GEOMETRIC METHOD FOR A PROBLEM OF SYNTHESIS OF THE ROBUST CONTROL ON LINEAR POLYHEDRONS

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One of the actual problems of the objectives of control is the problem of synthesizing the law of control, effective under the conditions of various indeterminacy and limitations on motion properties. At the same time the deriving of the robust laws of control is very important under the conditions of illegible data. In this work it is shown, that the solving of the mentioned problem comes to solution of some certain logarithmic correlations from the specially created connected phase plane. As far as smooth phase limitations are concerned, these correlations can be converted into algebraic equations, according to the synthesized law of control. The investigation in the cases of uneven limitations, which describe linear polyhedrons in state space, is performed in this work. It is shown, that in case of being symmetrical to set point of the coordinate system the solvability of synthesizing problem is all defined by the random apex of the polyhedron. If the required condition of the symmetry is not fulfilled, than the minimum amount of the random polyhedron apexes is concerned for analyzing the solvability of the objective. The description (symmetrization) of the phase plane is concerned with the help of the polyhedrons, symmetrical in any way. At the same time the system` indeterminacy is considered straight according to the algebraic correlations of the symmetric polyhedrons` apexes.

1. INTRODUCTION

The problem of controlling the complex objects under the conditions of various indeterminacy and limitations on motion properties is one of the most relevant problems. Many works, [1]-[5], are dedicated to its solution. Applying of known methods doesn't let to solve the problem effectively. It's concerned with complicated limitations set for this system. If, for example, the phase limitations are expressed directly in algebraic correlations of the space states, then to fulfill them especially set functionals should be applied or rather complicated functional equations should be solved, which means that usually the limitations are considered marginally. Besides that, under the conditions of indeterminacy in system's parameters or in external influence, it's not effective to use mentioned functional correlations as they get even more complicated in such case.

Though, one of the main problems in synthesis of regulators is working out a direct method of the synthesis, which allows considering the system's limitations straight away in conditions of various indeterminacy and forming effective robust control laws. A detailed investigation of solution of this problem, along with the case of linear system with the limitations set as phase polyhedron is performed in this work. The suggested approach is based on the method of variation of the phase limitations, instead of solving the problem of synthesis basing on forming the symmetrical phase limitations. This allows not only to expand the set of the permissible solutions, but also to simplify the procedure getting them.

2. STATEMENT OF THE TASK

Lets take into consideration a system close to a general system type:

$$\dot{x} = f(x,u),\ x(t_0) = x_0\ , t \ge t_0 \qquad (1)$$

Where x, u - $n\times 1$ $m\times 1$, $r\times 1$ are vectors of condition and control ; $f(.)$- $n\times 1$ vector-function satisfying a condition of existence and uniqueness of solution of the Cauchy problem.

The restrictions are defined in the following way:

$$x(t) \in Q(t), \quad t \ge t_{0,} \qquad (2)$$

$$where\ Q(t) = \left\{ x \in R^n : \psi_i(x,t) \le 0,\ i \in \overline{1,x} \right\} \qquad (3)$$

$\psi_i(x,t)$ - continuously differentiable functions.

Lets designate the border of the set Q(t) as ΓQ(t), and as ΓQ_i(t) - *hyper surfaces* of the type

$$\Gamma Q_i = \left\{ x \in R^n : \psi_i(x,t) = 0 \right\},\ i \in \overline{1,x} \qquad (4)$$

Then ΓQ_i(t) consists of x separated parts of type ΓQ(t) ∩ ΓQ_i(t).

Lets also consider the system in case of being prone to external disturbances and containing internal indeterminacies (parametric and structural). Then they could be defined as equation

$$\dot{x} = \varphi(z,x,u,\upsilon), \quad x(t_0) = x_0, t \ge t_0, \qquad (5)$$

Where $\upsilon - r\times 1$ is a vector of disturbance, z – px1 - vectorial parameter, which characterizes internal indeterminacies of the system. υ *and* z satisfy the conditions $\upsilon \in V, z \in Z$

Where V, Z – some given sets in R^r and R^p.

Let's consider $u = \tilde{u}(x)$ to be a control law, which has set form (i.e. it has any set structure). Lets say, that $\tilde{u}(x)$ has equivalent structure.

