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GENERIC CONTROLLABILITY PROPERTIES FOR THE BILINEAR SCHRÖDINGER EQUATION

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Abstract

In a recent paper we proposed a set of sufficient conditions for the approximate controllability of a discrete-spectrum bilinear Schrödinger equation on a fixed domain. These conditions are expressed in terms of the controlled potential and of the eigenpairs of the uncontrolled Schrödinger operator. The aim of this presentation is to show that these conditions are generic with respect to the uncontrolled or the controlled potential. The results are obtained by analytic perturbation arguments and through the study of asymptotic properties of eigenfunctions.

Key words

Generic conditions, Controllability, Schrödinger equation

1 Introduction

In this paper we consider controlled Schrödinger equations of the type

$$i\frac{\partial\psi}{\partial t}(t,x) = (-\Delta + V(x) + u(t)W(x))\psi(t,x), \quad (1)$$

where $u(t) \in U$, $\psi : I \times \Omega \to \mathbf{C}$ for some $\Omega \subset \mathbf{R}^d$ open bounded, I is a subinterval of $\mathbf{R}$, $\psi|_{I\times\partial\Omega} = 0$. Here V, W are suitable real valued functions and U is a nonempty subset of $\mathbf{R}$.

As proved in (Turinici, 2000), the control system (1) is never exactly controllable in $L^2(\Omega)$. Nevertheless, several positive controllability results have been proved in recent years. Among them, let us mention the exact controllability among regular enough wave functions for $d = 1$ and $V = 0$ (Beauchard, 2005; Beauchard and Coron, 2006) and the recently obtained L^2-approximate controllability (Nersesyan, 2008). The result we will consider for the discussion below is the L^2-approximate controllability obtained by the authors in (Chambrion *et al.*, 2008).

The scope of this paper is to establish that the sufficient conditions for controllability proposed in (Chambrion *et al.*, 2008) are robust and frequent enough. The mathematical framework for this analysis is provided by the standard notion of genericity.

Let us mention that the genericity question for the Schrödinger equation is already addressed in (Nersesyan, 2008), where some partial results are given. In particular, genericity for the case $d = 1$ is essentially proven in (Nersesyan, 2008, Lemma 3.12). Further genericity results on the controllability of a linearized Schrödinger equation can be found in (Beauchard *et al.*, 2008) and are further discussed in Section 6.

2 Notations and definition of solutions

We denote by $\mathbf{N}$ the set of positive integers, by A^* the adjoint of an operator A. We fix $d \in \mathbf{N}$ to denote the dimension of the space in which the Schrödinger equation is considered. We denote by Ξ the set of nonempty, open and bounded subsets of $\mathbf{R}^d$.

In the following we consider Equation (1) assuming that the potentials V, W are taken in $L^\infty(\Omega, \mathbf{R})$. Then, for every $u \in U$, $-\Delta + V + uW : H^2(\Omega, \mathbf{R}) \cap H_0^1(\Omega, \mathbf{R}) \to L^2(\Omega, \mathbf{C})$ is a skew-adjoint operator on $L^2(\Omega, \mathbf{C})$ with discrete spectrum. (See (Friedrichs, 1934).) In particular, $-\Delta + V + uW$ generates a group of unitary transformations $e^{it(-\Delta+V+uW)} : L^2(\Omega) \to$

$L^2(\Omega)$. Therefore, $e^{it(-\Delta+V+uW)}(\mathcal{S}) = \mathcal{S}$ where $\mathcal{S}$ denotes the unit sphere of $L^2(\Omega)$.

For every $u \in L^\infty([0,T],U)$ and every $\psi_0 \in L^2(\Omega)$ there exists a unique weak (and mild) solution $\psi(\cdot;\psi_0,u) \in \mathcal{C}([0,T],\mathcal{H})$. Moreover, if $\psi_0 \in D(A)$ and $u \in \mathcal{C}^1([0,T],U)$ then $\psi(\cdot;\psi_0,u)$ is differentiable and it is a strong solution of (1). (See (Ball *et al.*, 1982) and references therein.)

