Kalman Filtering in the Theory of Gyroscopic Systems ⋆

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Abstract: The substitution of equations of motion of gyroscopic systems for the so-called precessional equations is commonly used in mechanics. This procedure gives, in many cases, favorably compatible with the experiment results. This approach was justified with the help of the method of integral manifolds. But this very method reveals that the use of precessional equations instead of original ones within calculating the filtering error may lead to a intolerable error if the motion is performed by the action of random forces of the Gaussian white noise type. This conclusion is ascertained by the example analyzing the flow of the plane gyroscopic pendulum on the moving foundation.

Keywords: gyroscopic system, random inputs, optimal estimation

1. INTRODUCTION

The paper deals with the analysis of the equations of gyroscopic systems under the influence of random forces in the optimal estimation problem. The influence of random inputs on the motions of mechanical system was studied in many works, see, for example, Arato (1956); Gorelova (1997). Here, possibility of the replacement of the equations of motion by the corresponding precessional equations is investigated. This approach is widespread in mechanics and gives suitable results in numerous cases. But there are numerous examples when the substitution of the original equations by the precessional ones leads to inaccurate or qualitatively incorrect results. In this respect, there have been a few works studying either the reasoning behind such a procedure, or the conditions under which it gives an appropriate result Magnus (1971); Merkin (1974).

Formerly, this problem was solved by the method of integral manifolds Sobolev and Strygin (1978). The essence of this method is in the separation of the class of slow motions of the original system. The dimension of the system is reduced, but the system obtained, while of lower dimension, inherits the main features of its qualitative behavior. In this paper, the equations of motion of the gyroscopic system of the form suggested by Merkin (1974) are analyzed. It is shown that the method of integral manifolds can be applied to systems of this type.

Note that the equations of the flow along the integral manifold to the specified accuracy coincide with the corresponding precessional equations. In most applications the restrictions under which this slow integral manifold is stable are fulfilled. This means that any solution of the original equations, starting in the vicinity of the integral manifold, may be represented as a sum of some solution of the precessional equations and a small rapidly vanishing term. In this sense conversion to the precessional equations is permissible.

The main result of this work is concerned with the possibility of conversion to the precessional equations in the presence of random terms. It is shown that the use of precessional equations as the basis for equations of the filtering error in the problem of optimal estimation may provide inadmissible errors.

2. KALMAN FILTER EQUATIONS FOR GYROSCOPIC SYSTEM

Now, equations of the Kalman filter for gyroscopic systems will be derived. Consider the equations of motion of gyroscopic system in the non-stationary case under the action of random forces in the form in Merkin (1974)

$$\ddot{x} + [HG_0(t) + G_1(t)]\dot{x} + N(t)x = B(t)\dot{\omega}(t). \qquad (1)$$

Here x is the $n-$dimensional vector of the system state, $G_0(t)$ is a skew-symmetric matrix of gyroscopic forces, and possessing a bounded inverse for all $t \geq 0$, $G_1(t)$ is a symmetric matrix of damping forces, $N(t)$ is the matrix of potential and non-conservative forces, H is a large parameter proportional to the angular velocity of the proper rotation of the gyroscope and which is much larger than the values of all the other system parameters for many gyroscopic systems.

Let the observation take place in the presence of Gaussian white noise described by the equation

$$z = C(t)x + \dot{\xi}(t), \qquad (2)$$

where z is $m-$dimensional vector, $C(t)$ is $m\times n$ matrix. Let $\dot{\omega}(t)$ and $\dot{\xi}(t)$ be independent Gaussian white noise with

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zero expected values and correlation matrices $Q(t)\delta(t-s)$ and $R(t)\delta(t-s)$, respectively, where $Q(t)$ and $R(t)$ are symmetric positive semidefinite $m \times m-$matrices.

With $\varepsilon = 1/H$, equations (1) result in the system

$$\begin{cases} \dot{x} = y \\ \varepsilon\dot{y} = -[G_0(t) + \varepsilon G_1(t)]y - \varepsilon N(t)x + \varepsilon B(t)\dot{\omega}. \end{cases} \tag{3}$$

For simplicity of presentation $x_0 = x(0)$ and $y_0 = y(0)$ are assumed to be known vectors.