Then the problem of synthesis may be formulated as following:

For the system (1) or (5) it is necessary to synthesize a control law for a desired structure $\tilde{u}(.)$, providing the fulfillment of phase limitations (2) for all trajectories x(t), for which $x(t_0) \in Q(t_0)$ and possible influence of external and internal indeterminacies (6) is taken into consideration.

The solution of the stated problem is based on the usage of some geometric properties of system trajectory..

3. CREATING OF "TRANSPARENT" AND "OPAQUE" AREAS IN THE STATE SPACE

Let's investigate the system (1). For each continuously differentiable limitation function $\psi_i(x,t), i \in \overline{1,\chi}$ it is possible to form sets:

$$\rho_i^- = \left\{ x \in R^n : \left(\nabla_x \psi_i, f(.)\right) + \frac{\partial \psi_i}{\partial t} < 0 \right\} - \text{"}opaque\text{" } sets$$

$$\rho_i^+ = \left\{ x \in R^n : \left(\nabla_x \psi_i, f(.)\right) + \frac{\partial \psi_i}{\partial t} > 0 \right\} - \text{"}transparent\text{" } sets$$

$$\Gamma\rho_i = \left\{ x \in R^n : \left(\nabla_x \psi_i, f(.)\right) + \frac{\partial \psi_i}{\partial t} = 0 \right\} - border\ of\ sets\ \rho_i^-\ u\ \rho_i^+$$

$$где\quad \nabla_x \psi_i = \left[\partial\psi_i/\partial x_1, \partial\psi_i/\partial x_2, ..., \partial\psi_i/\partial x_n, \right]^T - function\ \psi_i(x,t)\, gradient,$$

$(d,g)-$ scalar product in Euclidean arithmetic space R^n

Furthermore, let's form an additional set:

$$Q_i^\delta(t) = \left\{ x \in R^n : \psi_i(x,t) \le \delta \right\}, \delta \ge \min_x \psi_i(x,t),\quad i \in \overline{1,\chi}$$

with a border

$$\Gamma Q_i^\delta(t) = \left\{ x \in R^n : \psi_i(x,t) = \delta \right\}, i \in \overline{1,\chi}$$

- which are the sets and surfaces of the functions' levels $\psi_i(.)$. Then

$$Q(t) = \bigcap_{i=1}^{x} Q_i(t), Q_i^0(t) = Q_i(t), \quad \Gamma Q_i^0(t),\ \delta = 0 \quad (7)$$

Lets say, that a set N$^+$ is "transparent" for a set Q$_i$(t), if along an arbitrary trajectory x(t), passing through N$^+$ at any t (i.e. at $x(t) \in \mathrm{N}^+$), function $\psi_i(.)$ is increasing (i.e. through the element of set N$^+$ trajectory x(t) is striving for exiting from Q$_i$(t) or for movin away from Q$_i$(t)).

So, the set N$^-$ - "transparent" for a set Q$_i$(t), if function $\psi_i(.)$ is decreasing along an arbitrary trajectory x(t) at $x(t) \in \mathrm{N}^-$ (i.e. through the element of set N$^-$ trajectory x(t) cannot exit from Q$_i$(t) or striving for getting into it).

It is clear, that for each set Q$_i$(t)

$$N^+ = \rho_i^+, \quad N^- = \rho_i^-, i \in \overline{1,\chi}$$

Is true, as along arbitrary x(t)

$$\frac{d}{dt}\psi_i(x(t),t) = \dot{\psi}_i = \left(\nabla_x \psi_i, f(.)\right) + \frac{\partial \psi_i}{\partial t}$$

It means, that inequalities $\dot{\psi}_i > 0$, $\dot{\psi}_i < 0$ mean increasing and decreasing of $\psi_i(.)$ along x(t) accordingly. And it's evident , that ρ_i^+ -"transparent", and ρ_i^- - "opaque" sets for Q$_i$(t).