Definition 2.1. *We say that the quadruple (Ω,V,W,U) is approximately controllable if for every $\psi_0,\psi_1 \in \mathcal{S}$ and every $\varepsilon > 0$ there exist $T > 0$ and $u \in L^\infty([0,T],U)$ such that $\|\psi_1 - \psi(T;\psi_0,u)\| < \varepsilon$.*

It is useful for the applications to extend the notion of approximate controllability from a single Schrödinger equation to a (possibly infinite) family of identical systems with different initial conditions, through the study of the evolution of the associated density matrix (see (Albertini and D'Alessandro, 2003)).

Let $(\varphi_j)_{j\in\mathbf{N}}$ be an orthonormal basis of $L^2(\Omega)$, $(P_j)_{j\in N}$ be a sequence of non-negative numbers such that $\sum_{j=1}^\infty P_j = 1$, and denote by ρ the *density matrix*

$$\rho = \sum_{j=1}^{\infty} P_j \varphi_j {\varphi_j}^*,$$

where $\psi^*(\cdot) = \langle\psi,\cdot\rangle$, for $\psi \in L^2(\Omega)$. In accord with the classical definition of density matrix, ρ is a non-negative, self-adjoint operator of trace class (see (Reed and Simon, 1978, Vol. I)). If each $\varphi_j = \varphi_j(t)$ is interpreted as the state of a Schrödinger equation of the form (1), each equation being characterized by the same potentials V and W and driven by the same control $u = u(t)$, then the time evolution of the density matrix $\rho = \rho(t)$ is described by

$$\begin{aligned}\rho(t) &= \mathbf{U}(t)\rho(0)\mathbf{U}^*(t)\\ &= \sum_{j=1}^{\infty} P_j \mathbf{U}(t)\varphi_j(0)(\mathbf{U}(t)\varphi_j(0))^* \end{aligned} \tag{2}$$

where the operator $\mathbf{U}(t)$ is defined by

$$\mathbf{U}(t)\psi_0 = \psi(t;\psi_0,u). \tag{3}$$

Definition 2.2. *Two density matrices ρ_0 and ρ_1 are said to be unitarily equivalent if there exists a unitary transformation $\mathbf{U}$ of $\mathcal{H}$ such that $\rho_1 = \mathbf{U}\rho_0\mathbf{U}^*$.*

For closed systems the problem of connecting two density matrices by a feasible trajectory makes sense only for pairs of density matrices that are unitarily equivalent. (The situation is different for open systems, see for instance (Altafini, 2003).)

Definition 2.3. *We say that the quadruple (Ω,V,W,U) is approximately controllable in the sense of its density matrices if for every pair ρ_0,ρ_1 of unitarily equivalent density matrices and every $\varepsilon > 0$ there exist $T > 0$ and $u \in L^\infty([0,T],U)$ such that $\|\rho_1 - \mathbf{U}(T)\rho_0\mathbf{U}(T)^*\| < \varepsilon$, where $\|\cdot\|$ denotes the operator norm on $\mathcal{H}$ and $\mathbf{U}$ is defined as in (3).*

It is clear that approximate controllability in the sense of its density matrices implies approximate controllability (just take $P_1 = 1$).

In order to state the approximate controllability result obtained in (Chambrion *et al.*, 2008), we need to recall the following two definitions.

Definition 2.4. *The elements of a sequence $(\mu_n)_{n\in\mathbf{N}} \subset \mathbf{R}$ are said to be $\mathbf{Q}$-linearly independent (equivalently, the sequence is said to be non-resonant) if for every $N \in \mathbf{N}$ and $(q_1,\dots,q_N) \in \mathbf{Q}^N \setminus \{0\}$ one has $\sum_{n=1}^N q_n\mu_n \neq 0$.*

Definition 2.5. *A $n\times n$ matrix $C = (c_{jk})_{1\le j,k\le n}$ is said to be connected if for every pair of indices $j,k \in \{1,\dots,n\}$ there exists a finite sequence $r_1,\dots,r_l \in \{1,\dots,n\}$ such that $c_{jr_1}c_{r_1r_2}\cdots c_{r_{l-1}r_l}c_{r_lk} \neq 0$.*