We are required to obtain an estimate $(\hat{x}(t), \hat{y}(t))^T$ of the state $(x(t), y(t))^T$, (^{T} stands for transposition), of system (3) in accordance with the vector-function $z(t)$ available for measurement at $t > 0$. The vector-function $x(t)$ is not available for measurement. The system which determines the vector $(\hat{x}(t), \hat{y}(t))^T$ is usually called the filter. Filters which are non-stationary linear systems of the form

$$\dot{\rho} = F(t)\rho + G(t)z,$$

are examined. Here $\rho(t)$ is a $2n$ dimensional vector, $F(t)$ is a $2n \times 2n$ matrix, $G(t)$ is a $2n \times m$ matrix. It is known Kalman and Bucy (1961); Roitenberg (1992) that the filter which provides an unbiased estimate

$$e(t) = (x(t), y(t))^T - (\hat{x}(t), \hat{y}(t))^T$$

for the system

$$\dot{x} = A(t)x + B(T)\dot{w},$$

with the observation (2), is defined by the differential equation

$$\frac{d\rho}{dt} = [A(t) - G(t)C(t)]\rho + G(t)z(t), \tag{4}$$

and satisfies the initial condition

$$\rho(0) = E[(x(0), y(0))^T].$$

Here $E[\cdot]$ is an expected value. Those filters which satisfy equation (4) contain the matrix $G(t)$ as a parameter, and it should be chosen to minimize the variance of the error $e(t)$. To ensure that the estimate is unbiased, we require that

$$E[(x(t), y(t))^T] = E[\rho(t)],$$

at all $t > 0$, whence $E[e(t)] = 0$.

Consequently, the correlation matrix $P(t)$ of the error $e(t)$ has the form

$$P(t) = E[e(t)e^T(t)].$$

It is clear that $P(t)$ is a symmetric matrix satisfying the initial condition

$$P(0) = E[e(0)e^T(0)] = P_0,$$

and the differential equation

$$\frac{dP}{dt} = [A(t) - G(t)C(t)]P + P[A(t) - G(t)C(t)]^T +$$
$$+B(t)Q(t)B^T(t) + G(t)R(t)G^T(t).$$

Note that matrix $G(t)$ is still unknown. Following Kalman and Bucy (1961) the filter is optimal if

$$G(t) = P(t)C^T(t)R^{-1}(t). \tag{5}$$

Hence, the equation for the correlation matrix of errors can be obtained in the form of the Riccati equation

$$\frac{dP}{dt} = A(t)P + PA^T(t) - PC^T R^{-1} CP + BQB^T, \tag{6}$$

$$P(0) = P_0. \tag{7}$$

It was shown in Kalman and Bucy (1961) that, if P_0 is a positive definite matrix, equation (6) can be solved uniquely for the matrix $P(t)$, which exists for all $t \geq 0$. Then the equation for the optimal filter, on using (4) and (5), takes the form

$$\frac{d\rho}{dt} = [A(t) - P(t)C^T(t)R^{-1}(t)C(t)]\rho + P(t)C^T(t)R^{-1}(t)z(t),$$

$$\rho(0) = E[(x(0), y(0))^T],$$

where $P(t)$ is the solution of the differential Riccati equation (6) satisfying the initial conditions (7).

Let $m_1(t,\varepsilon)$ and $m_2(t,\varepsilon)$ be the mathematical expectations of the vectors $x(t)$ and $y(t)$ of system (3), i.e.,

$$m_1(t,\varepsilon) = E[x(t)], m_2(t,\varepsilon) = E[y(t)].$$

Then vector $m(t,\varepsilon) = (m_1(t,\varepsilon), m_2(t,\varepsilon))^T$ satisfies the differential equation

$$\dot{m} = A(t)m + PC^T(t)R^{-1}(t)(z - C(t)m). \tag{8}$$

The above results may be applied to system (3). $A(t)$ is the matrix of linear terms of the system (3) and is defined by

$$A(t) = \begin{pmatrix} 0 & I \\ -N(t) & -\frac{1}{\varepsilon}G_0(t) - G_1(t) \end{pmatrix}.$$

Let $B_1(t)$ and $C_1(t)$ denote the block matrices

$$B_1(t) = \begin{pmatrix} 0 \\ -B(t) \end{pmatrix}, \; C_1(t) = \begin{pmatrix} C(t) & 0 \end{pmatrix}.$$

Then the Riccati equation for the correlation matrix $P(t,\varepsilon)$ of system (3) is

$$\frac{dP}{dt} = A(t)P + PA^T(t) - PC_1^T R^{-1} C_1 P + B_1 Q B_1^T. \tag{9}$$

The $n \times n$ blocks of the matrix $P(t,\varepsilon)$ can be designated as follows:

$$P(t,\varepsilon) = \begin{pmatrix} P_1(t,\varepsilon) & P_2(t,\varepsilon) \\ P_2^T(t,\mathrm{e}) & P_3(t,\varepsilon) \end{pmatrix}.$$

Then equation (9) implies the system

$$\dot{P}_1 = P_2^T + P_2 - P_1 S P_1, \tag{10}$$

$$\varepsilon\dot{P}_2 = \varepsilon P_3 - \varepsilon P_1 N^T - P_2(G_0 + \varepsilon G_1)^T - \varepsilon P_1 S P_2, \tag{11}$$

$$\varepsilon\dot{P}_3 = -\varepsilon(NP_2 + P_2^T N^T) - P_3(G_0 + \varepsilon G_1)^T$$
$$-(G_0 + \varepsilon G_1)P_3 - \varepsilon P_2^T S P_2 + \varepsilon L, \tag{12}$$

where $S = C^T R^{-1} C, L = BQB^T$.