Possible behavior of system trajectories in "transparent" and "opaque" areas is shown on Fig. 1

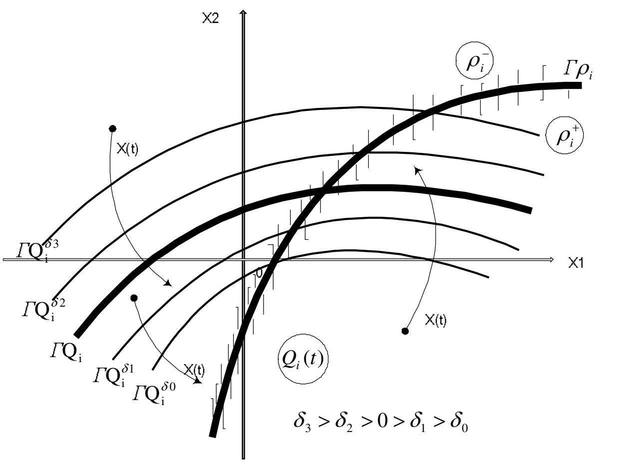

Fig. 1 Behavior of x(t) in ρ_i^-, ρ_i^+ areas

It is obvious, that if at some moment of time $t \ge t_0$ $x(t) \in \rho_i^- \bigcap Q_i(t)$, then x(t) will never get out Q$_i$(t) through the elements $\rho_i^- \bigcap \Gamma Q_i(t)$ .Obviously x(t) exceeds the borders of Q$_i$(t) through the elements of a set $\rho_i^+ \bigcap \Gamma Q_i(t)$.

Thus, to ensure phase limitations it is enough to fulfill the following conditions.

3.1 Statement 1.

Arbitrary trajectory x(t) of the system (1) at $x(t_0) = x_0 \in Q(t_0)$ will not exceed the bounds of Q(t) at t>t$_0$, if the following correlations are fulfilled

$$\Gamma \mathrm{Q(t)} \bigcap \Gamma \mathrm{Q_i}(t) \subset \rho_i^-, \quad i \in \overline{1,\chi}, t \ge t_0 \quad (8)$$

This statement is correct, which directly comes from a thesis, that x(t) cannot get out from $Q_i(t)$ through the elements of the set

$$\rho_i^- \bigcap \left[\Gamma Q(t) \bigcap \Gamma Q_i(t) \right] \subset \rho_i^- \bigcap \Gamma Q_i(t)$$
$$if \quad x_0 \in Q(t_0)$$

4. FORMING OF THE CORRELATIONS FOR SYNTHESIZING THE CONTROL LAW

Let's use conditions (8) for solving the stated problem. It is obvious, that if for some $i \in \overline{1,\chi}$

$$\Gamma Q \bigcap \Gamma Q_i \subset \rho_i^-$$

then the border $\Gamma\rho_i$ of the set ρ_i^- must not intersect a section $\Gamma Q \bigcap \Gamma Q_i$. So, this condition is equal to following:

$\Gamma\rho_i \bigcap \left[\Gamma Q \bigcap \Gamma Q_i \right] = \varnothing$ (Ø-designation of an empty set) (9)

or

$$\Gamma\rho_i \bigcap \left[\Gamma Q \bigcap \Gamma Q_i \right] = \{S^i\} \qquad (10)$$

Where S^i – is a point of contact of the surfaces $\Gamma\rho_i$ *and* $\Gamma Q \bigcap \Gamma Q_i$.

From (10) follows, that the gradients of contacting(tangent) surfaces in S^i have opposite directions as it is shown on Fig. 2.

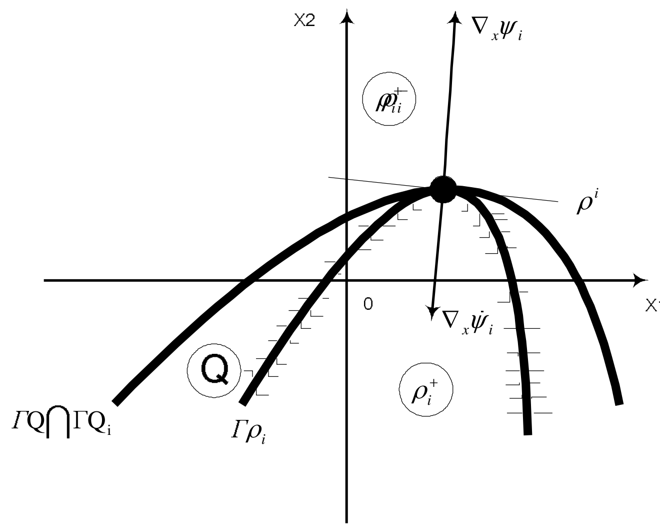

Fig.2 Implementation of conditions (10)

