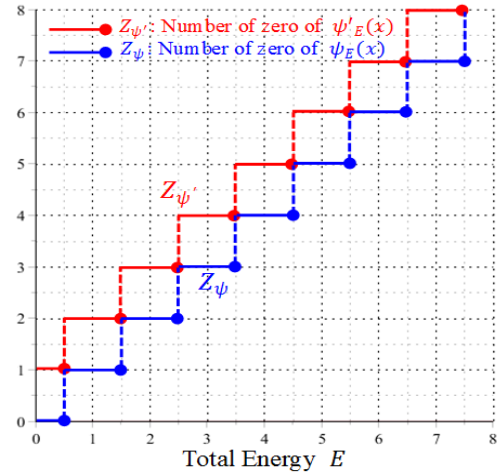


Fig. 1. The stair-like distributions of Z_ψ and $Z_{\psi'}$ determine, respectively, the quantization levels of $\langle E_k \rangle_T$ and $\langle E_k + Q \rangle_T$.

As can be seen from Fig. 1, when the total energy E increases monotonically, $Z_{\psi'}$ and Z_ψ exhibit a stair-like distribution in the form of

$$Z_{\psi'} = n_{\psi'} = n + 1, \quad Z_\psi = n_\psi = n, \quad (23)$$

where $n + 1/2 < E \leq n + 3/2$, $n = 0, 1, 2, \dots$. Based on the above distributions of $Z_{\psi'}$ and Z_ψ , the quantization laws derived in Eq. (12), Eq. (14) and Eq. (16) now become

$$\langle E_k \rangle_T = \frac{n_\psi}{2} = \frac{n}{2}, \quad \langle E_k + Q \rangle_T = \frac{n_{\psi'}}{2} = \frac{n+1}{2},$$

$$\langle Q \rangle_T = (n_{\psi'} - n_\psi)/2 = 1/2 \quad (24)$$

when the total energy E falls in the interval $n + 1/2 < E \leq n + 3/2$, $n = 0, 1, 2, \dots$. All the energies in Eq. (24) have been expressed in terms of

the multiples of $\hbar\omega$. Consequently, as we increase the total energy E monotonically, $\langle E_k \rangle_T$ and $\langle E_k + Q \rangle_T$ increase in a stair-like manner with the step levels given by Eq. (24).

The most noticeable point is that the step change of $\langle E_k \rangle_T$ and $\langle E_k + Q \rangle_T$ occurs at the specific energies $E_n = n + 1/2$, which coincide with the energy eigenvalues of the harmonic oscillator. In other words, the action of the eigenfunctions $\psi_n(x)$ amounts to determining the discrete energy E_n at which the numbers of zero of $\psi_E(x)$ and $\psi'_E(x)$ exhibit a step jump, while the action of the unbound wavefunctions $\psi_E(x)$ with $E \neq E_n$ is to form the flat parts of the stair-like distribution in Fig. 1, where the numbers of zero of $\psi_E(x)$ and $\psi'_E(x)$ keep unchanged.

5. Synchronization between Quantization and Bifurcation

As the total energy E increases, the wavefunction $\psi_E(x)$ transits repeatedly from an unbound state to a bound state, once E coincides with an eigen energy E_n . In this section, we will demonstrate that the encounter with an eigen energy not only causes a step jump of $\langle E_k \rangle_T$ and $\langle E_k + Q \rangle_T$, but also causes a nonlinear phenomenon - quantum bifurcation, where the number of equilibrium points of the quantum dynamics experiences an instantaneous change.

With $\psi_E(x)$ given by Eq. (19), the quantum dynamics (6) assumes the following dimensionless form

$$\begin{aligned} dx/dt &= -i\psi'_E(x)/\psi_E(x) \\ &= \frac{i}{2x}(2E + 1 - 2x^2) + \frac{2iW(E/2 + 1, 1/4, x^2)}{xW(E/2, 1/4, x^2)} \end{aligned} \quad (25)$$

where the total energy E is treated as a free parameter, whose critical values for the occurrence of bifurcation are to be identified.

As can be seen from Eq. (25), the equilibrium point x_{eq} of the quantum dynamics is equal to the zero of $\psi'_E(x)$, while the singular point x_s of the quantum dynamics is just the zero of $\psi_E(x)$. Hence the step changes of Z_ψ and $Z_{\psi'}$ shown in Fig. 1 also imply the step changes of the numbers of equilibrium points x_{eq} and singular points x_s , respectively. In other words, we can say that the following two phenomena occur synchronously as E increases monotonically: one phenomenon is the quantization of $\langle E_k \rangle_T$ and $\langle E_k + Q \rangle_T$ regarding the step changes of Z_ψ and $Z_{\psi'}$ as discussed previously, and the other is the bifurcation of the quantum dynamics (25) regarding the step changes of the

equilibrium points and singular points, as to be discussed below.