Equation (8) may also be rewritten as a system:

$$\begin{cases} \dot{m}_1 = m_2 + P_1 C R^{-1}(z - Cm_1), \\ \varepsilon\dot{m}_2 = -(G_0 + \varepsilon G_1)m_2 - \varepsilon N m_1 + \varepsilon P_2 C R^{-1}(z - Cm_1), \end{cases}$$

where $m_1(t,\varepsilon)$ and $m_2(t,\varepsilon)$ satisfy the initial conditions

$$m_1(0,\varepsilon) = x_0, \; m_2(0,\varepsilon) = y_0.$$

Now, some results concerning the use of precessional equations for gyroscopic system without random forces will be presented.

3. PRECESSIONAL EQUATIONS IN THE DETERMINISTIC CASE

Consider the equations of a gyroscopic system in the form

$$\ddot{x} + (HG_0 + G_1)\dot{x} + Nx = 0,$$

in the deterministic case. With $\varepsilon = 1/H$, these equations result in

$$\varepsilon\ddot{x} + (G_0 + \varepsilon G_1)\dot{x} + \varepsilon Nx = 0. \tag{13}$$

It is a commonly held view that equations (13) may be replaced by the corresponding precessional equations

$$(G_0 + \varepsilon G_1)\dot{x} + \varepsilon N x = 0. \tag{14}$$

Note that the dimension of (14) is half the dimension of (13). Equation (14) can be transformed into the first order system

$$\dot{x} = y,\ \varepsilon\dot{y} = -(G_0 + \varepsilon G_1)y - \varepsilon N x. \tag{15}$$

According to the integral manifolds approach, the flow on the integral manifold is described by an equation

$$y = h(x, \varepsilon). \tag{16}$$

The function $h(x, \mathrm{e})$ may be found as an asymptotic series

$$h(x, \varepsilon) = \sum_{i\geq 1} \varepsilon^i h_i(x) \tag{17}$$

from the equation

$$\varepsilon\frac{\partial h(x, \varepsilon)}{\partial x} = -(G_0 + \varepsilon G_1)h(x, \varepsilon) - \varepsilon N x. \tag{18}$$

Now, the usual technique of asymptotic analysis is applied. The expansion (17) is put into (18). Having equated the coefficients of powers of the small parameter ε, one can compute the approximate solution of (18) in the form

$$h(x, \varepsilon) = -(G_0 + \varepsilon G_1)^{-1}\varepsilon N x + O(\varepsilon^2).$$

Thus, equation (15) turns into

$$\dot{x} = -(G_0 + \varepsilon G_1)^{-1}\varepsilon N x + O(\varepsilon^3). \tag{19}$$

Evidently, equations (14) and (19) coincide to the accuracy of $O(\varepsilon^3)$. Consequently, the solutions of the system (15) and the solutions of the precessional equations (14) differ in the rapidly vanishing terms only, which correspond to the so-called nutational oscillations in the gyroscopic system. So it is quite correct to examine the precessional equation instead of the full equations of the gyroscopic system in the deterministic case.

Notice that the dimension of the slow integral manifold coincides with the dimension of vector of slow variables.

4. OPTIMAL FILTERING IN THE PRECESSIONAL EQUATIONS OF GYROSCOPIC SYSTEMS

Let us now examine optimal filtering in gyroscopic systems described by the precessional equations.

Physical aspects of obtaining the precessional equations are not discussed here. It should be noted only that such equations may be derived by neglecting the second derivative terms in (1). Consider precessional equations corresponding to (1) in the form

$$\dot{x} = -(G_0 + \varepsilon G_1)^{-1}\varepsilon N x + \varepsilon(G_0 + \varepsilon G_1)^{-1}B\dot{w}.$$

Denote the correlation matrix of the vector $x(t)$ by $\Phi(t)$. Then, according to (9), this matrix must satisfy the equation

$$(G_0+\varepsilon G_1)\dot{\Phi} = -\varepsilon N\Phi - \varepsilon(G_0+\varepsilon G_1)\Phi N^T((G_0+\varepsilon G_1)^{-1})^T$$
$$-(G_0+\varepsilon G_1)\Phi C^T R^1 C\Phi + \varepsilon^2 BQB^T((G_0+\varepsilon G_1)^T)^{-1}. \tag{20}$$

Notice that at $\varepsilon = 0$ equation (20) has much in common with equation (10). Still this similarity is not sufficient to consider the precessional equations (1) to be acceptable as the basis for Kalman filtering.

