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One-Dimensional Dynamics

and

Winnerless Competition of Patterns

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Building dynamical models to study the neural basis of behavior has long history. Recently a dynamical principle, called winnerless competition (WLC), was suggested [1]-[3]. In such models, given by multidimensional dynamical systems, spatio-temporal coding is realized in the form of deterministic trajectories of a moving along heteroclinic orbits that connect certain saddle fixed points and saddle limit cycles in the state space. The separatrices connecting these saddle states correspond to sequential switching from one active state – specific neurons or groups of neurons – to another.

For modelling information of this kind, we propose to use one-dimensional maps with positive topological entropy. Such maps have a countable set of saddle cycles and an uncountable set of homoclinic and heteroclinic orbits connecting these cycles.

In [2] P.Seliger, L.Tsimring, and M.Rabinovich write:

"*The ability to process sequential information is one of the most important functions of living and artificial intelligent systems. In spite of the long history of studies of sequential learning and memory, little is known about dynamical principles of learning and remembering of multiple events and their temporal order by neural systems.*

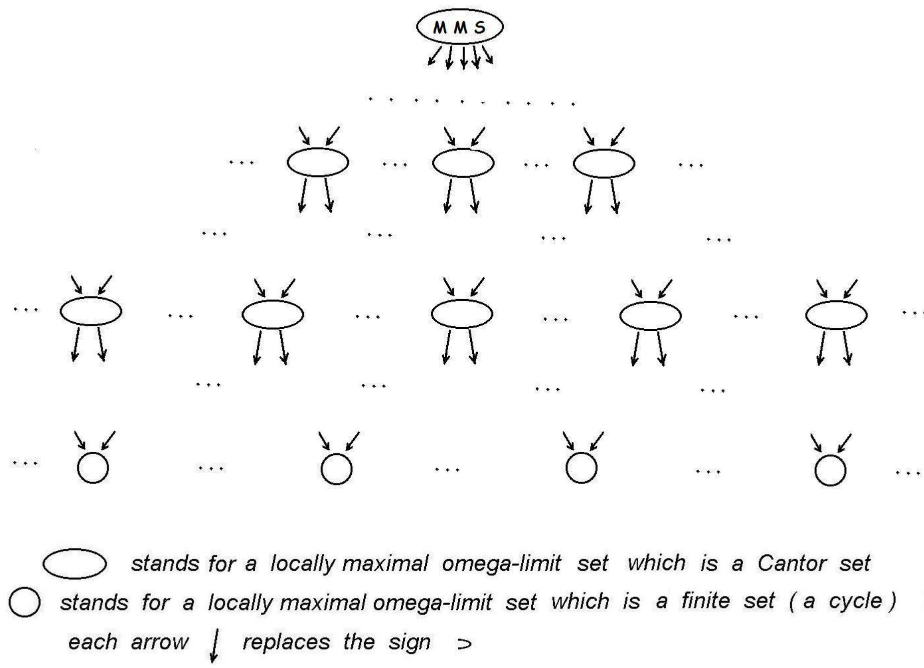
We propose a dynamical principle of winnerless competition WLC, that can be the basic mechanism of the sequential memory.

The essence of the idea is that the sequential memory is encoded in a multidimensional dynamical system with a complex heteroclinic trajectory connecting a sequence of saddle points. Each of the saddle points represents an event in a sequence to be remembered. The specific structure of the phase space is such that each saddle point can have many stable directions but only a single unstable direction. All saddle points are unidirectionally connected by these one-dimensional unstable separatrices. Once the state of the system approaches one fixed point representing a certain event, it is drawn along an unstable separatrix toward the next fixed point, and so on.

The existence of such heteroclinic structure is determined by specific connections between neurons within the WLC neural network. These connections are formed by the sensory inputs caused by sequential events in a sequence."