In the following we denote by $\sigma(V,\Omega) = (\lambda_j(V,\Omega))_{j\in\mathbf{N}}$ the non-decreasing sequence of eigenvalues of $-\Delta + V$ (on $H^2(\Omega,\mathbf{R}) \cap H_0^1(\Omega,\mathbf{R})$), counted according to their multiplicity and by $(\phi_j(V,\Omega))_{j\in\mathbf{N}}$ the corresponding sequence of eigenfunctions (unique up to the sign if the corresponding eigenvalue is simple). In particular $(\phi_j(V,\Omega))_{j\in\mathbf{N}}$ forms an orthonormal basis of $L^2(\Omega,\mathbf{C})$.

The theorem below recalls the controllability results obtained by the authors in (Chambrion *et al.*, 2008, Theorems 3.4, 5.2).

Theorem 2.6. *Let $\Omega \in \Xi$, V,W belong to $L^\infty(\Omega,\mathbf{R})$, and U contain the interval $(0,\delta)$ for some $\delta > 0$. Assume that the elements of $\big(\lambda_{k+1}(V,\Omega) - \lambda_k(V,\Omega)\big)_{k\in\mathbf{N}}$ are $\mathbf{Q}$-linearly independent and that for infinitely many $n \in \mathbf{N}$ the matrix*

$$B^{(n)}(\Omega,V,W) := \left(\int_\Omega W(x)\phi_j(V,\Omega)\phi_k(V,\Omega)\,dx\right)_{j,k=1}^n$$

is connected (i.e., $B^{(n)}(\Omega,V,W)$ is frequently connected). Then (Ω,V,W,U) is approximately controllable in the sense of its density matrices.

Remark 2.7. *In (Chambrion* et al.*, 2008) the case Ω unbounded is also considered. The potentials V and W are allowed to be unbounded as well, and Theorem 2.6 still holds, though the notion of solution of (1) gets more delicate. In this presentation we restrict our attention to the bounded case, although many of the results presented below admit suitable counterparts in the unbounded setting.*

We say that (Ω,V,W) is *fit for control* if $-\Delta + V$ is non-resonant and $B^{(n)}(\Omega,V,W)$ is frequently connected.

We say that the quadruple (Ω, V, W, U) is *effective* if $(\Omega, V + uW, W)$ is fit for control for some u in the interior of U, denoted by $\mathrm{int}\,(U)$. Theorem 2.6 states that being effective is a sufficient condition for controllability in the sense of the density matrices.

Let us recall some useful perturbation result describing the dependence on V of the spectrum of the operator $-\Delta + V$.

The first result recalls some well-know continuity properties. (See, for instance, (Henrot, 2006).)

Theorem 2.8. *Assume that $\Omega \in \Xi$, $V \in L^\infty(\Omega)$ and that the eigenvalue $\lambda_k(V)$ of the Schrödinger operator $-\Delta+V$ is simple. Then $\lambda_k(V+W,\Omega)$ depends continuously on $W \in L^\infty(\Omega)$ on a neighborhood of $W = 0$ and, analogously, the map from $L^\infty(\Omega)$ to $L^2(\Omega)$ that associates to W the corresponding k-th eigenvector of $-\Delta+V+W$ (up to the sign) is continuous on a neighborhood of $W = 0$.*

The second result concerns analytic perturbation properties. (See (Kato, 1966, Chapter VII), (Rellich, 1969, Chapter II).)