To justify this conclusion, system (10)–(12) is examined. It has a stable integral manifold of slow motions Sobolev and Strygin (1978). The flow along this manifold is governed by the regularly (not singularly) perturbed equations of this system. At first sight only equation (10) is regular, and (20), being quite similar to it, may replace the full system (10)–(12). But, in fact, there are more regular equations in the system (10)–(12). Though the matrix $G_0(t)$ has no zero eigenvalues for all $t \in R$, the linear operator

$$LY = YG - GY$$

has a nontrivial kernel, since differences $(\lambda_i(t)-\lambda_j(t))$, $i, j = 1, \ldots, n$, form its spectrum. That is why there are many regular scalar equations in (12), since this operator has many zero eigenvalues. Thus, the dimension of the slow integral manifold of (10)–(12) is greater than the dimension of the matrix $\Phi(t)$, and the use of equation (7) for filtering can give unacceptable results.

This situation has much in common with that in the gyroscopic systems with a degenerate matrix of gyroscopic forces, where one should use the so-called "full" precessional equations to obtain acceptable results instead of the system given by the traditional precessional equations.

An additional advantage of the approach used here is that it allows us to consider equation (7), and regular equations from (12), instead of the full system (10)–(12).

5. ONE MECHANICAL SYSTEM

The gyroscopic pendulum is the simplest apparatus for indicating the proper vertical line direction in a moving ship or aeroplane.

Consider the equations of the plane gyroscopic pendulum with the horizontal axis of a gimbal. This pendulum is provided with a gyroscope which can turn near the axis of its housing. The turning of the gyroscope housing is limited by a spring. The movement of a plane gyroscopic pendulum under the rolling of a ship is investigated. Assume that the system is supplied with an apparatus for radial correction. The latter imposes the moment proportional to the rotation angle of the gyroscope housing round the axis of the pendulum oscillation. Then the equations of motion of the plane gyroscopic pendulum are of the form

$$I_1\ddot{\alpha} + H\dot{\beta} + lp\alpha + M\beta + n\dot{\alpha} + b\dot{w} = 0,$$
$$I_2\ddot{\beta} - H\dot{\alpha} + E\dot{\beta} + \kappa\beta = 0. \tag{21}$$

Here α is the angle of the pendulum rotation around its axis; β is the angle of gyroscope rotation around its housing axis; I_1 and I_2 are the corresponding moments of inertia; H is a moment of momentum of the gyroscope; lp is the static moment of the pendulum; M is the steepness of the moment of the radial correction; κ is the rigidity of the spring connecting the gyroscope housing with the pendulum; E and n are the coefficients of the viscous friction; $\dot{w}$ is a stationary random process corresponding to the angle of roll of the ship. Let $\dot{w}$ be a Gaussian white noise process with zero mean value and correlation function $q\delta(t - s)$.

Let the variable $z = \beta+\dot{\xi}$ be observed. At first precessional equations for (21) are considered in the form

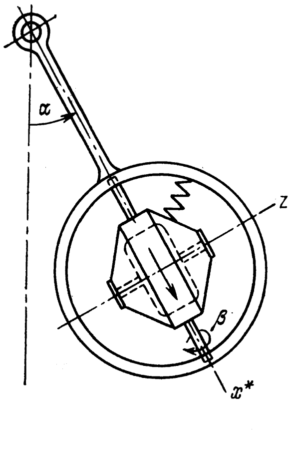

Fig. 1. The plane gyroscopic pendulum

$$H\dot{\beta} + lp\alpha + n\dot{\alpha} + M\beta + b\dot{w} = 0,$$
$$-H\dot{\alpha} + E\dot{\beta} + k\beta = 0. \qquad (22)$$

Having divided both parts of the equations (22) by H and set $1/H = \varepsilon$, $(\alpha\,\beta)^T = \omega$ we obtain:

$$\dot{\omega} = \varepsilon \begin{pmatrix} -\varepsilon Elp & -\varepsilon EM + \kappa \\ -lp & -M - \varepsilon n\kappa \end{pmatrix} \omega - \varepsilon \begin{pmatrix} \varepsilon Eb \\ b \end{pmatrix} + O(\varepsilon^3).$$

