ON SEPARABILITY OF SEMIGROUPS: APPLICATION TO MODELS OF PHYSICS AND DISCRETE OPTIMIZATION

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Abstract

Nilpotent semigroups are used in various fields of physics to model processes like system decay in quantum mechanics, state transitions in statistical mechanics, and simplifying dynamical systems. They also assist in optimizing control processes. The set of nilpotent matrices forms a semigroup and plays a key role in Jordan decomposition, which has many physical applications. In quantum mechanics, the Jordan form helps analyze linear operators on Hilbert spaces, especially with systems involving eigenvalue multiplicities. In vibration analysis, it aids in studying normal modes of mechanical systems with degenerate frequencies. Additionally, Jordan decomposition simplifies control theory for linear time-invariant systems, stability analysis in dynamical systems, and the behavior of coupled oscillators, as well as solving systems of linear differential equations.

Certainly, the semigroup theory has even more applications in theoretical computer science: suffice it to say, for example, that some authors identify semigroups and finite automata.

In this paper, we address the problem of P-separability of semigroups with respect to three key predicates: equality-separability, divisibility-separability, and subsemigroup-separability, achieved through homomorphisms into a nilpotent semigroup. We present some significant results that advance the understanding of these concepts.

Key words

Semigroup, separability, nilpotent semigroup, approximation, quantum mechanics, dynamical systems.

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1 Introduction

In this paper, we study application of special properties of semigroups in some areas of physics, in particular application of so called P-separability of semigroups in quantum mechanics. At the same time, we should immediately note that this property can be applied in other areas of physics, as well as in the theoretical cybernetics, in particular in the theory of finite automata, where some authors even identify semigroups and deterministic finite automata without inputs and outputs.

The theory of approximations of algebraic structures was first introduced in the research work of Academician A. I. Maltsev [Maltsev, 1958]. In this work, he established the connection between the finite approximation of an algebraic structure, corresponding to a given predicate, and the solvability of that predicate. This can be seen as the first example of applying the theory of approximations to algebraic structures. Although this area has practical applications, it has received the limited attention in research, except the papers [Kublanovski, 2000; Kostyrev, 2010].

M. M. Leshokhin and his followers have actively studied the problem of semigroup approximation from the 1960s to the present. Several results have been published in this research direction, mostly in Russian and in specialized journals of the Soviet Union and Russia. Some papers are also published in [Hall, 1973; Lesokhin and Popyrin, 1986; Lesokhin and Rasulov, 1991].

The semigroup approximation problem comprises three main components.

I) Classes of semigroups. This includes:

- commutative, cancellative semigroups;
- commutative, cancellative, periodic semigroups;
- commutative, regular, and periodic semigroups;

- inverse, regular semigroups;
- semigroups that satisfy the following property: if the N_e class contains an idempotent, then it is a commutative group;
- commutative semigroups characterized by the following property: if an Archimedean component [Clifford and Preston, 1961] contains an idempotent, then it is a subgroup;
- etc.

II) Set of predicates. This includes some important predicates in semigroup theory such as:

- the equality of two elements;
- an element belonging to a subsemigroup;
- an element belonging to an ideal;
- divisibility of two elements;
- an element belonging to a maximal subgroup;
- each item of Green's relations, [Howie, 1995; Schein, 1999] etc.;
- an element belonging to a simple subgroup;
- etc.

III) Set of mappings. This includes:

- characters;
- bicharacters;
- generalized characters;
- homomorphisms;
- continuous characters;
- etc.

The semigroup approximation problem is often stated as follows:

> *given a predicate and a set of mappings (such as semigroup homomorphisms), determine the necessary and sufficient conditions for a given semigroup to approximate with respect to the predicate through these mappings.*

A lot of predicates in semigroup theory have been analyzed, and the necessary and sufficient conditions have been established. The problem of *finding the smallest approximating subsemigroup* for a class of semigroups corresponding to different predicates was posed by M. M. Leshokhin, and several results have been published in [Lesokhin and Rasulov, 1991; Dang et al., 2017b; Dang et al., 2018; Vinh et al., 2021; Dang et al., 2023a; Dang et al., 2023b] etc.