Theorem 2.9. *Let U be an open interval containing zero. Assume that $\Omega \in \Xi$, $V \in L^\infty(\Omega)$ and $\mu \mapsto W_\mu$ is an analytic function from U into $L^\infty(\Omega)$. Then, there exist two families of analytic functions $(\Lambda_k : U \to \mathbf{C})_{k\in\mathbf{N}}$ and $(\Phi_k : U \to L^2(\Omega))_{k\in\mathbf{N}}$ such that for any μ in U the sequence $(\Lambda_k(\mu))_{k\in\mathbf{N}}$ is the family of eigenvalues of $-\Delta + V + W_\mu$ counted according to their multiplicities, $(\Phi_k(\mu))_{k\in\mathbf{N}}$ is an orthonormal basis of corresponding eigenfunctions and, moreover, $\Lambda_k(0) = \lambda_k(V,\Omega)$ and $\Phi_k(0) = \phi_k(V,\Omega)$ for every $k \in \mathbf{N}$.*

3 Genericity: topologies and definitions

Let us recall that every complete metric space X is a Baire space, that is, any intersection of countably many open and dense subsets of X is dense in X. The intersection of countably many open and dense subsets of a Baire space is called a *residual* subset of X. Given a Baire space X and a boolean function $P : X \to \{0,1\}$ we say that P is a *generic property* if there exists a residual subset Y of X such that every x in Y satisfies property P, that is, $P(x) = 1$.

In the following the role of X will be played by $L^\infty(\Omega) \times L^\infty(\Omega)$ or $L^\infty(\Omega)$.

4 The triple (Ω, V, W) is generically fit for control with respect to the pair (V, W)

Here below we prove that, given $\Omega \in \Xi$, for a generic pair $(V,W) \in L^\infty(\Omega) \times L^\infty(\Omega)$ the triple (Ω, V, W) is fit for control.

Let us start by recalling a known result on the generic simplicity of eigenvalues (see (Albert, 1975; Uhlenbeck, 1976)).

Proposition 4.1 (Albert). *Let $\Omega \in \Xi$. For every $k \in \mathbf{N}$ the set*

$$\mathcal{R}_k = \{V \in L^\infty(\Omega) \mid \lambda_1(V,\Omega), \dots, \lambda_k(V,\Omega) \text{ simple}\} \tag{4}$$

is open and dense in $L^\infty(\Omega)$. Hence, the spectrum $\sigma(V,\Omega)$ is, generically with respect V, simple.

We generalize Proposition 4.1 as follows.

Proposition 4.2. *Let $\Omega \in \Xi$. For every $K \in \mathbf{N}$ and $q = (q_1, \dots, q_K) \in \mathbf{Q}^K \setminus \{0\}$, the set*

$$\mathcal{O}_q = \left\{ V \in L^\infty(\Omega) \mid \sum_{j=1}^{K} q_j \lambda_j(V,\Omega) \neq 0 \right\} \tag{5}$$

is open and dense in $L^\infty(\Omega)$. Hence, the spectrum $\sigma(V,\Omega)$ forms, generically with respect V, a non-resonant family.

Proposition 4.1 is clearly a special case of Proposition 4.2, since $\mathcal{R}_k = \cap_{j=1}^k \mathcal{O}_{e_{j+1}-e_j}$, where $e_1, \dots, e_{k+1}$ denotes the canonical basis of $\mathbf{R}^{k+1}$.

The proof of Proposition 4.2 is based on the following lemma.

Lemma 4.3. *Let $\Omega \in \Xi$ and ω be a nonempty, open subset compactly contained in Ω and whose boundary is Lipschitz. Let v belong to $L^\infty(\omega)$ and $(V_k)_{k\in\mathbf{N}}$ be a sequence in $L^\infty(\Omega)$ such that $V_k|_\omega \to v$ in $L^\infty(\omega)$ as $k \to \infty$ and $\lim_{k\to\infty} \inf_{\Omega\setminus\omega} V_k = +\infty$. Then, for every $j \in \mathbf{N}$, $\lim_{k\to+\infty} \lambda_j(V_k,\Omega) = \lambda_j(v,\omega)$. Moreover, if $\lambda_j(v,\omega)$ is simple then (up to a choice of sign) $\lim_{k\to+\infty} \phi_j(V_k,\Omega) = \phi_j(v,\omega)$ in $L^2(\Omega)$, where $\phi_j(v,\omega)$ is identified with its extension by zero outside ω. When both $\lambda_j(v,\omega)$ and $\lambda_m(v,\omega)$ are simple, then*

$$\lim_{k\to+\infty} \int_\Omega V_k \phi_j(V_k,\Omega)\phi_m(V_k,\Omega) = \int_\omega v\phi_j(v,\omega)\phi_m(v,\omega). \tag{6}$$