Then the equations of the Kalman filter for the correlation matrix P of the errors in the angles take the form

$$\dot{P} = \varepsilon \begin{pmatrix} \varepsilon Elp & -\varepsilon EM + \kappa \\ -lp & -M - \varepsilon n\kappa \end{pmatrix} P +$$
$$\varepsilon P \begin{pmatrix} -\varepsilon Elp & -lp \\ -\varepsilon EM + \kappa & -M - \varepsilon n\kappa \end{pmatrix}$$
$$- P^T SP + \varepsilon^2 q \begin{pmatrix} \varepsilon^2 E^2 b^2 & b^2 E \\ \varepsilon E b^2 & b^2 \end{pmatrix} + O(\varepsilon^3), \qquad (23)$$

where

$$S = \begin{pmatrix} 0 & 0 \\ 0 & 1/r \end{pmatrix}.$$

We seek a solution of (23) as a series:

$$P(\varepsilon) = D_0 + \varepsilon D_1 + O(\varepsilon^2).$$

From (23) it follows that $D_0 = 0$, and D_1 satisfies the equation

$$\dot{D}_1 = \begin{pmatrix} 0 & \kappa \\ -lp & -M \end{pmatrix} D_1 + D_1 \begin{pmatrix} 0 & -lp \\ \kappa & -M \end{pmatrix} - D_1 S D_1 + \begin{pmatrix} 0 & 0 \\ 0 & qb^2 \end{pmatrix}.$$

It should be noted, that this mechanical system (plane gyroscopic pendulum) was examined in Roitenberg (1992) by means of the precessional theory of gyroscopes, provided that $n = E = 0$. Under such assumptions, Equation (23) does not contain $O(\varepsilon^3)$ terms and, in coordinate form, is as follows:

$$\dot{d}_1 = 2\frac{\kappa}{H} d_2 - \frac{d_2^2}{r},$$
$$\dot{d}_2 = -\frac{lp}{H} d_1 - \frac{M}{H} d_2 + \frac{\kappa}{H} d_3 - \frac{d_2 d_3}{r},$$
$$\dot{d}_3 = -2\frac{lp}{H} d_2 - 2\frac{M}{H} d_3 - \frac{d_3^2}{r} + \frac{qb^2}{H^2}.$$

Here d_1, d_2 and d_3 denote the elements of the symmetric correlation matrix D. But we cannot compare these equations with those obtained on the basis of the theory of integral manifolds, since, for $n = E = 0$, the equations of motion of the plane gyroscopic pendulum may have no attracting integral manifold.

Next the full equations (21) are considered in the form

$$\varepsilon\ddot{\alpha} + \frac{\beta}{I_1} + \varepsilon\frac{n}{I_1}\dot{\alpha} + \varepsilon\frac{lp}{I_1}\alpha + \varepsilon\frac{M}{I_1}\beta + \varepsilon\frac{b}{I_1}\dot{w} = 0,$$
$$\varepsilon\ddot{\beta} - \frac{1}{I_2}\alpha + \varepsilon\frac{E}{I_2}\beta + \varepsilon\frac{\kappa}{I_2}\beta = 0,$$

or, in the more convenient form,

$$\varepsilon \begin{pmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{pmatrix} + \begin{pmatrix} 0 & 1/I_1 \\ -1/I_2 & 0 \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} + \varepsilon \begin{pmatrix} n/I_1 & 0 \\ 0 & E/I_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$
$$+ \varepsilon \begin{pmatrix} lp/I_1 & M/I_1 \\ 0 & \kappa/I_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\varepsilon \begin{pmatrix} b/I_1 \\ 0 \end{pmatrix} \dot{w}. \qquad (24)$$

We use the following notation:

$$G_0 = \begin{pmatrix} 0 & 1/I_1 \\ -1/I_2 & 0 \end{pmatrix}, \quad G_1 = \begin{pmatrix} n/I_1 & 0 \\ 0 & E/I_2 \end{pmatrix},$$
$$N = \begin{pmatrix} lp/I_1 & M/I_1 \\ 0 & \kappa/I_2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} b/I_1 \\ 0 \end{pmatrix}.$$

Then the equations of the Kalman filter, according to (24), may be written as follows:

$$\dot{P}_1 = P_2^T + P_2 - P_1 S P_2, \qquad (25)$$

$$\varepsilon\dot{P}_2 = \varepsilon P_3 - \varepsilon P_1 N^T - P_2 (G_0 + \varepsilon G_1)^T - \varepsilon P_1 S P_2, \quad (26)$$

$$\varepsilon\dot{P}_3 = -\varepsilon(N P_2 + P_2^T N^T) - P_3 (G_0 + \varepsilon G_1)^T$$
$$-(G_0 + \varepsilon G_1) P_3 - \varepsilon P_2^T S P_2 + \varepsilon B_2 Q B_2^T. \qquad (27)$$