Recently, a new research group introduced alternative terms for approximation concepts [Miller et al., 2021; O'Reilly et al., 2022]. For example:

- a semigroup approximated by the predicate "equality of two elements" is defined as "a residual finite semigroup";
- the predicate "belonging of an element to a finitely generated subsemigroup" is termed "weakly subsemigroup separable";
- the predicate "belonging of an element to a subsemigroup" is called as "strongly subsemigroup separable";
- the predicate "belonging of an element to a subset" is referred to as "a completely separable semigroup";
- etc.

See also [East et al., 2024] and other works available from the links cited.

Our research builds on the existing work to explore P-separability in semigroups, where P represents different predicates. We do not limit ourselves to finite semigroups but include a broader range, for example infinite ones. Specifically, we focus on identifying a class K of semigroups (including infinite ones) such that a given class S of semigroups can be made P-separable through mappings, such as homomorphisms, characters, generalized characters, and others, from S to K. We also consider the minimization of P-separability.

As we already said, we study application of P-separability of semigroups in this paper. Here are some possible areas of practical application of these results, although, of course, it is not possible to mention all such areas.

- They appear in perturbation theory for approximations in finite-dimensional models, see [Bogolyubov, 1967; Maslov et al., 1972; Nayfeh, 1981; Zheng and Zhang, 2017] etc.
- In supersymmetry (in particular, when developing the idea of our erroneous understanding of gravity[1]), nilpotent operators are the part of the algebraic structure connecting different types of particles; see [Guillemin et al., 1999; Hikasa, 2024; Yesiltas, 2024] etc.
- Additionally, in gauge theories and classical mechanics, they help describe constraints and effective field interactions; see [Altarelli and Wells, 2017] etc.
- Overall, nilpotent operators are crucial for modeling and understanding physical systems.

We can say for all these areas, that we study P-separability using a special type of mapping, exactly homomorphisms into a semigroup of nilpotent operators. Such operators are used in physics to simplify complex systems and describe various phenomena. For example, in quantum mechanics, they include ladder operators that

[1] Let us explain this item in a little more detail, based on information from several scientific and popular science sources. The nonzero Higgs field has a size of about 250 GeV, which gives a mass of particles W and Z equal to about 100 GeV. But it follows from quantum mechanics that such a size of the Higgs field is unstable. It follows from the so-called "quantum jitter" that there must be two natural values for the Higgs field. It turns out that the Higgs field must be either zero or comparable in size to the Planck energy, 10^{16} times the observed value. Why does its value turn out to be non-zero and so small? To study such contradictions, physicists apply the theory of separability of semigroups, and, in particular, an analogue of the semigroup approximation we are considering.

change states and Jordan blocks for systems with degenerate energy levels, see [O'Reilly et al., 2022].

We have already briefly discussed the possible application of the obtained results in some areas of theoretical cybernetics, namely in the theory of finite automata, where some authors even identify semigroups and deterministic finite automata without inputs and outputs, see [Skornyakov, 1991] etc. According to the authors of this paper, among the areas of theoretical cybernetics, the described results related to the approximation of semigroups should be applied precisely in the theory of finite automata[2]; for more information, see also our previous publications [Melnikov and Melnikova, 2001; Melnikov, 2006; Melnikov, 2010]

Here are the contents of the paper by sections.

Section 2 is preliminaries, we consider some definitions given in previous publications and necessary for further presentation of the results of the current paper.

Section 3 presents the main algebraic results of this paper.

In Section 4, we consider some examples. In particular, a simple example of a specific application of one of the results of the paper in quantum mechanics will be considered.