Here: $\nabla_x \psi_i, \nabla_x \dot{\psi}_i$ - function $\psi_i(.)$ *and* $\dot{\psi}_i(.)$ gradients. Obviously, (10) implements if and only if, the next equation is right:

$$\left. \begin{array}{l} \nabla_x \psi_i + \lambda_i \nabla_x \dot{\psi}_i = 0, \lambda_i > 0 \\ x \in \Gamma Q(t) \bigcap \Gamma Q_i(t) \end{array} \right\} \qquad (11)$$

In general case with an arbitrary in some way statement of parts $\Gamma Q \bigcap \Gamma Q_i$, $i \in \overline{1,\chi}$ conditions (10) or(11) will fulfill more rarely, than conditions (9). It is possible to find such functions $\psi_i(.)$, for which (11) will never fulfill. Must be admitted, that condition (11) can be rationally used in case of smooth bound ΓQ(t), which is described by only one function of limitation. So in future we will consider, that, if Q(t) is given as some phase polyhedron (set with uneven bound), then the general conditions for providing phase limitations are the correlations (9).

Obviously, the conditions (9) can be represented by

$$\left. \begin{array}{l} \psi_i(x,t) = 0 \\ \dot{\psi}_i(x,u,t) = 0 \\ \exists \nu \in \overline{1,\chi} \backslash i \quad , that \ \psi_\nu(x,t) > 0 \end{array} \right\} \qquad (12)$$

The result leads to the correctness of the following statement.

4.1 Statement 2.

For ensuring the phase limitations on the system (1) using the control law $\tilde{u}(x)$ the fulfillment of the correlation (12) is enough at $\forall i \in \overline{1,\chi}$ for the stated law $\tilde{u}(x)$.

In some cases correlations (12) may be a direct solution in analytical form regarding the parameters of control law of the desired structure. However, the analytical solution is not necessary in general case. It is enough to shown, when at given $\tilde{u}(x)$ correlations (12) are solvable, i.e. when solutions set x of an equation

$$\psi_i(x,t) = \dot{\psi}_i(x, \tilde{u}(x), t) = 0$$

does not intersect the section $\Gamma Q \bigcap \Gamma Q_i$. Stated case is shown on the Fig. 3.

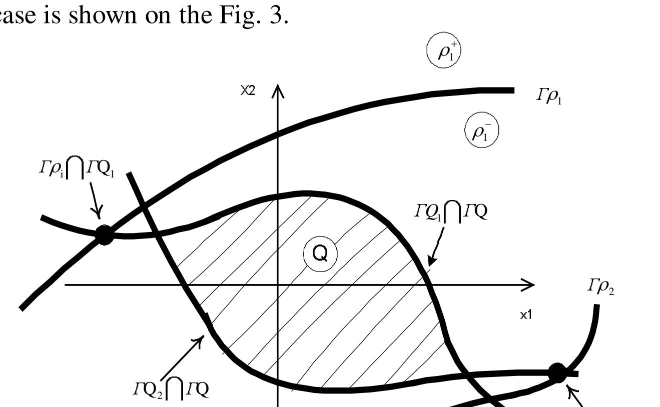

Fig.3. Fulfillment of conditions (9)

Depending on a control system type and on properties of the phase polyhedron Q various methods of analysis of the equations (9), (12) are possible. Lets investigate one of them, which represent a case, when system (1) or (5), and a polyhedron Q, are linear.

5. REALIZATION OF SYNTHESIS ALGORITHM FOR LINEAR SYSTEMS

Let system (1) or (5) to be linear and can be represented by following equations:

$$\dot{x} = Ax + Bu \, , \, x(t_0) = x_0 \, , t \geq t_0 \tag{13}$$

or

$$\dot{x} = Ax + Bu + D\upsilon \, , \, x(t_0) = x_0 \, , t \geq t_0 \tag{14}$$

Where matrices A, B, D have formats, concerted with vectors x, u, υ accordingly. And for (14)

A=A(z) , B=B(z) , D=D(z).

Besides, the limitation functions $\psi_i(x,t), i \in \overline{1,\chi}$, are also linear and are described by correlations

$$\psi_i(x,t) = (\alpha^i, x) - q_i(t) \leq 0 \, , i \in \overline{1,\chi} \tag{15}$$

Where α^i - given nx1-vectors , $q_i(t)$ - given scalar and continuously differentiable functions.