The matrices $B_2 Q B_2^T$ and $CR^{-1}C = S$ can be computed easily in the form

$$B_2 Q B_2^T = \begin{pmatrix} b^2 q/I_1^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ 0 & 1/r \end{pmatrix}.$$

We designate the elements of the 2×2 matrices P_1, P_2, P_3 as follows:

$$P_1 = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}, P_2 = \begin{pmatrix} p_4 & p_7 \\ p_5 & p_8 \end{pmatrix}, P_3 = \begin{pmatrix} p_6 & p_9 \\ p_9 & p_{10} \end{pmatrix}.$$

Then equation (27) may be transformed into a system of three scalar equations:

$$\varepsilon\dot{p}_6 = -\frac{2}{I_1} p_9 - 2\varepsilon \left(\frac{lp}{I_1} p_4 + \frac{M}{I_1} \right) p_5 - 2\varepsilon\frac{n}{I_1} p_6 + \varepsilon\frac{b^2}{I_1^2} q - \varepsilon\frac{p_5^2}{r},$$
$$\varepsilon\dot{p}_9 = \frac{p_6}{I_2} - \frac{p_{10}}{I_1} - \varepsilon \left(\frac{\kappa}{I_2} p_5 + \frac{M}{I_1} p_8 + \frac{lp}{I_1} p_7 + \left(\frac{E}{I_2} + \frac{n}{I_1} \right) p_9 \right)$$
$$-\varepsilon\frac{p_5 p_8}{r}, \qquad (28)$$
$$\varepsilon\dot{p}_{10} = 2\frac{1}{I_2} p_9 - 2\varepsilon\frac{\kappa}{I_2} p_8 - 2\varepsilon\frac{E}{I_2} p_{10} - \varepsilon\frac{p_8^2}{r}.$$

In (28) we introduce the change of variables

$$(p_6\ p_9\ p_{10})^T = T(\omega_6\ \omega_9\ \omega_{10})^T, \qquad (29)$$

where T is the matrix

$$T=\begin{pmatrix} I_2 & I_2 & I_2\\ 0 & \sqrt{I_1I_2} & -\sqrt{I_1I_2}\\ I_1 & -I_1 & -I_1\end{pmatrix},$$

with the inverse

$$T^{-1}=\begin{pmatrix} \frac{1}{2I_2} & 0 & \frac{1}{2I_1}\\ \frac{1}{4I_2} & \frac{1}{2\sqrt{I_1I_2}} & -\frac{1}{4I_1}\\ \frac{1}{4I_2} & -\frac{1}{2\sqrt{I_1I_2}} & -\frac{1}{4I_1}\end{pmatrix}.$$

This matrix T transforms the matrix of linear terms of the system (28) to the skew-symmetric matrix

$$\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & \frac{2}{\sqrt{I_1I_2}}\\ 0 & -\frac{2}{\sqrt{I_1I_2}} & 0\end{pmatrix}.$$

It may be easily checked that, after this transformation of variables, system (28) becomes

$$\begin{aligned}\dot\omega_6 = &-\left(\frac{n}{I_1}+\frac{E}{I_2}\right)\omega_6+\left(-\frac{n}{I_1}-\frac{E}{I_2}\right)\omega_9+\left(-\frac{n}{I_1}+\frac{E}{I_2}\right)\omega_{10}\\ &-\left(\frac{lp}{I_1I_2}p_4+\frac{M}{I_1I_2}p_5\right)-\frac{\kappa}{I_1I_2}p_8-\frac{p_5^2}{2I_2r}-\frac{p_8^2}{2I_1r}+\frac{b^2q}{I_1^2I_2},\end{aligned} \tag{30}$$

$$\begin{aligned}\varepsilon\dot\omega_9 = &\,\varepsilon\left(-\frac{n}{2I_1}+\frac{E}{2I_2}\right)\omega_6-\varepsilon\left(\frac{n}{I_1}+\frac{E}{I_2}\right)\omega_{10}+\frac{2}{\sqrt{I_1I_2}}\omega_{10}-\\ &\varepsilon\frac{1}{2I_2}\left(\frac{lp}{I_1}p_4+\frac{M}{I_1}p_5\right)\\ &-\frac{\varepsilon}{\sqrt{I_1I_2}}\left(\frac{\kappa}{I_2}p_5+\frac{M}{I_1}p_8+\frac{lp}{2I_1}p_7+\frac{1}{2r}p_5p_8\right)\\ &+\varepsilon\frac{\kappa}{2I_1I_2}p_8-\varepsilon\frac{1}{4I_2r}p_5^2+\varepsilon\frac{1}{4I_1r}p_8^2+\varepsilon\frac{b^2}{4I_1^2I_2}q,\end{aligned} \tag{31}$$