In Conclusion (Section 5), we summarize the results of the paper, i.e. we formulate them once again. In addition, we shall give some more areas of possible application of the results of the paper on P-separability in physics models and discrete optimization problems.

2 Preliminaries

In this section, we consider some definitions given in previous publications and necessary for further presentation of the results of the current paper.

We define P-separability of an algebraic structure corresponding to a predicate P. When the predicate P changes, we have a problem of separability of the algebraic system with respect to this predicate P. Definitions such as residual finite semigroup, weakly subsemigroup separable, strongly subsemigroup separable, completely separable, etc., are specific cases of P-separability.

In the following definitions,

- let A and B be two algebraic structures as the same type;
- let Φ be the set of all mappings between A and B;
- let P be a predicate defined on the set that consists of the following objects:
  - A,
  - all subsets $\delta(A)$ of A,
  - and all images of A and $\delta(A)$ under the mappings from Φ.

Definition 1. *An algebraic structure A is said to be P-separable by mappings from Φ with respect to P, if for a pair of subsets A_1, A_2 from A such that $P(A_1, A_2)$ is false, there exists $\varphi \in \Phi$ such that $P(\varphi(A_1), \varphi(A_2))$ is also false.*

Definition 2. *A semigroup A with zero element 0_A is said to be a nilpotent semigroup if for any $a \in A$ there exists a natural number n such that $a^n = 0_A$.*

The smallest natural number n with this property for the semigroup A is called the step (sometimes class) of nilpotency.

Definition 3. *Denote A^n be the set of all elements from A that can be decomposed into the product of n factors.*

Definition 4. *Let A be a semigroup and B be a nilpotent semigroup. Denote Θ as the set of all homomorphisms from the semigroup A into the nilpotent semigroup B.*

Definition 5. *[1], page 145. Let Γ be some nonempty collection of nonempty subsets of some set $\mathbb{M}$.*

A set $S \in \Gamma$ is said to be minimal in Γ if none of its proper subsets belong to Γ.

A set $S \in \Gamma$ is said to be universally minimal in Γ if it is a subset of every set belonging Γ.

3 Results

This section presents the main algebraic results of this paper.

Let P_1, P_2 and P_3 be predicates, where P_1 represents the equality of two elements, P_2 represents an element belonging to a subsemigroup, and P_3 represents a divisibility of two elements.

Theorem 1. *A semigroup A is P_1-separable by a homomorphism from Θ if and only if the set*

$$J = \cap_{n=1}^{\infty} A^n$$

- *is either empty,*
- *or contains only the zero element of the semigroup A.*

Proof. 1) Necessity.
Suppose that A is a P_1-separable. Let

$$a_0 \in J = \cap_{n=1}^{\infty} A^n.$$

[2] Just as in the previous footnote, where we indicated in a little more detail the possible application of the obtained results in physics, here we shall explain their possible application in theoretical cybernetics. Indeed, the minimal approximation semigroup and minimal nondeterministic finite automaton is very related to the minimality of the finite language for which the following iteration property holds:

$$(\forall u \in A^*)\,(\exists v \in B^*)\,(u \in \operatorname{pref} v),$$

and vice versa, with the exchange of A and B. Exactly, we choose the minimum language by some criterion among all finite languages satisfying this property for the given language A.

The last property is important, among other things, for practical issues: for example, in [Melnikov and Kashlakova, 2000] we showed a connection between possible equivalence checking algorithms in subclasses of the class of context-free languages, and such algorithms are used in compiler automation systems.

Then we show that a_0 must be the left zero of the semigroup A.