We are determining synthesized control law in linear form, i.e.

$$\tilde{u}(x) = Kx$$

Then equations (13), (14) lead to:

$$\dot{x} = \tilde{A}x \, , \, x(t_0) = x_0 \, , t \geq t_0 \tag{16}$$

$$\dot{x} = \tilde{A}x + D\upsilon \, , \, x(t_0) = x_0 \, , t \geq t_0 \tag{17}$$

Where $\tilde{A} = A + BK$

Therefore lets investigate system (16) for a distinctness

If

$$\nabla_x \psi_i = \alpha^i \quad , \partial\psi_i / \partial t = -\dot{q}_i \, , \quad then$$

$$\dot{\psi}_i = \left(\alpha^i, \tilde{A}x\right) - \dot{q}_i = \left(\tilde{A}^T\alpha^i, x\right) - \dot{q}_i \, , i \in \overline{1,\chi} \tag{18}$$

Lets use the circumstance, that for an arbitrary hyper plane $\Gamma \subset \mathrm{R}^n$: $(\alpha, x) - q = 0$ - vector α is orthogonal, i.e. $\alpha \perp \Gamma$

Then considering equations for hyper planes ΓQ_i , $\Gamma\rho_i$ we will come to $\alpha^i \perp \Gamma Q_i, \tilde{A}^T\alpha^i \perp \Gamma\rho_i$ (19)

Lets determine conditions, under which correlations (12) will fulfill.

Lets designate arbitrary apex of polyhedron Q, belonging to a part $\Gamma Q_i \bigcap \Gamma Q$, as M_j^i i.e. $M_j^i \in \Gamma Q_i \bigcap \Gamma Q$

M_i^j is considered to be formed by hyper planes ΓQ_{ij} $i_j \in \overline{1,\chi_j^i}$. Considering statement 2 for ensuring phase limitations it is enough for each hyper plane $\Gamma\rho_i$ to intersect an *facet* ΓQ_i in such way, that intersection points do not belong to polyhedron Q. This condition is equal to a statement, that M_j^i must be in a half-space ρ_i^- , as it is shown on Fig. 4.

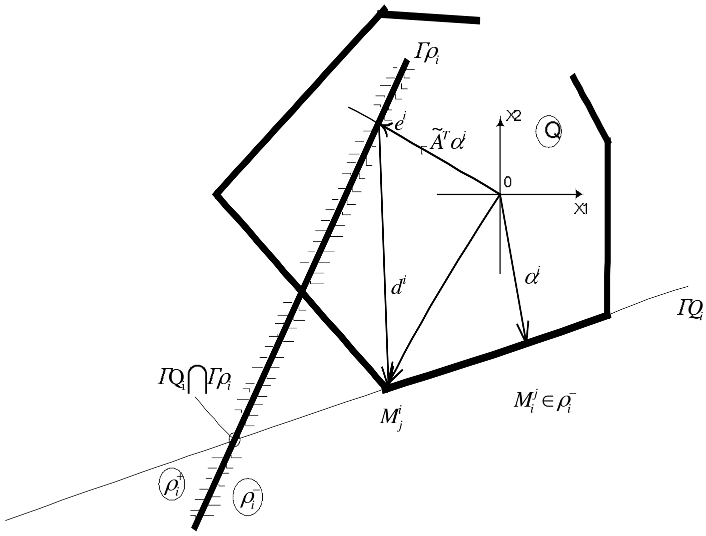

Fig.4 Required arrangement of hyper plane $\Gamma\rho_i$

Here:

$$d^i = M_i^j - e^i$$

e^i – vector, which has the same direction as vector $\tilde{A}^T\alpha^i$ (i.e. $e^i \perp \Gamma\rho_i$), but at the same time its endpoint belongs to $\Gamma\rho_i$. Thus, the length of e^i , i.e. $\left\| e^i \right\|$, corresponds with the distance between 0 and $\Gamma\rho_i$ and can be determined from an equation

$$\left\| e^i \right\| = \frac{|\dot{q}_i|}{\left\| \vec{\tilde{A}}^T \alpha^i \right\|}$$

and for a vector e^i itself the following correlation is right

$$\left\| e^i \right\| = \left\| e^i \right\| \frac{\vec{\tilde{A}}^T\alpha^i}{\left\| \tilde{A}^T\alpha^i \right\|} = \frac{\dot{q}_i}{\left\| \tilde{A}^T\alpha^i \right\|^2} \cdot \tilde{A}^T\alpha^i \tag{22}$$