$$\begin{aligned}\varepsilon\dot\omega_{10} = &-\varepsilon\left(-\frac{n}{2I_1}+\frac{E}{2I_2}\right)\omega_6-\varepsilon\left(\frac{n}{I_1}+\frac{E}{I_2}\right)\omega_9-\frac{2}{\sqrt{I_1I_2}}\omega_9\\ &+\varepsilon\frac{1}{2I_2}\left(\frac{lp}{I_1}p_4+\frac{M}{I_1}p_5\right)-\frac{\varepsilon}{\sqrt{I_1I_2}}\left(\frac{\kappa}{I_2}p_5+\frac{M}{I_1}p_8+\frac{lp}{2I_1}p_7+\frac{1}{r}p_5p_8\right)\\ &+\varepsilon\frac{\kappa}{2I_1I_2}p_8-\varepsilon\frac{1}{4I_2r}p_5^2+\varepsilon\frac{1}{4I_1r}p_8^2+\varepsilon\frac{b^2}{4I_1^2I_2}q.\end{aligned} \tag{32}$$

Now system (25), (26), (30)–(32) is considered. There are four regular equations, hence four slow variables. This system has a four-dimensional slow integral manifold, and it is stable. We search for this manifold as an asymptotic series

$$\begin{aligned}P_2 &= P_2^{(0)}(P_1,\omega_6)+\varepsilon P_2^{(1)}(P_1,\omega_6)+O(\varepsilon^2),\\ \omega_j &= \omega_j^{(0)}(P_1,\omega_6)+\varepsilon\omega_j^{(1)}(P_1,\omega_6)+O(\varepsilon^2), j=9,10.\end{aligned} \tag{33}$$

These expansions are substituted into the singularly perturbed equations (25), (26), (30)–(32). Let $\partial P_2/\partial P_1\dot P_1$ denotes the matrix

$$\frac{\partial P_2}{\partial P_1}\dot P_1=\begin{pmatrix} \sum_{i=1}^{3}\frac{\partial p_4}{\partial p_i}\dot p_i & \sum_{i=1}^{3}\frac{\partial p_5}{\partial p_i}\dot p_i\\ \sum_{i=1}^{3}\frac{\partial p_7}{\partial p_i}\dot p_i & \sum_{i=1}^{3}\frac{\partial p_8}{\partial p_i}\dot p_i\end{pmatrix},$$

where the notation $\partial\omega_k/\partial P_1\dot P_1,\ k=1,2,$ is interpreted as

$$\frac{\partial\omega_k}{\partial P_1}\dot P_1=\sum_{i=1}^{3}\frac{\partial\omega_k}{\partial p_i}\dot p_i.$$

Note that, after the change of variables (29), the matrix P_3 becomes

$$\begin{aligned}P_3=&\begin{pmatrix} I_2 & 0\\ 0 & I_1\end{pmatrix}\omega_6+\begin{pmatrix} I_2 & \sqrt{I_1I_2}\\ \sqrt{I_1I_2} & -I_1\end{pmatrix}\omega_9+\\ &\begin{pmatrix} I_2 & -\sqrt{I_1I_2}\\ -\sqrt{I_1I_2} & -I_1\end{pmatrix}\omega_{10}.\end{aligned}$$

Hence the equations from which the slow integral manifold (33) is calculated are:

$$\begin{aligned}&\varepsilon\frac{\partial P_2}{\partial P_1}\dot P_1+\varepsilon\frac{\partial P_2}{\partial\omega_6}\dot\omega_6=\\ &\varepsilon P_3-\varepsilon P_1N^T-P_2G_0^T-\varepsilon P_2G_1^T-\varepsilon P_1SP_2,\end{aligned} \tag{34}$$

$$\varepsilon\frac{\partial\omega_9}{\partial P_1}\dot P_1+\varepsilon\frac{\partial\omega_9}{\partial\omega_6}\dot\omega_6=\frac{2}{\sqrt{I_1I_2}}\omega_{10}+O(\varepsilon), \tag{35}$$

$$\varepsilon\frac{\partial\omega_{10}}{\partial P_1}\dot P_1+\varepsilon\frac{\partial\omega_{10}}{\partial\omega_6}\dot\omega_6=-\frac{2}{\sqrt{I_1I_2}}\omega_9+O(\varepsilon). \tag{36}$$