In order to simulate the dynamics of processes in the brain it makes sense to use dynamical systems containing strange attractors or, more general, mixing attractors. (We say that a set $\mathbb{A}$ of the phase space of a dynamical system is a mixing attractor if $\mathbb{A}$ is an attractor (i.e., an invariant closed set attracting all points from some its neighborhood) such that any two subsets of $\mathbb{A}$, open relatively $\mathbb{A}$, intersect with time.) Using such dynamical systems, we get great opportunity for modelling the information processing: mixing attractors usually contain a countable set of saddle cycles and an uncountable set of homoclinic and heteroclinic orbits connecting these cycles.

Even one-dimensional maps allow us to show the advantage of such possibility[4]-[6]. In typical situation every mixing set $\mathbb{A}$ contains a huge number of local repellors, in particular, periodic orbits that form a dense subset of $\mathbb{A}$. The most important representers of such repellors are so-called locally maximal ω-limit sets (or basis sets): *an ω-limit set $\mathbb{B}$ is called locally maximal set (LMS) if there exists its neighbourhood which does not contain any ω-limit set $\widetilde{\mathbb{B}} \supset \mathbb{B}$.* For one-dimensional dynamical systems, each LMS is always a finite set (i.e., a cycle) or a Cantor set with a dense subset of periodic orbits (these Cantor sets are mutually disjoint or are nested, one within the other).



For our goals, of importance are the following properties of one-dimensional dynamical systems:

- *Each LMS, being a local repellor, is a saddle (a set of saddle type) because of the presence of the feedback phenomenon.*
- *Any two LMS are connected by a huge number of heteroclinic orbits, and moreover, each such heteroclinic orbit "reachs" the LMS in a finite time.*
- *For any collection of LMS, there is a heteroclinic contour (closed path) connecting all these LMS (and in an arbitrary order).*
- *And finally, by the use of local perturbations only, it is possible to transform each LMS (which is a repellor) into an attractor.*

Below we are restricted ourselvs to the simple difference equation

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, ...,$$

i.e., we consider the case of discrete time. Moreover, we assume that x is a scalar, i.e., $x \in R^1$, and the function f is unimodal, i.e., f has one maximum only. Such functions also generate dynamical systems with mixing attractors but they are essentially simpler both for mathematical study and in their physical realization. For simlicity, we can think that f is a quadratic function, for example, the "chaotic parabola" $f(x) = 4x(1-x)$ or the tent map

$$f(x) = \begin{cases} 2x, & 0 \le x < 1/2, \\ 2(1-x), & 1/2 \le x \le 1. \end{cases}$$

Let $\mathcal{M} = \{p_1, p_2, ..., p_k\}$, $1 < k < \infty$, be a set of periodic points belonging to k cycles of f, which represent "stored episodes (events)" of the "stored information" in the memory.

For the tent map or the "chaotic parabola" and for other similar (for example, topologically conjugated to them) maps, all periodic orbits are local repellors. If there are no input signals, the random walk by "stored episodes" happens in the memory. This walk is subordinated to the probability distribution generated by the invariant measure of f. So, each "episode" $p_s \in \mathcal{M}$ can "be lifted" from the memory in a short time interval determined by orbit's instability (for example, the Lyapunov exponent) and some "identification threshold" for p_s.

Another situation takes place when a certain input signal arises, for example, in a response to the question "Do you know Mr X?". If "Mr X" corresponds to p_s, then this signal should change its associated unstable cycle $P_s = \{f^i(p_s), i = 0, 1, ...\}$ to a cycle which is attractive on the certain time interval. It is an easy metter to get this desirable attractive cycle instead the repulsive one. For example, we can assume that the impulse

$$g(x) = \begin{cases} f'(p_s)(x - p_s), & |x - p_s| < \delta_s, \\ 0, & |x - p_s| \ge \delta_s, \end{cases}$$

is just the answering reaction. Herein δ_s is the "identification threshold" of p_s. In this case, for the difference equation

$$x_{n+1} = \overline{f}(x_n) = f(x_n) + g(x_n),$$

the cycle P_s is attractive because for the (discontinuous) map $\overline{f}$ the multiplicator of P_s is equal to 0. Moreover, when f is an arbitrary quadratic (or tent) map, the cycle P_s is a "global" attractor for $\overline{f}$, and almost each orbit is attracted by this cycle very quickly.

