BEAM DYNAMICS OPTIMIZATION IN A LINEAR ACCELERATOR

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Abstract
The paper deals with the problem of dynamics optimization for charged particles beams in accelerators. The problem of simultaneous minimization of an integral and minimax functionals is considered. The variation for the combined functional and the necessary optimality condition are obtained. Application for a RFQ structure is suggested.

Key words
control, optimization, minimax, beam, RFQ accelerator

1 Introduction

In this paper we propose a new approach to the problem of control and optimization of an ensemble of trajectories.

The problem of optimization of the ensemble of trajectories was considered in [D.A. Ovsyannov, 1990], [D.A. Ovsyannikov, 1997].

Simultaneous optimization of some program motion and an ensemble of trajectories was studied in [A.D. Ovsyannikov, 2006]. Application of that approach to the problems of charged particles beam dynamics optimization was studied in [D.A. Ovsyannikov, 2012], [A.D. Ovsyannikov et al., 2015]. Those papers dealt only with smooth functionals.

Minimax functionals were initially introduced for the problem of optimization and control of an ensemble of trajectories in [Demyanov, 1973], and later applied in the field of the charged particles beam dynamics optimization [D.A. Ovsyannov, 1990].

The present work considers optimization using a combination of integral and minimax functionals, that was introduced in [Mizintseva, Ovsyannikov, 2015], [Mizintseva, Ovsyannikov, 2016].

The latest version of the combined functional contains a density variable, which allows to take particles distribution density into consideration [D.A. Ovsyannov, 1990].

In the current paper we consider applications of that approach to the problem of optimization of the longitudinal motion of the charged particles in a RFQ (radio frequency quadrupole) accelerator.

2 Statement of the optimization problem

Let us consider the equation of the longitudinal motion of the charged particles in a RFQ structure [Kapchinsky, 1982].

$$\frac{d\beta}{d\tau} = \frac{4eUT}{W_0 L} cos(Kz) cos(\tilde{\omega}\tau + \phi). \quad (1)$$

Here $\tau = ct$ is the independent variable (t — time, c — the speed of light), z is the longitudinal coordinate of a particle in the beam, β is the reduced speed of a particle, $\tilde{\omega} = 2\pi\omega/c$, — the effective frequency of the accelerating RF field, , U is the voltage on the electrodes, T is the acceleration effectiveness, W_0 and e are the rest energy and the charge of the particle, $K = 2\pi/L$, where L is the lenght of the period, ϕ is the phase of the synchronous particle. We also assume, that $L = \beta_s \lambda$, where λ is the wave length of the accelerating field and β_s is the reduced velocity of the synchronous particle.

We will consider the longitudinal motion of the particles in an equivalent travelling wave and take into account only the accelerating half-wave, so that the motion equation (1) for the synchronous particle can be rewritten as follows [Bondarev, Durkin, Ovsyannikov, 1999], [D.A. Ovsyannikov et al., 2006]

$$(\Lambda^2)' = 2k\eta cos\phi. \quad (2)$$

And the equation in deviations from the synchronous particle will be

$$\psi'' + 2\frac{\Lambda'}{\Lambda}\psi' + \frac{\Lambda''}{\Lambda}\psi - \frac{\eta}{\Lambda^2}(cos\phi - cos(\phi+\psi)) = 0. \quad (3)$$

Here $\psi = K(z_s - z)$, z_s is the coordinate of the synchromous particle, $\Lambda = \beta_s/\beta_0$ (β_0 is the initial reduced velocity of the synchronous particle), $\eta = UT/(UT)_{max}$, $k = \Omega/\tilde{\omega}$, $s = \Omega\tau \in [0, T_s]$ is the new independent variable, Ω is defined by the following expression

$$\Omega^2 = \frac{4(UT)_{max}}{W_0 L_0^2}, \tag{4}$$

where $L_0 = \beta_0 \lambda$.
In equations (2) and (3) derivatives are taken with respect to the new independent variable s.
For the construction of the quality functionals let us introduce the following penalty function

$$h(p,a) = \begin{cases} (p-a)^2, & p > a \\ 0, & p \leqslant a. \end{cases} \tag{5}$$

Using (5), we introduce a smooth functional, that evaluates the kinetic energy of the synchronous particle at the end of the accelerating structure and also consideres the defocusing factor restrictions for $s \in [0, T_s]$

$$J_1(u) = \int_0^{T_s} h(p_{def}, a_d)\, ds + (\Lambda^2(T_s)) - a_E)^2. \tag{6}$$