Here the expressions for $\dot P_1$ and $\dot\omega_6$ should be substituted into (34)–(36) from (24) and(29). From (31) and (32) it immediately follows that

$$\omega_9^{(0)}=\omega_{10}^{(0)}=0.$$

The terms $\omega_9^{(1)}$ and $\omega_{10}^{(1)}$ satisfy the equations

$$\begin{aligned}&\left(-\frac{n}{2I_1}+\frac{E}{2I_2}\right)\omega_6+\frac{2}{\sqrt{I_1I_2}}\omega_{10}^{(1)}+\frac{b^2}{4I_1^2I_2}q=0,\\ &\left(-\frac{n}{2I_1}+\frac{E}{2I_2}\right)\omega_6-\frac{2}{\sqrt{I_1I_2}}\omega_9^{(1)}+\frac{b^2}{4I_1^2I_2}q=0.\end{aligned} \tag{37}$$

Now, equation (34) yields $P_2^{(0)}=0$ and

$$P_2^{(1)}=\begin{pmatrix} p_2\kappa & I_1I_2\omega_6-p_1lp-p_2M\\ p_3\kappa-I_1I_2\omega_6 & -p_2lp-p_3M\end{pmatrix}.$$

The next approximation ${P_2}^{(2)}$ is

$${P_2}^{(2)}=\begin{pmatrix} p_4^{(2)} & p_7^{(2)}\\ p_5^{(2)} & p_8^{(2)}\end{pmatrix},$$

where

$$\begin{aligned}p_4^{(2)}&=-0.5(nI_2^2+EI_1I_2+I_1I_2^2p_1)\omega_6-E(p_1lp+p_2M)\\ &\quad+I_2b^2\frac{q}{2I_1}+\frac{1}{r}(I_2lpp_2^2+I_2Mp_2p_3+I_2\kappa p_1p_3),\\ p_7^{(2)}&=-p_2\kappa n+\frac{I_1^2I_2}{r}p_2\omega_6,\\ p_5^{(2)}&=(p_2lp+p_3M)\left(\frac{I_2(p_1+p_3)}{r}-E\right),\\ p_8^{(2)}&=I_1^2I_2\left(-\frac{3}{2}\frac{n}{I_1}\omega_6-\frac{E}{2I_2}\omega_6\right)+\frac{3}{4}b^2q\\ &\quad-p_3\kappa n+\frac{I_1}{r}\left(\kappa p_3^2+p_2^2lp+p_2p_3M\right).\end{aligned} \tag{38}$$

The approximations derived above permit to follow how equation (23), derived on the basis of precessional equations, differs from the equations which describe the flow along the attracting integral manifold of the system (24)–(26).

Consider the system describing the flow on the slow integral manifold of (24)–(26). According to the results of the theory of integral manifolds, this flow is determined by the regular equations of this system, namely, by the equations (24) and (29). To derive the equations of this flow one should substitute the asymptotic expansions (32) into the right-hand sides of these equations:

$$
\begin{aligned}
\dot{p}_1 &= 2\frac{\kappa}{H}p_2 - \frac{p_2^2}{r} + \frac{I_2 b^2 q}{I_1 H^2} \\
&\quad - (nI_2^2 + EI_1 I_2^2 p_1)\omega_6/H^2 + O(1/H^3), \\
\dot{p}_2 &= -\frac{lp}{H}p_1 - \frac{M}{H}p_2 + \frac{\kappa}{H}p_3 - \frac{p_2 p_3}{r} + O(1/H^3), \\
\dot{p}_3 &= -2\frac{lp}{H}p_2 - 2\frac{M}{H}p_3 - \frac{d_3^2}{r} + \frac{qb^2}{H^2} \\
&\quad + \frac{2}{H^2}\left(I_1^2 I_2\left(-\frac{3n}{2I_1}\omega_6 - \frac{E}{2I_2}\omega_6 + I_1 I_2 n\omega_6\right) + \frac{3}{4}b^2 q\right) \\
&\quad + O(1/H^3).
\end{aligned} \tag{39}
$$

The calculations may be carried to any desired accuracy. We compare the results obtained in this Example for the full equations of motion, and those got on the basis of the precessional equations. The 4×4 correlation matrix $P(t, \mathrm{e})$, corresponding to the full equations, is calculated from (25)–(27). This system has a 4 dimensional stable integral manifold of slow motions. So, every solution of this system, starting in the vicinity of this manifold, tends to it as $t \to \infty$. Consequently, to investigate the trajectories of (25)–(27) it is enough to follow the trajectories lying on the manifold. Equations (25) and (30) describe this motion. The solutions of (25), (30) and of (23) were compared numerically. Both systems were solved numerically with the same initial conditions (Dormand-Prince method), and the solutions obtained differed in the $O(\varepsilon^3)$ terms. By this is meant that the use of the 4 dimensional stable integral manifold of slow motions gives an accurate account of the behavior of the original system, whereas the use of precessional equations, instead of the original ones, for calculating the filtering error may lead to an intolerable error if the motion is performed under the action of random forces of Gaussian white noise type.

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