Proof of Proposition 4.2. The second part of the statement clearly follows from the first one, since

$$\{V \in L^\infty(\Omega) \mid \sigma(V,\Omega) \text{ non-resonant}\} = \cap_{q\in\cup_{k\in\mathbf{N}}(\mathbf{Q}^k\setminus\{0\})} \mathcal{O}_q.$$

Fix $K \in \mathbf{N}$ and $q = (q_1, \dots, q_K) \in \mathbf{Q}^K \setminus \{0\}$. Let $\mathcal{O}_q$ be defined as in (5). The openness of $\mathcal{O}_q$ in $L^\infty(\Omega)$ follows directly from the continuity of the eigenvalues on V. (See Theorem 2.8.)

The density of $\mathcal{O}_q$ is obtained by an analytic perturbation argument. Fix $V \in L^\infty(\Omega)$. Let ω be a d-orthotope compactly contained in Ω and W a measurable bounded function on ω such that $(\lambda_k(W,\omega))_{k\in\mathbf{N}}$

is a $\mathbf{Q}$-linearly independent family. (The existence of such ω and W is obtained in (Chambrion *et al.*, 2008, Section 6.3) for $d = 3$ and the proof extends with no extra difficulty to the general case $d \in \mathbf{N}$.)

For every $t \in \mathbf{R}$ let V_t be defined as $V + t$ on $\Omega \setminus \omega$ and as

$$\frac{1}{1+t}V + \frac{t}{1+t}W$$

on ω. Notice that $V_0 = V$ and that $t \mapsto V_t$ is an analytic function from $[0, \infty)$ to $L^\infty(\Omega)$. It follows from Theorem 2.9 that there exists a family $(\Lambda_k(\cdot))_{k\in\mathbf{N}}$ of analytic functions such that $\sigma(V_t, \Omega) = \{\Lambda_k(t) \mid k \in \mathbf{N}\}$ for every $t \in [0, \infty)$ and $\Lambda_k(0) = \lambda_k(V, \Omega)$.

Notice that, as $t \to +\infty$, V_t converges uniformly to $+\infty$ on $\Omega \setminus \omega$ and to W on ω. Therefore, according to Lemma 4.3, for every $k \in \mathbf{N}$ the function $\Lambda_k(t)$ tends to some $\lambda_{h(k)}(W, \omega)$ as $t \to +\infty$, where $h : \mathbf{N} \to \mathbf{N}$ is a bijection.

Since $(\lambda_k(W, \omega))_{k\in\mathbf{N}}$ is a $\mathbf{Q}$-linearly independent family, we have that for every injective map $\varphi : \{1, \ldots, K\} \to \mathbf{N}$,

$$\sum_{j=1}^{K} q_j \lambda_{h(\varphi(j))}(W, \omega) \neq 0.$$

Therefore, for every φ, $t \mapsto \sum_{j=1}^{K} q_j \Lambda_{\varphi(j)}(t)$ is an analytic function with nonzero limit as $t \to +\infty$. As a consequence, $\sum_{j=1}^{K} q_j \lambda_j(V_t, \Omega) \neq 0$ for all but a countable subset of t. In particular, there exists $t > 0$ arbitrarily small such that $\sum_{j=1}^{K} q_j \lambda_j(V_t, \Omega) \neq 0$. The proof is concluded, since $V_t \to V$ in $L^\infty(\Omega)$ as $t \to 0$. ■

The following theorem extends the analysis from V to the pair (V, W), combining the generic non-resonance of the spectrum of $-\Delta + V$ with a genericity connectedness condition on the matrices $B^{(n)}(\Omega, V, W)$.