Obviously:

$$M_j^i \in \rho_i^- \, , \, i_j \in \overline{1,\chi_j^i} \tag{23}$$

if and only if, when the inequality (24) is fulfilled

$$\left(\tilde{A}^T\alpha^i, d^i\right) < 0 \tag{24}$$

Considering (20),(22)

$$\left(\tilde{A}^T\alpha^i, d^i\right) = \left(\tilde{A}^T\alpha^i, M_j^i\right) - \dot{q}_i \tag{25}$$

The result leads to the correctness of the following statements.

5.1 Statement 3.

For an arbitrary apex $M_j^i \in \Gamma Q_i \bigcap \Gamma Q$, $i_j \in \overline{1,\chi_j^i}$ the condition (23) fulfills if and only if, when the following inequality is right

$$\left(\tilde{A}^T\alpha^i, M_j^i\right) - \dot{q}_i < 0,\ i \in \overline{1,\chi_j^i} \qquad (26)$$

5.2 Statement 4.

For ensuring the correlation (8) the fulfillment of the following inequalities is needed and enough

$$\left(\tilde{A}^T\alpha^i, M_j^i\right) - \dot{q}_i < 0,\ j \in \overline{1,N_i} \qquad (27)$$

where N_i is total number of apexes on the edge of $\Gamma Q_i \bigcap \Gamma Q$

Correlations (27) allow get simple enough solution and analysis for solvability concerted with matrix K, which is linearly in (27) at given phase limitations. Already known methods of solving linear inequalities can be used for this.

Let (27) to fulfill for all hyperplanes ΓQ_{i_j}, $i_j \in \overline{1,\chi_j^i}$, forming M_j^i apex. Then, if Q is a polyhedron, symmetrical to point of origin, then the correlations (27) will be valid for an apex $\bar{M}_j^i \in Q$ too, where $\bar{M}_j^i = -M_j^i$ because of the stated symmetry of Q. And $\bar{M}_j^i$ is forming hyperplanes $\Gamma \bar{Q}_{i_j}$, $i_j \in \overline{1,\chi_j^i}$, which are symmetrical t ΓQ_{i_j}, $i_j \in \overline{1,\chi_j^i}$ accordingly. If the polyhedron Q contains only the edges ΓQ_{i_j} and $\Gamma \bar{Q}_{i_j}$, $i_j \in \overline{1,\chi_j^i}$, which contain M_j^i and $\bar{M}_j^i$, then for synthesizing desired regulator correlations (27) are enough. In case of random phase set it is possible to implement a procedure of its symmetrization as it is shown on Fig.5.

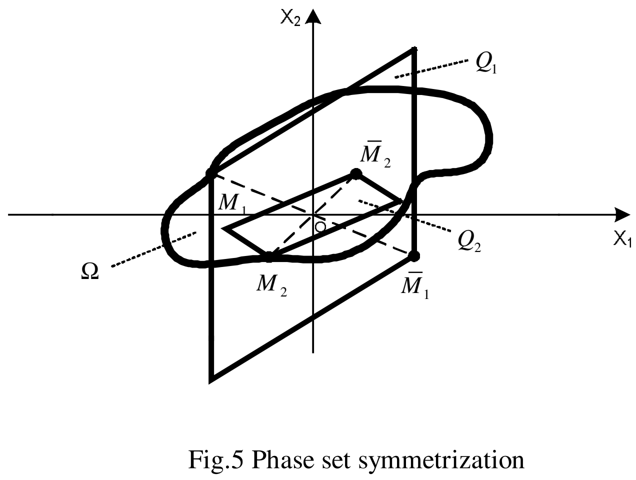

Fig.5 Phase set symmetrization

And then, on the base of (27), in apexes M_1, M_2 we synthesize the regulator.

6. CONCLUSION

The proposed method allows synthesizing a regulator of the desired structure effectively enough. This method is rather simple and the analysis of synthesis problem solvability is also simple, because the system of linear inequalities concerted with the regulator parameters is investigated. If symmetrization of initial phase set is used, then the problem of synthesis may be simplified by reducing its dimension.

7. REFERENCE

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