The bifurcations of the equilibrium points x_{eq} and the singular points x_s of the quantum dynamics (25) occur alternatively as E increases. To gain a global picture of the bifurcation pattern, we solve the equilibrium points $x_{eq}(E)$ and the singular points $x_s(E)$ from Eq. (22) and Eq. (21), respectively, and then plot them as functions of E . The resulting plots generate two sequences of pitchfork bifurcation diagram as shown in Fig. 2 and Fig. 3 for $x_{eq}(E)$ and $x_s(E)$, respectively. It can be seen that the bifurcations of $x_{eq}(E)$ and $x_s(E)$ occur alternatively at the critical energies $E_n = n + 1/2$ in such a way that the branches of $x_{eq}(E)$ bifurcate sequentially at $E = 1/2, 5/2, 9/2, \dots$, while the branches of $x_s(E)$ bifurcate sequentially at $E = 3/2, 7/2, 11/2, \dots$.

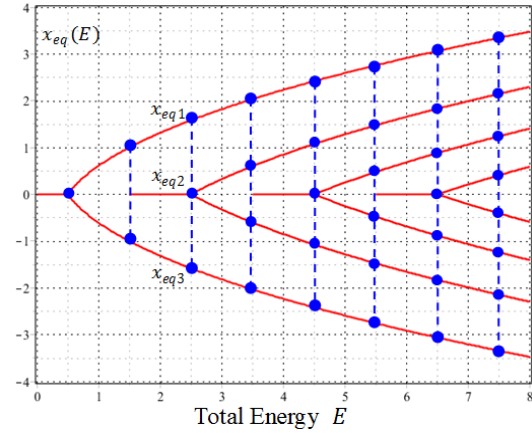


Fig. 2. A sequence of pitchfork bifurcation curves shows the variation of equilibrium points $x_{eq}(E)$ with respect to the total energy E . The number of $x_{eq}(E)$ at each E , denoted by the blue dots, is equal to the energy level $Z_{\psi'}$ as plotted in Fig. 1. The branches of the $x_{eq}(E)$ curves start sequentially at $E = 0, 3/2, 7/2, \dots$, and bifurcate sequentially at $E = 1/2, 5/2, 9/2, \dots$. Except for the bifurcation points (the blue dots), the entire sequential bifurcation diagram is formed by the unbound wavefunctions $\psi_E(x)$ with $E \neq n + 1/2$.

On the other hand, we recall that the number of $x_{eq}(E)$ at each E is just the number of zero of $\psi'_E(x)$, which gives the quantization level of $\langle E_k + Q \rangle_T$. This relation indicates that the bifurcation of the equilibrium point $x_{eq}(E)$ and the quantization of $\langle E_k + Q \rangle_T$ occur synchronously. Similarly, because the number of $x_s(E)$ at each E is the number of zero of $\psi_E(x)$, which gives the

quantization level of $\langle E_k \rangle_T$, the bifurcation of the singular point $x_{eq}(E)$ is thus synchronous with the quantization of $\langle E_k \rangle_T$. Furthermore, it is worth noting that except for the bifurcation points (the blue dots in Fig. 2 and the red dots in Fig. 3), the sequential bifurcation diagram is constructed entirely by the unbound wavefunctions $\psi_E(x)$ with $E \neq n + 1/2$. Without the participation of the unbound wavefunctions, adjacent eigenfunctions lose their interconnection and a continuous description of the bifurcation sequence becomes impossible.

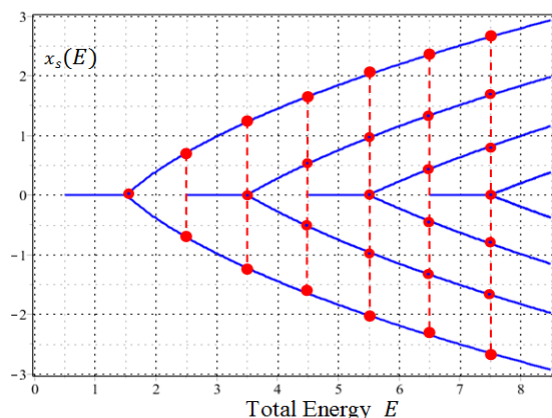


Fig. 3. A sequence of pitchfork bifurcation curves shows the variation of singular points $x_s(E)$ with respect to the total energy E . The number of $x_s(E)$ at each E , denoted by the red dots, is equal to the energy level Z_ψ as plotted in Fig. 1. The branches of the $x_s(E)$ curves start sequentially at $E = 1/2, 5/2, 9/2, \dots$, and bifurcate sequentially at $E = 3/2, 7/2, 11/2, \dots$. Except for the bifurcation points (the red dots), the entire sequential bifurcation diagram is formed by the unbound wavefunctions $\psi_E(x)$ with $E \neq n + 1/2$.

6. Conclusions

In this paper we have given a renewed interpretation for energy eigenvalues E_n and considered unbound solutions $\psi_E(x)$ to the Schrödinger equation by releasing the square-integrable requirement. The release of this requirement leads to two significant findings. (1) The origin of energy quantization: Energy quantization in a confined system originates from the discrete change of the numbers of zero of $\psi_E(x)$ and $\psi'_E(x)$, whose values determine the quantization levels of $\langle E_k \rangle_T$ and $\langle E_k + Q \rangle_T$. (2) Concurrence of Quantization and bifurcation: Bifurcations of equilibrium points and singular points of the quantum dynamics are shown to be synchronous, respectively, with the quantization process of $\langle E_k \rangle_T$ and $\langle E_k + Q \rangle_T$, as the total energy E increases monotonically.

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