Theorem 4.4. *Let $\Omega \in \Xi$. Then, generically with respect to $(V, W) \in L^\infty(\Omega) \times L^\infty(\Omega)$ the triple (Ω, V, W) is fit for control and, in particular, (Ω, V, W, U) is approximately controllable in the sense of its density matrices for every $U \subset \mathbf{R}$ with nonempty interior.*

Proof. Recall that $\mathcal{R}_k$, defined in (4), is open and dense in $L^\infty(\Omega)$. If V belongs to $\mathcal{R}_k$, then the eigenfunctions $\phi_1(V, \Omega), \ldots, \phi_k(V, \Omega)$ are uniquely defined in $H_0^1(\Omega)$ up to sign. It makes sense, therefore, to define

$$\mathcal{U}_k = \{(V, W) \in \mathcal{R}_k \times L^\infty(\Omega) \mid \int_\Omega W \phi_{j_1}(V, \Omega)\phi_{j_2}(V, \Omega) \neq 0 \text{ for every } 1 \leq j_1, j_2 \leq k\}.$$

As it follows from the unique continuation theorem, for every $1 \leq j_1, j_2 \leq k$ the product $\phi_{j_1}(V, \Omega)\phi_{j_2}(V, \Omega)$ is a nonzero function on Ω. Therefore, $\mathcal{U}_k$ is dense in $L^\infty(\Omega) \times L^\infty(\Omega)$. Its openness follows, moreover, from the continuity of $V \mapsto \{\phi_j(V, \Omega), -\phi_j(V, \Omega)\}$ on $\mathcal{R}_k$ for $j = 1, \ldots, k$ (see Theorem 2.8).

The proof is concluded by noticing that (Ω, V, W) is fit for control if (V, W) belongs to

$$(\cap_{k\in\mathbf{N}}\mathcal{U}_k) \cap \left(\cap_{q\in\cup_{k\in\mathbf{N}}\mathbf{Q}^k\setminus\{0\}}\mathcal{O}_q \times L^\infty(\Omega)\right),$$

which is a countable intersection of open and dense subsets of $L^\infty(\Omega) \times L^\infty(\Omega)$.

5 Generic controllability with respect to one single argument

The following technical result will be useful in the discussion below.

Lemma 5.1. *Let $\Omega \in \Xi$ and V a non-constant absolutely continuous function on Ω. Then there exist $\omega \in \Xi$ compactly contained in Ω with Lipschitz boundary such that $\sigma(0, \omega)$ is simple and a reordering $h : \mathbf{N} \to \mathbf{N}$ such that*

$$\int_\omega V \phi_{h(l)}(0, \omega)\phi_{h(l+1)}(0, \omega) \neq 0$$

for every $l \in \mathbf{N}$.

5.1 Generic controllability with respect to W

We shall prove in this section that for a fixed potential V, generically with respect to $W \in L^\infty(\Omega)$, (Ω, V, W, U) is effective. Notice that (Ω, V, W) cannot be fit for control if the spectrum of V is resonant, independently of V. In this regard the result is necessarily weaker than Proposition 4.2, where the genericity of the fitness for control was proved. The precise statement of our result is given by the following proposition.

Proposition 5.2. *Let $\Omega \in \Xi$, V an absolutely continuous function on Ω and $U \subset \mathbf{R}$ with nonempty interior. Then, generically with respect to W, (Ω, V, W, U) is effective.*

Proof. Given a reordering h of $\mathbf{N}$, we will denote by $\mathcal{R}_l^h$ the set of potentials $\tilde{V} \in L^\infty(\Omega)$ such that $\lambda_{h(l)}(V, \Omega)$ and $\lambda_{h(l+1)}(V, \Omega)$ are simple.

We prove the proposition by showing that, for a suitable reordering h, for each Z (playing the role of $V + uW$ for some fixed $u \in U \setminus \{0\}$) in an open and dense subset of $\mathcal{R}_l^h$,

$$\int_\Omega (V - Z)\phi_{h(l)}(Z, \Omega)\phi_{h(l+1)}(Z, \Omega) \neq 0, \quad (7)$$

for every $l \in \mathbf{N}$. Define

$$\mathcal{A}_l^h = \{Z \in \mathcal{R}_l^h \mid (7) \text{ holds true}\},$$

where h has to be fixed later.