Assume that a_0 is not the left zero of A. Then there exists an element $a \in A$ such that

$$a_0 a \neq a_0.$$

By the condition of P_1-separability of A, there is a homomorphism φ from the semigroup A into the semigroup $\varphi(A)$ with the nilpotency k and zero element O, such that

$$\varphi(a_0 a) \neq \varphi(a_0).$$

Since $a_0 \in J$, there exist

$$a_1, a_2, ..., a_k \in A \quad (\text{where } k \in \mathbb{N})$$

such that $a_0 = a_1 a_2 ... a_k$. Thus, we have:

$$\varphi(a_0 a) = \varphi(a_0)\varphi(a) = \varphi(a_1 a_2 ... a_k)\varphi(a) = \\ \varphi(a_1)\varphi(a_2)...\varphi(a_k)\varphi(a) = O\varphi(a) = O.$$

On the other hand,

$$\varphi(a_0) = \varphi(a_1)\varphi(a_2)...\varphi(a_k) = O.$$

Thus,

$$\varphi(a_0 a) = O = \varphi(a_0).$$

This is a contradiction.

Therefore, a_0 must be the left zero of the semigroup A. Similarly, we can show that a_0 is also the right zero of the semigroup A. Thus, a_0 is the zero of the semigroup A.

2) Sufficiency.

Let $a, b \in A, a \neq b$. There are two possibilities: either $a \notin J$ or $b \notin J$.

Without the loss of generality, assume that $a \notin J$. Since

$$J = \cap_{n=1}^{\infty} A^n$$

is either empty or contains only the zero element of the semigroup A, there exists $n \in \mathbb{N}$ such that $a \notin A^n$.

Because of the fact that A^n is an ideal and

$$A/_{A^n}$$

is a nilpotent semigroup, there exists a natural homomorphism φ from A into $A/_{A^n}$ such that $\varphi(a) \neq \varphi(b)$. □

Theorem 2. *A semigroup A is P_2-separable by a homomorphism from Θ if and only if*

- *either A is a nilpotent semigroup with a zero element O_A such that*

$$J = \cap_{n=1}^{\infty} A^n = \{O_A\},$$

- *or A has no zero element and the set J is empty.*

Proof. 1) Necessity.

Case 1. The semigroup A contains a zero element O_A.

Assume that A is not periodic semigroup. Hence there exists an element $a \in A$ such that $[a]$ is an infinite subsemigroup. Clearly that $O_A \notin [a]$.

By the condition of P_2-separability of the semigroup A, there exists a homomorphism $\varphi \in \Theta$ from A into a nilpotent semigroup B with the nilpotency k and a zero element O_B such that

$$\varphi([O_A]) \notin \varphi([a]).$$

Since

$$O_B = \varphi(a^k) \in \varphi([a]),$$

then

$$\varphi(O_A) = O_B \in \varphi([a]).$$

It contradicts the assumption. Thus, the semigroup A is periodic.

The semigroup A is a P_2-separable, then A is an P_1-separable by homomorphisms from Θ. Follow by the theorem 1,

$$J = \cap_{n=1}^{\infty} A^n = O_A.$$

Because of every idempotent of the semigroup A belongs to J and J contains only zero element, then the semigroup A is a nilpotent semigroup.

Case 2. The semigroup A has no zero element. Follow by the theorem 1,

$$J = \cap_{n=1}^{\infty} A^n$$

is a empty set.

For both cases, the necessity is proved.

2) Sufficiency.

Case 1. The semigroup A contains a zero element O_A. Let a be an element of A and A' be a subsemigroup of A such that $a \notin A'$. Since A' is a subsemigroup, $O_A \in A'$, hence $a \neq O_A$.

Because of

$$J = \cap_{n=1}^{\infty} A^n = O_A,$$

there exists a natural number n, such that $a \notin A^n$.

Let φ be the homomorphism from A into the quotient semigroup $A/_{A^n}$, we obtain $\varphi(a) \neq \varphi(O_A)$. It follows that

$$\varphi(a) \notin \varphi(A').$$

Case 2. The semigroup A does not contain a zero element. Suppose that $a \notin A'$. There exists a natural number n such that $a \notin A^n$.