Here a_d, a_E are some fixed values, defocusing factor p_{def} is defined as follows [A.D. Ovsyannikov et al., 2009]

$$p_{def} = \frac{2k^2|sin\phi|}{\Lambda^2}. \tag{7}$$

We will also introduce a minimax functional considering $\rho = \rho(s, \psi, \psi')$ — the particles' distribution density along the beam of trajectories on the set of terminal positions Y of the system (3) in the normal form

$$J_2(u) = \max_{(\psi_{T_s}, \psi'_{T_s}) \in Y} p_w^2 \rho(T_s, \psi_{T_s}, \psi'_{T_s}). \tag{8}$$

Here parameter $p_w = (W_k - W_k^s)/W_k^s$ refers to the deviations of the energies of the particles in the beam from the energy of the synchronous particle, which in terms of Λ and ψ can be written as follows

$$p_w = (p_\beta + 1)^2 - 1, \quad p_\beta = -k\left(\psi' + \psi\frac{\Lambda'}{\Lambda}\right). \tag{9}$$

Functional (8) allows to consider the most deviating particles in the process of optimization.

3 Mathematical optimization

Let us generalize the problem stated in the previous section and consider the following systems of differential equations

$$\frac{dx}{dt} = f(t, x, u), \qquad x(0) = x_0. \tag{10}$$

$$\frac{dy}{dt} = F(t, x, y, u), \qquad y(0) = y_0 \in M_0. \tag{11}$$

$$\frac{d\rho}{dt} = -\rho \cdot div_y F(t, x, y, u), \quad \rho(0) = \rho_0(y_0). \tag{12}$$

Here $t \in [0, T]$ — independent variable; x — n–dimensional phase-vector that refers to Λ^2; $u = u(t)$ — r–dimentional piecewise continuous control vector-function from a class D that takes value in a compact set U, in the described model of particle dynamics in a RFQ: $u = (\eta(s), \phi(s))$; $f(t, x, u)$ — n–dimentional reasonably smooth vector-function; y — n–dimensional phase-vector that refers to the phase vector (ψ, ψ') from the previous section; $F(t, x, y, u)$ — n-dimensional reasonably smooth vector-function; M_0 — a compact set.
Equations (10)–(11) represent equations (2)–(3) rewritten in the normal form using new notations.
The solution of sub-system (10) is called program motion and the trajectories of system (11) are called disturbed motions or the ensemble of trajectories. Equation (12) describes the particles' distribution density $\rho = \rho(t, y(t)$ on the trajectories of sub-system (11).
On the solutions of sub-system (10) we will consider an integral functional, that repeats the structure of functional (6)

$$I_1(u) = \int_0^T \varphi_1(x(t, x_0, u))dt + g(x(T)). \tag{13}$$

And on the trajectories of system (11) we introduce a generalized minimax functional (8), that takes particles' distribution density into consideration

$$I_2(u) = \max_{y_T \in Y} \varphi_2(y_T, \rho(y_T)), \tag{14}$$

where Y is the set of terminal positions of the subsystem (11), defined the following way

$$\begin{aligned} Y = \{&y(T, x_0, y_0, u) \mid u \in D, \\ &x(0) = x_0, y(0) = y_0 \in M_0\}. \end{aligned} \tag{15}$$

Functions φ_1 and φ_2 in the expressions for the functionals (13) and (14) are non-negative smooth functions.
We will consider a combination of $I_1(u)$ and $I_2(u)$

$$I(u) = I_1(u) + I_2(u). \tag{16}$$

The combined functional allows to carry out simultaneous optimization of the program motion and the ensemble of disturbed motions using integral and minimax quality criteria.

4 Variation of the functional

Let us write down the variations equations correspodding to systems (10)–(12) [A.D. Ovsyannikov, 2006]

$$\begin{aligned}
&\frac{d\delta x}{dt} = \frac{\partial f}{\partial x}\delta x + \Delta_u f,\\
&\delta x(0) = 0;\\
&\frac{d\delta y}{dt} = \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y + \Delta_u F,\\
&\delta y(0) = 0;\\
&\frac{d\delta\rho}{dt} = -\delta\rho\cdot div_y F - \rho\frac{d(div_y\delta y)}{dt},\\
&\delta\rho(0) = 0.
\end{aligned} \tag{17}$$

Also let us introduce the variation equation for $div_y\delta y$

$$\begin{aligned}
&\frac{d(div_y\delta y)}{dt} = \frac{\partial(div_y F)}{\partial x}\delta x + \frac{\partial(div_y F)}{\partial y}\delta y +\\
&\qquad + \Delta_u div_y F,\\
&div_y\delta y(0) = 0.
\end{aligned} \tag{18}$$

Here and further operator Δ_u of some function f is defined the following way

$$\Delta_u f\,(t,x,u) = f(t,x,u+\Delta u) - f\,(t,x,u)\,. \tag{19}$$

The variation of the functional represented by a smooth function is

$$\delta I_1 = \int_0^T \frac{\partial\varphi_1}{\partial x}\delta x dt + \frac{\partial g(x(T))}{\partial x}\delta x(T). \tag{20}$$