Each $\mathcal{A}_l^h$ is open in $L^\infty(\Omega)$ due to Theorem 2.8. In order to prove their density fix $Z_0 \in L^\infty(\Omega)$. We want to prove that Z_0 belongs to the closure of $\mathcal{A}_l^h$ for every $l \in \mathbf{N}$ for a suitable h.

Consider first the case in which V is constant. Then

$$\int_\Omega (V - Z)\phi_j(Z,\Omega)\phi_m(Z,\Omega) = - \int_\Omega Z\phi_j(Z,\Omega)\phi_m(Z,\Omega) \qquad (8)$$

for every $Z \in \mathcal{R}_{\max(j,m)}$.

Fix a nonempty, open subset ω compactly contained in Ω, whose boundary is Lipschitz and such that the spectrum $\sigma(0,\omega)$ is simple. For instance, ω can be taken as an orthotope whose side's length are non-resonant. (The simplicity of the spectrum of the Laplace-Dirichlet operator on ω is actually generic among sufficiently smooth domains, as proved in (Micheletti, 1972; Uhlenbeck, 1976).)

Let $z \in L^\infty(\omega)$ and assume that z is not $L^2(\omega)$-orthogonal to $\phi_\gamma(0,\omega)\phi_\mu(0,\omega)$ for every $\gamma, \mu \in \mathbf{N}$. (Such z exists because each product $\phi_\gamma(0,\omega)\phi_\mu(0,\omega)$ is not identically equal to zero and the intersection of countably many open nonempty subspaces of the Baire space $L^\infty(\omega)$ is nonempty.) Then, each derivative of

$$\varepsilon \mapsto \int_\omega \varepsilon z \phi_\gamma(\varepsilon z,\omega)\phi_\mu(\varepsilon z,\omega)$$

at $\varepsilon = 0$ is equal to

$$\int_\omega z\phi_\gamma(0,\omega)\phi_\mu(0,\omega) \neq 0.$$

By Theorem 2.9, there exists $\bar\varepsilon$ such that the spectrum $\sigma(\bar\varepsilon z,\omega)$ is simple and

$$\int_\omega \bar\varepsilon z\phi_\gamma(\bar\varepsilon z,\omega)\phi_\mu(\bar\varepsilon z,\omega) \neq 0$$

for every $\gamma, \mu \in \mathbf{N}$.

Consider now an analytic curve $t \mapsto Z_t$ in $L^\infty(\Omega)$ for $t \geq 0$ such that $Z_t|_\omega \to \bar\varepsilon z$ in $L^\infty(\omega)$ as $t \to \infty$ and $\lim_{t\to\infty} \inf_{\Omega\setminus\omega} Z_t = +\infty$. (The curve Z_t can be constructed as in the proof of Proposition 4.2.) By analyticity we have that for almost every $t \geq 0$ the spectrum of $-\Delta + Z_t$ is simple and

$$\int_\Omega Z_t\phi_\gamma(Z_t,\Omega)\phi_\mu(Z_t,\Omega) \neq 0$$

for every $\gamma, \mu \in \mathbf{N}$. In particular this is true for some t arbitrarily small, implying that Z_0 belongs to the closure of $\mathcal{A}_l^h$ for every reordering h and every $l \in \mathbf{N}$.

Let now V be non-constant. Let $\omega \subset \Omega$ and h be as in the statement of Lemma 5.1.