The natural homomorphism φ from A into the quotient semigroup $A/_{A^n}$ separates the image of the element a from the image of the subsemigroup A', meaning that $\varphi(a) \notin \varphi(A')$. □

Theorem 3. *A semigroup A is P_3-separable by a homomorphism from Θ if and only if*

- *either the set*

$$J = \cap_{n=1}^{\infty} A^n$$

is empty,
- *or the set J is a universally minimal two-sided ideal of the semigroup A.*

Proof

1) Necessity.

Assume that $a \in J = \cap_{n=1}^{\infty} A^n$ and a is not divisible by an element $b \in A$ on the right. By the condition of P_3-separability of the semigroup A, there is a homomorphism $\varphi \in \Theta$ from A into a nilpotent semigroup B with the nilpotency k and a zero element O_B such that $\varphi(a)$ is not divisible by $\varphi(b)$ to the right. Since $a \in J$, there exist elements

$$a_1, a_2, ..., a_k \in A \quad \text{such that} \quad a = a_1 \cdot a_2 \cdots a_k.$$

Hence,

$$\varphi(a) = \varphi(a_1 \cdot a_2 \cdots a_k) =$$
$$\varphi(a_1) \cdot \varphi(a_2) \cdots \varphi(a_k) = O_B$$
$$\Rightarrow \quad \varphi(a) = O_B \cdot \varphi(b).$$

It contradicts the assumption. It follows that the element a is divisible to the right by any element of the semigroup A.

Similarly, we obtain that the element a is divisible to the left by any element of the semigroup A.

Because of elements of the set J are divisible by any element of the semigroup A to the left and to the right, the set J belongs to the minimal ideal A_e of the semigroup A. Since the set J is also an ideal of A, then $J = A_e$.

According to [1, p. 158], a minimal two-sided ideal of a semigroup is always a universally minimal two-sided ideal. Therefore, the set J is a universally minimal two-sided ideal of the semigroup A.

2) Sufficiency. Let a and b be two elements of the semigroup A such that a is not divisible by b on the right. Then $a \notin J$. Hence there exists a natural number n such that $a \notin A^n$. Let us consider the natural homomorphism φ from A into ${}^{A}\!/\!_{A^n}$ with a zero element $\mathbf{O}$. Because of $a \notin A^n$, we obtain that $\varphi(a) \neq \mathbf{O}$.

Assume that $\varphi(a) = \varphi(c)\varphi(b) = \varphi(cb)$ for some $\varphi(x) \in \varphi(A)$.

If $xb \in A^n$, then $\varphi(a) = \mathbf{O}$. What is impossible. Hence, $xb \notin A^n$ and $a = xb$. It contradicts the assumption. Thus $\varphi(a)$ is not divisible by $\varphi(b)$ on the right. Similarly, we can prove for the case of divisibility on the left.

Some results of the problem of P-separability of semigroups using homomorphisms into a nilpotent semigroup corresponding to various predicates are presented below.

Let P_4, P_5 and P_6 be predicates, where P_4 represents an element belonging to a monogenic subsemigroup, P_5 represents an element belonging to a maximal subsemigroup, and P_6 represents an element belonging to an ideal.

Proposition 1. *The following statements hold:*

- *If the semigroup A is P_6-separable, then A is P_3-separable.*
- *If the semigroup A is P_5-separable, then A is P_3-separable.*
- *If the semigroup A is P_4-separable, then A is P_1-separable.*

4 Some examples

Example 1. Let $A = \{1; 2; ...,\}$ with the operation of addition. Then A is a semigroup and

$$J = \cap_{n=1}^{\infty} A^n$$

is empty.

Let m and k be two distinct elements of A such that $m < k$. Then $m \notin A^{m+1}$ but $k \in A^{k+1}$. The quotient semigroup ${}^{A}\!/\!_{A^{m+1}}$ has exactly $m+1$ distinct elements (classes).