Variation of the functional $I_2(u)$ considering the particles distribution density can be obtained as follows [D.A. Ovsyannov, 1990]

$$\delta I_2 = \max_{y_0\in R_T(u)}\left[\frac{\partial\varphi_2}{\partial y}\delta y(T) + \frac{\partial\varphi_2}{\partial\rho}\delta\rho(T)\right], \tag{21}$$

where $R_T(u)$ is a set defined by the following expression

$$\begin{aligned}
R_T(u) = \{&\bar{y}_0 : \bar{y}_0 \in M_0, \varphi_2(y(T,x_0,\bar{y}_0,u),\rho) =\\
&= \max_{y_0\in M_0}\varphi_2(y(T,x_0,y_0,u),\rho)\}.
\end{aligned} \tag{22}$$

Then the variation of the functional (16) is

$$\delta I = \delta I_1 + \delta I_2. \tag{23}$$

Let us choose auxiliary vector-functions ψ, λ and scalar function χ so that

$$\begin{aligned}
&\psi^{*'} + \psi^*\frac{\partial f}{\partial x} =\\
&= \frac{\partial\varphi_1}{\partial x} - \lambda^*\frac{\partial F}{\partial x} + \chi^*\rho\frac{\partial(div_y F)}{\partial x},\\
&\psi^*(T) = -\frac{\partial g(x(T))}{\partial x},\\
&\lambda^{*'} + \lambda^*\frac{\partial F}{\partial y} = \chi^*\rho\frac{\partial(div_y F)}{\partial y},\\
&\lambda^*(T) = -\frac{\partial\varphi_2(y_T,\rho_T)}{\partial y},\\
&\chi' = \chi div_y F,\\
&\chi(T) = -\frac{\partial\varphi_2(y_T,\rho_T)}{\partial\rho}.
\end{aligned} \tag{24}$$

Here and further symbol * stands for the operation of transposition of a vector or matrix.
The variation of the functional (23) using expressions (24) can be written as follows

$$\begin{aligned}
\delta I(u) = \max_{y_0\in R_T(u)} - \int_0^T &(\psi^*\Delta_u f + \lambda^*\Delta_u F -\\
&-\chi\rho\Delta_u div_y F)\,dt.
\end{aligned} \tag{25}$$

5 Optimality conditions

Let us intriduce Hamilton's function

$$\begin{aligned}
&H(t,x,y,\rho,\psi,\lambda,\chi,u) = \psi^* f(t,x,u) +\\
&+ \lambda^* F(t,x,y,u) - \chi\rho div_y F.
\end{aligned} \tag{26}$$

Using (26) we can rewrite the expression for the variation (23), so that

$$\begin{aligned}
\delta I(u) = \max_{y_0\in R_T(u)} - \int_0^T &\big(H(t,x,y,\rho,\psi,\lambda,\chi,\tilde{u}) -\\
- H(t,x,y,\rho,\psi,\lambda,\chi,u)\big)dt.&
\end{aligned} \tag{27}$$

Optimal control $u^0 = u^0(t)$, optimal trajectories $x_t^0 = x^0(t)$, $y_t^0 = y^0(t)$ and distribution density on the optimal trajectories $\rho_t^0 = \rho^0(t, y_t^0)$ comprise the so-called optimal process.

Theorem If $u^0 = u^0(t)$ is the optimal control, then for all $t \in [0, T]$ except for the discontinuity points of the control function we have

$$\begin{aligned} \min_{u \in U} \max_{y_0 \in R_T(u^0)} &(H(t, x_t^0, y_t^0, \rho_t^0, \psi_t^0, \lambda_t^0, \chi_t^0, u) - \\ &- H(t, x_t^0, y_t^0, \rho_t^0, \psi_t^0, \lambda_t^0, \chi_t^0, u^0)) = 0, \end{aligned} \quad (28)$$

where $\rho_t^0, \psi_t^0, \lambda_t^0, \chi_t^0$ can be found from equations (24) alongside the optimal process.

6 Conclusion

Simultaneous use of smooth and non-smooth functional in the problem of optimal control allows to perform optimization not only for the averaged values, but also considering the most deflecting particles.

Notably, not only the beam of trajectories is affected by the dynamics of the program motion, but also in the process of optimization $x(t)$ is affected by $y(t)$ through the auxiliary functions, defined by (24).

The obtained expression for the variation of the functional (27) can be used for directional methods of minimization. The proposed approach can be applied for various electro-physical structures, in this particular paper application for the mathematical model of the charged particle beam dynamics in RFQ structures was considered.

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