Similarly to what done above, take an analytic curve $t \mapsto Z_t$ in $L^\infty(\Omega)$ for $t \geq 0$ such that $Z_t|_\omega \to 0$ in $L^\infty(\omega)$ as $t \to \infty$ and $\lim_{t\to\infty} \inf_{\Omega\setminus\omega} Z_t = +\infty$. According to Lemma 4.3,

$$\int_\Omega (V - Z_t)\phi_{h(l)}(Z_t,\Omega)\phi_{h(l+1)}(Z_t,\Omega) \to \int_\omega V\phi_{h(l)}(0,\omega)\phi_{h(l+1)}(0,\omega) \neq 0,$$

for every $l \in \mathbf{N}$. Moreover, by analyticity, for almost every $t \geq 0$ the spectrum of $-\Delta + Z_t$ is simple and

$$\int_\Omega (V - Z_t)\phi_{h(l)}(Z_t,\Omega)\phi_{h(l+1)}(Z_t,\Omega) \neq 0$$

for every $l \in \mathbf{N}$. This implies that Z_0 belongs to the closure of $\mathcal{A}_l^h$. ■

5.2 Generic controllability with respect to V

This section is devoted to the proof of the following result.

Proposition 5.3. *Let $\Omega \in \Xi$ and $U \subset \mathbf{R}$ with nonempty interior. Fix W non-constant and absolutely continuous on Ω. Then, generically with respect to $V \in L^\infty(\Omega)$, (Ω, V, W, U) is effective.*

Proof. Let $k \in \mathbf{N}$. We shall prove that there exists a reordering h such that the set of $V \in \mathcal{R}_l^h$ such that

$$\int_\Omega W\phi_{h(l)}(V,\Omega)\phi_{h(l+1)}(V,\Omega) \neq 0 \qquad (9)$$

is open and dense in $L^\infty(\Omega)$ for every $l \in \mathbf{N}$. Its openness follows directly from Theorem 2.8.

As for its density, apply Lemma 5.1 with W playing the role of V. Then there exist $\omega \in \Xi$ compactly contained in Ω and a reordering h such that $\sigma(0,\omega)$ is simple and

$$\int_\omega W\phi_{h(l)}(0,\omega)\phi_{h(l+1)}(0,\omega) \neq 0$$

for every $l \in \mathbf{N}$. Let $(V_t)_{t\geq 0}$ be an analytic curve in $L^\infty(\Omega)$ converging to 0 in ω and such that $\inf_{\Omega\setminus\omega} V_t$ converges to ∞ for $t \to \infty$.

Then, Theorem 2.9 and Lemma 4.3 imply that, for almost every t, $\sigma(V_t,\Omega)$ is simple and

$$\int_\Omega W\phi_{h(l)}(V_t,\Omega)\phi_{h(l+1)}(V_t,\Omega) \neq 0$$

for every $l \in \mathbf{N}$.

6 Conclusion

In this paper we proved that once (Ω, V) or (Ω, W) is fixed, the bilinear Schrödinger equation on Ω having V as uncontrolled and W as controlled potential is generically approximately controllable with respect to the other element of the triple (Ω, V, W).

A natural question is whether a similar property holds with respect to the dependence on Ω. It makes sense to conjecture that it does but the proof of this fact seems hard. Fix V and W absolutely continuous on $\mathbf{R}^d$ with W nowhere locally constant. One important remark is that the dependence of $\lambda_k(\Omega, V)$ is not necessarily analytic with respect to Ω, as it would be the case if V was analytic. (A genericity non-resonance result for the spectrum in the case $V = 0$, for instance, has been proved along these lines in (Privat and Sigalotti, 2008).) Similarly, the quantities $\int_\Omega W\phi_k(\Omega, V)\phi_j(\Omega, V)$ do not in general vary analytically with respect to Ω. Hence, the pattern of the proofs seen in the previous sections could not be followed. A partial result going in the right direction can be found in (Beauchard *et al.*, 2008), where the authors prove that for $V = 0$ and W regular enough, for a generic $\mathcal{C}^3$ domain $\Omega \subset \mathbf{R}^2$ one has $\int_\Omega W\phi_1(\Omega, 0)\phi_j(\Omega, 0) \neq 0$ for every $j \in \mathbf{N}$. The proof of this fact in (Beauchard *et al.*, 2008) is very technical and ingenious. Its extension to less regular domains and potentials and to the case of higher dimensions looks an extremely hard task.

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