Let us consider the natural homomorphism

$$\varphi : A \longrightarrow {}^{A}\!/\!_{A^{m+1}}.$$

$\varphi(m) = [m]$, $\varphi(k) = [m+1]$. Therefore $\varphi(m) \neq \varphi(k)$.

Example 2. Annihilation operators, used in quantum mechanics, reduce the number of particles in a given state. They are essential in describing particle interactions, state transitions, and quantum fields, particularly in harmonic oscillators and quantum field theory. In this context, we study the problem of P- separability in the semigroup A of nilpotent operators. Let

$$A = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in \mathbb{R} \right\}$$

be the semigroup of all annihilation operators in 3-dimensional space, including the zero operator (all strictly upper triangular real matrices of order 3) with the operation of multiplication of two matrices.

Directly calculation shows that A is a nilpotent semigroup with the nilpotency is 3 and

$$J = \cap_{n=1}^{\infty} A^n$$

contains only zero matrix.

Let us consider two elements

$$a = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $a \neq b$ and $a \notin J$.

There exists a natural number $n = 2$ such that $a \notin A^2$. Then let us consider the natural homomorphism

$$\varphi : A \longrightarrow {}^{A}\!/\!_{A^2}.$$

Since $a \neq b$, then $\varphi(a) \neq \varphi(b)$.

Example 3. Let p, q be two distinct prime numbers, and let

$$A = \{p^m q^n \mid m, n \in \mathbb{N}, m, n \neq 0\}$$

with the operation of multiplication.

Direct calculation shows that:

- A is a commutative semigroup;
- $(\forall k \in \mathbb{N})\ (A^k = \{p^m q^n \mid m + n \geq k\})$;
- $J = \cap_{n=1}^{\infty} A^n$ is empty.

Let a, b be two distinct elements of A. Since

$$J = \cap_{n=1}^{\infty} A^n$$

is empty, then $a \notin J$, and there exists $n \in \mathbb{N}$ such that $a \notin A^n$.

Because of the fact that A^n is an ideal and

$${}^{A}\!/\!_{A^n}$$

is a nilpotent semigroup, there exists a natural homomorphism φ from A into ${}^{A}\!/\!_{A^n}$ such that $\varphi(a) \neq \varphi(b)$.

If we add zero element to the semigroup A, then A becomes a semigroup with zero, and the set J contains only the zero element.

Example 4. (This example is based on the material of the monograph [Clifford and Preston, 1961].)

- Let P be the predicate the occurrence of an element in the given subsemigroup B of the semigroup A.
- Let K be the class of all semilattices (i.e., idempotent commutative semigroups).

Then let us consider two following semigroups of the defined class:

- 2-element chain S_2, let (a, b) below,
- and 3-element chain S_3, let (x, y, z) below.

S_2 is not a minimal approximation semigroup for P-separability, because of the following. In the semigroup S_3, the predicate $P(y, B = \{x, z\})$ is false, and therefore we cannot construct a homomorphism $\phi : S_3 \to S_2$ such that $P(\phi(y), \phi(B))$ is false.

On the other hand, S_3 is the minimal approximation semigroup for P-separability.

Example 5. Let

$$T = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \middle| a, b, c \in \mathbb{F}_2 \right\}$$

be the semigroup consisting of all annihilation operators in 3-dimensional space, including the zero operator (i.e., all strictly upper triangular matrices of order 3 over $\mathbb{F}_2$), with the operation of multiplication of two matrices.

Direct calculations show that T is a nilpotent semigroup with nilpotency index 3, and

$$J = \bigcap_{n=1}^{\infty} T^n$$

contains the zero matrix only. Then the semigroup T has 8 following elements only:

$$t_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$$

$$t_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$$

$$t_7 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Below is the Cayley table for the semigroup T.