Put

$$x_{n+1} = \overline{f}(x_n, n) = f(x_n) + g(x_n, n),$$

where

$$g(x, n) = \begin{cases} \begin{cases} f'(p_{s_i})(x - p_{s_i}), & |x - p_{s_i}| < \delta_{s_i}, \\ 0, & |x - p_{s_i}| \ge \delta_{s_i},\ p_{s_i} \in \mathcal{M}, \end{cases} & n_i \le n < \overline{n}_i, \\ 0, & \overline{n}_i \le n < n_{i+1},\ i = 1, 2, ..., \end{cases}$$

$n_i, \overline{n}_i \in \mathbb{N}$, $n_1 < \overline{n}_1 \le n_2 < ...$, δ_{s_i} is the "identification threshold" for p_{s_i}. It is easy to see that the previously unstable cycle $P_{s_i} = \{p_{s_i}, f(p_{s_i}), f^2(p_{s_i}), ...\}$ is transformed into a stable one, and moreover, for the tent map or the "chaotic parabola", this cycle will be a *temporary attractor* of the map $\overline{f}$ *on the time interval* $[n_i, \overline{n}_i]$.

We say that a set $\mathcal{A}$ *is a temporary attractor on the time interval* $[m_1, m_2]$ *for the (nonauthonomous) DS generated by the equation* $x_{n+1} = h(x_n, n)$ *if* $h(x, n)$ *is independent of* n *for* $n \in [m_1, m_2]$ *and* $\mathcal{A}$ *is an attractor of DS generated by the equation* $x_{n+1} = \widetilde{h}(x_n)$, *where* $\widetilde{h}(x) = h(x, n)$, $n \in [m_1, m_2]$.

By the same way, one can transform a Cantor set which is a locally maximal ω-limit set for the original map f into a temporary attractor of some map $\overline{f}$ ("excited" by a corresponding exterior impulse). Let $p'_s = f^{-j}(p_s) \notin P_s$, $1 \leq j < \infty$, and $g(x, n)$ be as above but p_s be replaced with p'_s. In this case, instead the cycle P_s, some Cantor set $\mathcal{K}_s$ containing P_s will be a temporary attractor of the map $\overline{f}$ on the corresponding temporal interval. This set $\mathcal{K}_s$ has to be just a locally maximal ω-limit set for the original map f.

Of course, similar considerations are also applicable to multidimensional maps.

It is important to study main properties of dynamical systems under such or similar perturbations. It is desirable to know the average time in which a trajectory reaches an LMS and the dependence of this time on the system parameters, the output signal, and etc. We can remark that the average time of reaching an LMS in the models considered above should be $\approx 1/\delta$, where δ is the "identification threshold" of the LMS.

References

[1] M.Rabinovich, *Spatio-Temporal Neural Coding and WLC Principle: From Experiments to Understanding*, Powerpoint presentation: *http://inls.ucsd.edu/documents/misha_pres.ppt*.

[2] P.Seliger, L.S.Tsimring, and M.I.Rabinovich, *Dynamics-based sequential memory: Winnerless competition of patterns*, Phys. Rev. E 67, 2003.

[3] V.S.Afraimovich, M.I. Rabinovich, and P.Varona, *Heteroclinic Contours in Neural Ensembles and the Winnerless Competition Principle*, Int. J. Bifurcation and Chaos, **14**, 4, 2004.

[4] A.N.Sharkovsky, *On ω-limit sets of discrete dynamical systems*, DSc Thesis, Kiev, 1966 (in Russian).

[5] A.N.Sharkovsky, *Classification of One-Dimensional Dynamical Systems*, Proc. ECIT-87, World Scientific Publisher, Singapore, 1989.

[6] A.N.Sharkovsky, *One-Dimensional Dynamics and Information Processing*, Abstracts of ECIT'2006 (Gargnano, Italy, September 10-16, 2006).