$\cdot$	t_1	t_2	t_3	t_4	t_5	t_6	t_7	O
t_1	O	t_5	t_5	O	O	t_5	t_5	O
t_2	O	t_5	t_5	O	O	t_5	t_5	O
t_3	O	O	O	O	O	O	O	O
t_4	O	t_5	t_5	O	O	t_5	t_5	O
t_5	O	O	O	O	O	O	O	O
t_6	O	O	O	O	O	O	O	O
t_7	O	t_5	t_5	O	O	t_5	t_5	O
O	O	O	O	O	O	O	O	O

Table 1. Cayley table for the semigroup T

From the Cayley table, we obtain:

- T is a nilpotent semigroup with a nilpotency index of 3, and the zero element is O;
- $T^2 = \{t_5, O\}$, $T^3 = \{O\}$;
- $J = \bigcap_{n=1}^{\infty} T^n = \{O\}$.

Let us consider the subsemigroup $T_1 = \{t_1, t_5, O\}$ and the element $t_2 \in T$. Obviously, $t_2 \notin T_1$. By Theorem 2, there exists a natural number 2 such that $t_2 \notin T^2$. The natural homomorphism φ from T into the quotient semigroup ${}^{T}\!/\!_{T^2}$ separates $\varphi(t_2)$ from $\varphi(T_1)$.

5 Conclusion

Let us summarize the results of the paper.

We presented the necessary and sufficient conditions for the P-separability of semigroups, corresponding to three key predicates: equality of two elements, an element belonging to a subsemigroup, and the divisibility of two elements using homomorphisms into a nilpotent semigroup. Other important predicates in semigroup theory, such as an element belonging to a subgroup, an element belonging to an ideal, Green's relations, etc., are currently under investigation. In the future, we are going to study the problem of P-separability of semigroups using homomorphisms into important classes of semigroups. Additionally, the problem of finding a minimal semigroup for P-separability is also considered.

As we wrote in Introduction, we shall add a few more possible applications of our results that arise in the study of physical models. Most of these examples are related to the examples discussed above, in which the main role is played by nilpotent and annihilation operators and matrices (at first, Examples 2 and 5).

Firstly, let us note the article [Frydryszak, 2006]; the very title of the paper indicates its relation to our topic. It considers several variants of the occurrence of nilpotent elements, both at the quantum and classical levels. Among the models noted in the paper are the models arising from the theory of cohomology. No less interesting are various models of self-organization, which have applications both in quantum physics and far beyond; in these models, the nilpotent structure is also very important, see [Marcer and Rowlands, 2017] etc. Note that a lot of Internet resource are also interesting for the applications under consideration; see `https://www.physicsforums.com/threads/nilpotent-matrix-proof-problem.225161/` etc.

It is clear that in these examples, as in many other models related to nilpotent operators and matrices, the possibility of any approximation option and the study of separability, in particular the application of the relevant results of this work, makes it possible to consider simpler situations, simpler models, etc.

Now we present the corresponding models related to theoretical cybernetics, and note in advance that the noted simplification when applying the results on separability is possible for each of these models. Thus, let us firstly list the following subject areas for cybernetic applications of Nilpotent matrices:

- finite-time settling systems;
- deadbeat controllers;
- system observability;
- stability analysis in discrete-time systems

(we shall not provide specific links). It is clear that the possibility of applying the results considered in the paper is not limited to the listed subject areas. Secondly, let us list the following subject areas for cybernetic applications homomorphisms of semigroups:

- finite automata theory (we briefly mentioned this application above);
- signal processing and filtering;
- abstraction in system modeling;
- error-correcting codes;
- decomposition of complex systems;
- synchronization in distributed systems.

Thus, we have shown not so much the application of theoretical cybernetics issues in some areas of physics, as the application of various algebraic results related to the approximation of semigroups in these two sciences, that is, both in theoretical cybernetics and in physics. At the same time, according to the authors, the application of the results proved in the work will make it possible to simplify consideration in the above-mentioned areas related to both theoretical physics and theoretical cybernetics.

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